# Overt Subspaces of $\mathbb{R}^{\mathbf{n}}$

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## 1 Introduction

**1.1** Overtness has been introduced in a number of accounts as a dual to compactness that is invisible in classical topology. Here we shall try to be more helpful to general mathematicians by relating this unfamiliar idea to others that they have actually known for a long time.

We shall prove some simple theorems that characterise *overt subspaces* in locally compact metric spaces such as $\mathbb{R}^{\mathbf{n}}$. These characterisations turn out to be very similar to (but not the same as) the well known Newton–Raphson algorithm for solving equations.

When it is required to *find* a solution to an equation, it is of absolutely no help to postulate the *set of all* solutions. The theme of our investigation is that an overt "subspace" is a *problem that is amenable to solution* rather than a subset of points.

This concept has arisen in several different *constructive* formulations of topology and analysis, with definitions that dissolve into nothing when read verbatim in classical point–set topology. The significance comes from the unification of these disciplines with the theory of computation.

We want to convey the importance of overtness across classical, numerical, constructive and computable mathematics, so we shall write our technical development in traditional notation, instead of that of the various constructive disciplines. We begin with a brief survey of the many different roots of this idea, but it will not be necessary to understand all or any of the heterodox systems or their notation. We shall require no more knowledge of topology, metric spaces and computability than is to be found in any elementary text.

The prerequisite will instead be a willingness to take mathematical ideas *au naturel* and not try to shoe-horn them into a classical setting. It is the *explanation* of this notion on the boundaries of topology and computation, rather than the proof of theorems, that is the objective of this paper.

**1.2** We start from topology by thinking of an overt subspace $A \subset X$ as one that arises as *a fibre of some open map*. Recall that an ***open map*** is a continuous function $f : X \to Y$ for which the direct image $f_! U \equiv \{f(x) \mid x \in U\}$ of any open subspace $U \subset X$ is open in $Y$, whilst a ***fibre*** is simply the inverse image $f^{-1}(y) \equiv \{x \mid f(x) = y\} \subset X$ of a point $y \in Y$.

If your next step is to characterise such *fibres* classically then you will find yourself in a conceptual dead end. The following argument is constructed to get us around that obstacle by using continuous functions, maybe even classical ones.

We will show in Theorem 5.10 that a function $f : X \to Y$ between locally compact metric spaces is an open map iff the expression

$$d_y(x) \;\equiv\; d\big(x, f^{-1}(y)\big) \;\equiv\; \mathsf{inf}\,\{d(x,a) \mid f(a) = y\}$$

defines a continuous function $d_{(-)}(-) : X \times Y \to \mathbb{R}$. In this case, $f(x) = y \iff d_y(x) = 0$.

This is best illustrated by a *non*-example. For the squaring map $f : \mathbb{R} \to \mathbb{R}$,

$$f(x) \;=\; x^2 \qquad \text{and} \qquad d_y(x) \;=\; \begin{cases} \big|\,|\,x\,| - \sqrt{y}\,\big| & \text{if } y \geq 0 \\ \infty & \text{if } y < 0, \end{cases}$$

but $d_y(x)$ is not continuous in $y$ near 0. For an example where $d_y(x)$ has jumps but no infinities, you may like to work this out with $f(x) \equiv x^3 - x$.

**1.3** Any function $d_y(x)$ that arises from an open map in this way has the ***convergence property*** that

$$d_y(x) < r \implies \exists x'r'.\ d_y(x') < r' < \tfrac{1}{2}r \quad \wedge \quad d(x, x') < r - r',$$

which invites iteration towards some $a$ with $f(a) = y$.

The famous Newton–Raphson algorithm is (almost) an example of this situation. If $f : \mathbb{R} \to \mathbb{R}$ is differentiable with $\dot{f}(x) \neq 0$ for all $x$ then $f$ is an open map. In favourable circumstances (which we set out in Notation 6.3), $x' \equiv x + g(x)$ gives an improved approximation, where $g(x) \equiv (y - f(x))/\dot{f}(x)$, and $2|\,g(x)\,|$ is an (over)estimate of $d_y(x)$.

**1.4** Abstractly, $d_y(x)$ says how far any given point $x \in X$ is from the *nearest* solution $a \in X$ of the equation $f(a) = y$. The Newton–Raphson method, when it works, does this too, approximately.

We may also regard $d_y(x)$ as the distance between $x$ and the *set* $f^{-1}(y)$ of solutions, or $\infty$ if there are none. When the function $d_y(x)$ is continuous in ($x$ and) $y$, we may think of this parametric set $f^{-1}(y)$ as varying continuously in $y$ too. Theorem 5.10 says that this happens exactly when $f$ is an open map. However, considering $f^{-1}(y)$ in terms of set theory gets us nowhere, if our aim is to *find* a solution of a numerical problem.

For the squaring function, $f^{-1}(y)$ has two elements $\pm\sqrt{y}$ when $y > 0$. These get closer together as $y$ decreases, becoming the singleton $\{0\}$ when $y \equiv 0$, and they vanish when $y < 0$. Because of the last transition, $f^{-1}(y)$ fails to be a "continuously varying set" in the sense that interests us. In the cubic case $f^{-1}(y)$ may have 1, 2 or 3 elements, from which the distance function $d_y(x)$ "chooses" one, in a necessarily discontinuous way.

On the other hand, for a differentiable function $f$ with $\dot{f} \neq 0$, the solutions are isolated and (individually) define a continuous inverse (partial) function $f^{-1}$.

**1.5** Continuity or otherwise in a *parameter* is the way in which the issues that we want to discuss manifest themselves when we work in classical point–set topology. This is because excluded middle and choice prevent them from appearing in the non-parametric case. However, they appear in different forms in other disciplines.

The reason why you have never seen this definition of an overt subspace before is that, in *classical* point–set topology, *any* closed subspace $A \subset X$ can be expressed as a fibre of some open map. There are other definitions of overtness, but they all trivialise in the classical setting. So the idea, or at least the word, is useless there.

It was therefore in *constructive* and *computable* topology and analysis that this concept arose. There are several different approaches to these subjects, each with its own name and definition for the idea, but, as these separate disciplines have interacted more with one another, the word *overt* has begun to gain acceptance amongst them. Note that in everyday English this word is not a synonym of *open* derived from French, but means *explicit.* We shall find that the name is apt because overtness is associated with having *evidence* or even an *algorithm* for something, in particular for *solving* an equation.

As a matter of elementary general topology, overtness also fills a conceptual gap as the counterpart of *compactness* in the lattice duality that matches open with closed subspaces.

Since these disciplines also have different logical strengths, our parallel treatment of them has to be ambiguous about certain foundational issues, to which we will return in the Conclusion. However, it is better to understand the topological and computational issues before worrying about this. The *corollary* of this is that our characterisation of overtness is in fact a *different theorem* for each system, one extreme of which is that overtness trivialises in the classical case.

Let's survey these disciplines.

**1.6** The study of mathematical constructions containing a parameter $y$ that varies over a topological space $Y$ is called **Sheaf Theory**. In this subject, we may either

(a) always retain the explicit parameter $y$ and therefore have to consider different behaviours at different points $y \in Y$ of the space, or

(b) develop a way of describing the mathematical situation "simultaneously" throughout the space, so that the "truth value" of a statement is the *subspace* of $Y$ on which it is valid.

The second view shifts the parameter from the mathematical formulae to the logical ones. So, if we want to eliminate it altogether, we have to be willing to modify our logical operations to accommodate subspaces as truth values. This is most easily done by restricting attention to *open* subspaces and those operations that take and return open subspaces.

However, the open subspaces of a space such as $\mathbb{R}^{\mathbf{n}}$ do not form a Boolean algebra, so these truth values do not obey the principles of classical mathematics such as excluded middle ($P \vee \neg P$). Therefore, although the mathematicians who developed sheaf theory had a classical education, they found themselves forced to use the logic of the *Intuitionism* of L.E.J. Brouwer [Hey56, McL90].

**1.7** The evolution of sheaf theory led to the notions of *topos* and *locale*. **Locale theory** studies topology entirely in terms of the lattice of open subspaces, eschewing points altogether. Its classic text [Joh82] demonstrates that the apparently ubiquitous use of the Axiom of Choice in topology actually occurs at the stage where *points* need to be found to translate Choice-*free* theorems of locale theory into those of point–set topology.

It was Marshall Stone who first stressed the importance of the duality between algebra and topology. Traditionally, the spectrum of an algebra consisted of "points" that were identified using prime ideals or similar notions, and had a topology re-imposed on it. But often there are not *enough* abstract points to recover the original algebraic structure.

However, one can often define a lattice like that of open subspaces *simply* and *directly* from an algebraic structure, without needing to identify the points. For example, the Zariski topology for a commutative ring is (isomorphic to the opposite of) the lattice of its radical ideals, *i.e.* those $I$ for which $r^2 \in I \Rightarrow r \in I$. Johnstone's book surveys examples like this and shows how general topology can be developed without using points, working just with the lattices.

**1.8** Open maps $f : X \to Y$ and overt subspaces of $X$ can be defined quite easily in terms of open subspaces $U \subset X$ and $V \subset Y$ instead of points. The direct and inverse image operations, $f_!$ and $f^{-1}$ respectively, are related by

$$U \subset f^{-1}(f_! U), \qquad f_!(f^{-1}V) \subset V \qquad \text{and} \qquad f_! U \cap V \;=\; f_!(U \cap f^{-1}V),$$

of which the first two state the *adjunction* $f_! \dashv f^{-1}$. The third is known in categorical logic as the ***Frobenius law***, although its connection with Ferdinand Georg Frobenius and the similarly named property in Algebra is tenuous; the containment $\supset$ follows from the adjunction.

**Notation 1.9** For a given open map $f : X \to Y$ and point $y \in Y$, we shall write

$$\Diamond U \;\equiv\; (y \in f_! U) \qquad \text{and} \qquad \mathcal{M} \equiv \{U \subset X \mid y \in f_! U\}.$$

These both convey the same information; whilst the second ("family of open subsets") may be more familiar to many mathematicians, the ***modal operator*** $\Diamond$ is less cumbersome. We will ignore the dependency on $y$ (and $f$) until it becomes relevant.

We may see from the adjunction or directly that $f_!$ preserves unions, so these satisfy

$$\Diamond \bigcup_i U_i \iff \exists i.\ \Diamond U_i \qquad \text{and} \qquad \bigcup_i U_i \in \mathcal{M} \iff \exists i.\, U_i \in \mathcal{M}.$$

The so-called Frobenius law becomes

$$\Diamond U \ \wedge\ y \in V \Longrightarrow \Diamond(U \cap f^{-1}V) \qquad \text{and} \qquad U \in \mathcal{M} \ \wedge\ y \in V \Longrightarrow (U \cap f^{-1}V) \in \mathcal{M}.$$

However, we shall take just the ***join-preserving*** property as our *working definition* of an overt subspace in terms of open subspaces. That is, we ignore the Frobenius law for the time being, but we shall find in Corollary 7.11 that it is not needed.

**1.10** Overt locales were first studied by André Joyal and Miles Tierney [JT84] and by Peter Johnstone [Joh84a, Joh84b].

The category of locales defined in the topos of sheaves on a base locale $B$ is equivalent to the category whose objects are continuous maps $X \to B$ and whose morphisms are commutative triangles. It is therefore natural to match properties of locales with those of maps, which is a similar question to ours of looking at fibres [Joh91, page 101].

In particular, a locale $X$ is overt iff the map $X \to \mathbf{1}$ is open, so these authors called $X$ an ***open locale***. However, their name needs to be changed when we start talking about subspaces because any *fibre* of a map between Hausdorff spaces is *closed* (in the usual sense).

Function-spaces of the form $Y^X$ (for both point–set topology and locales) obey the appropriate universal property iff $X$ is locally compact, but to form such exponentials over a base space $B$ we need to use separation properties such as Hausdorffness. These in turn depend on closed inclusions, which are preserved by $(-)^X$ exactly when $X$ is overt [Joh84a].

**1.11** The study of overt locales involves a particularly delicate grasp of intuitionistic lattice theory, but we shall see that it is possible to appreciate overtness more easily in other formulations of topology. In particular, it is not really necessary to use the full lattice of *all* open subspaces. For example, the *open ball*

$$B_r(x) \;\equiv\; \{y \in X \mid d(x,y) < r\}$$

is ubiquitous in elementary analysis, where *general* open subspaces are hardly needed, and this paper will use them in a similar way.

**Formal Topology**, which was introduced by Giovanni Sambin [Sam87] based on an idea of Per Martin-Löf, works with systems of *basic* open subspaces such as balls. For such a system to determine a space, we must specify its ***cover relation***

$$a \lhd U, \qquad \text{meaning} \qquad O_a \;\subset\; \bigcup\{O_b \mid b \in U\},$$

where $O_a$ is the basic open subspace named by the symbol $a$ and $U$ is a set of such symbols. (This use of $a$ and $U$ is typical in Formal Topology but conflicts with our custom in this paper.)

A general open subspace is a union of basic ones, but formal topologists prefer to avoid discussing arbitrary unions. In particular, since the operator $\lozenge$ must preserve unions, we only need to define it on *basic* open subspaces: Adapting a notation that had been used in locale theory, Sambin writes $\mathsf{Pos}\,(a)$ for $\lozenge\, O_a$ and calls this a ***positivity predicate***. It has to satisfy the condition that

$$\mathsf{Pos}\,(a) \implies \exists b \in U.\, \mathsf{Pos}\,(b) \qquad \text{whenever} \qquad a \lhd U.$$

**1.12** Cantor Space provides a good example of the use of Formal Topology and the computational meaning of overt subspaces. Mathematicians know this space through the *middle-third* construction but computer scientists use infinite streams of binary digits instead.

A basic open subspace is named by a finite sequence of digits or a nested third of a particular resolution; such a subspace is also compact. Each one is covered by the two smaller basic opens that are named by adding a single extra digit or repeating the middle-third operation once. For example,

$$011 \quad \lhd \quad \{\ 0110, \quad 0111\ \}.$$

An overt subspace of Cantor space is then defined by declaring certain of the finite sequences to be "positive", with the requirement that some sequence is positive if and only if at least one of its single-digit extensions is too:

$$\mathsf{Pos}\,(011) \iff \mathsf{Pos}\,(0110) \;\vee\; \mathsf{Pos}\,(0111).$$

We say that such a predicate defines an ***inhabited*** overt subspace if the empty sequence (encoding the whole space) is positive. In that case, Dependent Choice allows us to select an infinite sequence all of whose finite initial segments are positive. We regard the subspace of

Cantor Space consisting of sequences that arise from such a process as the *extent* of the overt subspace that is defined abstractly by the $\Diamond$ operator.

Section 3 generalises this to the extent of any $\Diamond$ operator on a locally compact metric space and our Tangency Theorem will find points in a similar way.

**1.13** Cantor Space is a very simple example of a Formal Topology because the covers are disjoint, but when we present the real line in this way we need to say, for example,

$$(1,3) \quad \lhd \quad \{(0,2), \quad (1,4)\},$$

where now tbe basic open subspaces are the open intervals.

**Errett Bishop** developed Brouwer's Intuitionistic ideas into an account of elementary real analysis [BB85]. As a rule, it is as challenging a piece of mathematical research to strip excluded middle from traditional proofs as it was to find the original classical versions. Nevertheless, Bishop's skilful account is both easy to follow and gets on with the business of analysis without dwelling on counterexamples. The reader needs to learn little more than to stop assuming that equality of real numbers is decidable ($x = y \vee x \neq y$), rather as many classical mathematicians have learned not to use the Axiom of Choice. Indeed, the Bishop school is sometimes criticised by logicians for *not* setting out its formal system.

**1.14** Bishop's followers have extended his work to functional analysis. In so doing, they have often found themselves making use of the same notion of distance of a subspace $A \subset X$ from a point $x \in X$,

$$d(x, A) \;\equiv\; \mathsf{inf}\,\{d(x, a) \mid a \in A\},$$

that we used in our theorem about open maps between metric spaces. (Formulae for the distance between a point and a subspace go back to Felix Hausdorff [Hau14] and have been exploited by numerous authors since then.)

In order to make the classical form of many theorems valid constructively, $d(x, A)$ must be a *Euclidean* real number, rather than just a *lower* one (paragraph 2.5). When this is the case for all $x$, the subspace $A$ is called ***located***.

For example, it is well known in *classical* functional analysis that any surjective continuous linear map $f : X \to Y$ between Banach spaces is an open map. The kernel $K \equiv f^{-1}(0) \subset X$ is a closed linear subspace, and this is enough classically, *i.e.* we may define $f : X \twoheadrightarrow X/K$ and recover $K$ as its kernel. Constructively, however, $K$ has to be located. This agrees with our theorem above: the kernel is the fibre of an open map over 0 and must therefore be overt.

**1.15** Unfortunately, whilst we make several links in this paper with the ideas of the Bishop school, it is not one of the disciplines in which our results can be given a direct formal interpretaion. This is because we shall use the "finite open sub-cover" definition of compactness, which Bishop does not accept [BR87], whereas it is valid in Locale Theory and Formal Topology.

Specifically, we shall need local compactness of $X$ in order to make $d_y(x)$ lower semicontinuous in $y$ and upper semicontinuous *jointly* in $x$ and $y$ when $f$ is open. Without this assumption, our characterisation of open maps using continuous distance functions becomes this less interesting statement:

**Theorem** A continuous function $f : X \to Y$ is an open map with respect to the metric topology on $X$ iff $d_y(x)$ is upper semicontinuous in $y$.

**Proof** Both of these properties say that $f_! B_r(x) \subset Y$ is open. $\square$

Another dissonance with Bishop is that some of his definitions are conjunctions of properties that we prefer to consider separately. For example, he uses the word *compact* for a subspace that is closed and totally bounded, making it overt as well as compact in our terminology, total boundedness being one of the manifestations of overtness (Definition 7.9).

For examples of compact subspaces of $\mathbb{R}$ that are not overt, see Section J 11.

**1.17** The disciplines that we have mentioned so far were motivated primarily by mathematical questions. However, consideration of the underlying logic is not just a matter of piety but has practical mathematical consequences in itself.

One powerful result in Logic is Gerhard Gentzen's ***existence property*** [Gen35]. It says that, if we have a *proof* of an existentially quantified predicate $\exists x.\, \phi(x)$ then we may extract from this proof a *term* (value) $a$ and a proof of $\phi(a)$. This metatheorem (theorem about theorems) is only valid for *intuitionistic* logic and not for classical logic — giving another reason to develop mathematics constructively.

This subject, Proof Theory, treats proofs as mathematical objects in themselves, understanding each *logical* connective as a *mathematical* one. For example, a proof of an implication $A \Rightarrow B$ is a *function* that turns a proof of $A$ into one of $B$. Along with the interpretations of the other symbols, this analogy is known as the ***Curry–Howard correspondence***. One of the formal systems that exploits it is the Type Theory of Per Martin-Löf [ML84] that is used in Formal Topology.

In general, the computation that is involved in this "extraction" process can be *spectacularly* infeasible — when it is applied to a logic that is as general as mathematicians commonly use. However, there are weaker systems of logic and practical applications of them for which these ideas actually make Logic Programming a reasonable way of solving problems.

**1.18** The significance of such methods to us is that overtness is the topological embodiment of the *existential quantifier*, whilst an open subspace $U$ is a predicate $\phi(x)$ in a suitable weak logic.

Therefore, our notation $\Diamond U$ may alternatively be written as $\exists x \in A.\, \phi(x)$ (Corollary 3.6). From any proof of this, Gentzen's theorem obtains a point $a \in A$ satisfying $\phi(a)$, that is, $a \in A \cap U$. Intuitively, $\Diamond U$ says that the open subspace $U$ contains a solution to the problem that is encoded in the operator $\Diamond$ corresponding to an overt subspace. However, to define $\Diamond$ in this way would be to beg the question.

We saw with Cantor space that an overt subspace is inhabited if it satisfies the simple condition that the whole space is designated as a positive open subspace, $\mathsf{Pos}\,(X)$. In numerical analysis, some "boundary condition" typically provides this property. In either case, some *computation* needs to be done to find the solution. The intuition behind $\Diamond$ is therefore captured in a version of the Existence Theorem that we call the Tangency Theorem.

Our proof of this result is ostensibly topological, but it relies on a Choice Principle. Someone who thinks in purely mathematical terms may just take this principle as an article of faith, *i.e.* as an axiom. However, for a proof theorist or a logic programmer, it is a consequence of the syntactic analysis of the formal language in which the mathematical ideas are expressed and proved.

**1.19** In order to make the Existence or Tangency Theorem a natural part of general topology and tame the infeasibility of the computation, we need to cut the classical logic of point–set topology down to one that only deals in open subspaces and operations on them. The calculus of Abstract Stone Duality does this [J].

Even then, the Tangency Theorem is by no means a magic bullet, or even an explicitly given algorithm. We claim that the solution of problems can be embodied in the search for a point of an inhabited overt subspace. However, the other side of this coin is that the process of finding such a point potentially involves the solution of arbitrarily difficult numerical problems.

**1.20** The significance of the concept of overtness is therefore that

(a) pure mathematicians can use it as a notion in general topology, just as they would compactness, whilst

(b) to logicians and programmers it is the specification for a computation.

We will characterise overtness in metric spaces using ideas from the ancient history of analysis and show how to use it in the same idiom of other ideas in general topology. This requires slightly better logical hygiene but no major change to our habits of presenting mathematics. The reward is that mathematicians from the newer, more syntactic, disciplines that are nowadays found in Computer Science departments can take the *proofs* that we have found and turn them directly into *algorithms* for finding solutions to mathematical problems.

**1.21** The technical content of this paper is a characterisation of an overt subspace of a locally compact metric space in terms of

(a) an operator $\Diamond$ that says whether any open subspace $U$ touches $A$, so $\Diamond U$ means that $U \cap A$ is inhabited, which makes overtness a property of general topology like compactness;

(b) a distance function $d(x, A) \equiv \mathsf{inf}\,\{d(x, a) \mid a \in A\}$;

(c) an existential quantifier $\exists a\colon A.\ a \in U$; or

(d) a dense net ("computably representable" set of points), which is useful for combinatorics and computation.

Such connections have been identified before, in particular in [Spi10], but assuming that the subspace is closed, which we shall not do.

## 2 Overtness in metric spaces

In this section we show that any join-preserving operator $\Diamond$ on the open subspaces of a locally compact metric space, such as arises from each fibre of an open map, can be characterised in terms of a distance-like function.

**Remark 2.1** We shall work in a locally compact complete metric space $(X, d)$ such as $\mathbb{R}^{\mathbf{n}}$ or Cantor space. However, our objective is to *explain* overtness in metric spaces, rather than to give the most general result. We shall therefore occasionally make other assumptions that hold in $\mathbb{R}^{\mathbf{n}}$ but not in general metric spaces.

By convention, the variables $w, x, y, z$ will range over the space $X$, whilst $r, s, t, \epsilon, \delta$ denote strictly positive rational numbers.

We write $B_r(x) \equiv \{y \mid d(x, y) < r\}$ for the open ball with centre $x \in X$ and radius $r > 0$. We also write $\overline{B}_r(x)$ for is its closure, and assume that this is compact and that any open subspace $U \subset X$ has a ***basis expansion***

$$U = \bigcup_{x,r} \{B_r(x) \mid \overline{B}_r(x) \subset U\}.$$

Beware that this is stronger than just asking that the topology induced by the metric be locally compact. We also assume that

$$q < r \implies \overline{B}_q(x) \subset B_r(x).$$

**Notation 2.2** Given any predicate $\Diamond$ on open sets such that $\Diamond(\bigcup U_i) \iff \exists i.\ \Diamond U_i$, or a family $\mathcal{M}$ of open sets such that $(\bigcup U_i) \in \mathcal{M} \iff \exists i.\ (U_i \in \mathcal{M})$, we write

$$d(x) < r \quad \equiv \quad \Diamond B_r(x) \quad \equiv \quad B_r(x) \in \mathcal{M}.$$

We need to justify using a strict inequality and the same letter as the metric:

**Lemma 2.3** This notation satisfies

$$\begin{array}{lll l}
d(x) < r' < r & \implies & d(x) < r & \text{monotonicity} \\
d(x) < r & \implies & \exists r'.\, d(x) < r' < r & \text{roundedness} \\
d(x) < r \ \wedge\ d(x, y) < s & \implies & d(y) < r + s & \text{triangle law} \\
d(x) < r & \implies & \forall \epsilon.\, \exists y.\, d(x, y) < r \ \wedge\ d(y) < \epsilon & \text{convergence}
\end{array}$$

and the operator $\Diamond$ is recovered as

$$\Diamond U \iff \exists xr.\ d(x) < r \ \wedge\ B_r(x) \subset U.$$

**Proof** Replacing $d(-)$ by $d(-,z)$, all four properties hold for the metric: roundedness because $<$ on $\mathbb{R}$ satisfies it and convergence by putting $y \equiv z$.

Re-writing these statements about the metric using balls, we have

$$B_r(x) \;=\; \bigcup_{0<r'<r} B_{r'}(x),$$

$$d(x,y) < s \implies B_r(x) \;\subset\; B_{r+s}(y)$$

and
$$B_r(x) \;\subset\; \bigcup\{B_\epsilon(y) \mid y \in B_r(x)\},$$

whilst any open subspace $U$ satisfies the basis expansion

$$U \;=\; \bigcup_{x,r}\{B_r(x) \mid B_r(x) \subset U\}.$$

Applying $\lozenge$ to each of these expressions gives the stated results. $\square$

**Proposition 2.4** The function $d$ defined by

$$d(x) \;\equiv\; \mathsf{inf}\,\{r \mid d(x) < r\}$$

satisfies
$$\mathsf{inf} < r \iff \exists r'.\, d(x) < r' < r \iff d(x) < r$$

and is ***upper semicontinuous***.

**Proof** The (simplest) definition of upper semicontinuity is that $\{x \mid d(x) < r\} \subset X$ is open for each $r > 0$. Indeed, by roundedness and the triangle law,

$$\begin{aligned} d(x) < r &\implies \exists\epsilon.\, d(x) < r - \epsilon \\ &\implies \exists\epsilon.\, \forall x'.\, \big(d(x,x') < \epsilon \Rightarrow d(x') < r\big), \\ &\implies \exists\epsilon.\, B_\epsilon(x) \subset \{x' \mid d(x') < r\}. \end{aligned}$$

Beware that this way of defining $d(x)$ only says which positive rational numbers $r$ are *upper bounds* for it. $\square$

**Remark 2.5** There are plenty of classical functions that are upper semicontinuous but not continuous, such as the step function defined by $f(x) \equiv 0$ if $x < 0$ and $+1$ if $x \geq 0$. However, as will become apparent, these examples do not satisfy the other properties of our $d$. We have to take a more explicitly constructive view to see why we only have semicontinuity in general.

Beware that *semi*continuity of a function $d$ has nothing to do with continuity from the left or right in a real *argument* $x$, but is about its *values* $d(x)$. Constructively, the issue is that these values need not be Euclidean real numbers (in $\mathbb{R}$), but are weaker things called ***upper*** or ***descending reals***, the space of which is called $\overline{\mathbb{R}}$.

**Example 2.6** To find upper reals that are not Euclidean, we need to think in terms of computability. Consider a program whose termination is undecidable, such as a search for a proof that $0 = 1$ in the ambient logic. Let $g_n \equiv 0$ if the program has halted by time $n$ and $g_n \equiv 1$ if is still running. Then put

$$g \;\equiv\; 1 - \sum_{n=1}^{\infty} 2^{-n} g_n,$$

so $g > 0$ if the program ever terminates or 0 otherwise, but we cannot know this. However, it is easy to test whether $g < r$, by considering a *finite* sum whose length is easily determined by $r$.

The issue here is that we only know about $g$ or $d(x)$ *from above.* More complicated forms of the same situation often arise in analysis, as domains of definition that are obtained by a process of "continuation", such as when solving a differential equation. Such a problem tells us about the domain *from the inside.* Many papers have been written to say that the boundary of some domain of this nature is "not computable" — but this is to be expected, since we usually have no knowledge of it from the *outside*. The remarkable thing is when we have some other information saying that the region does have a computable boundary.

**Remark 2.7** Topologically, the space $\overline{\mathbb{R}}$ of upper reals may be unfamiliar, because it is *not Hausdorff*. In particular, singletons such as $\{0\}$ are not closed in it. Classically, one may view $\overline{\mathbb{R}}$ as $\mathbb{R} + \{-\infty, +\infty\}$ with a topology whose open subsets are *downwards-closed* open intervals $[-\infty, r)$, together with $\emptyset$ and $\overline{\mathbb{R}}$.

However, this is clumsy and wrecks the point that we have just made about computability. It is much better to represent the points $U \in \overline{\mathbb{R}}$ as *rounded upper* sets of *rationals*,

$$U \subset \mathbb{Q} \qquad \text{such that} \qquad r \in U \iff \exists r'.\, r > r' \in U,$$

which was the roundedness property for $d(x) < r$. Spaces such as $\overline{\mathbb{R}}$ that are determined by an order relation have been used in theoretical computer science since the 1970s to study the semantics of programming languages, but pure mathematicians have resisted incorporating them into their own curriculum.

**Definition 2.8** Ordinary ("Euclidean") real numbers are defined by giving both their upper and lower rational bounds. These sets are respectively an upper real $U$ and a lower real $D$ (the latter being defined in the same way but with the opposite order) that are *disjoint* sets of rationals also satisfying the condition

$$q < r \implies q \in D \ \vee \ r \in U,$$

which (somewhat confusingly) is called ***locatedness***. Such a pair is called a ***Dedekind cut***.

Note that, according to the definition that we adopt, if the real number that $D$ and $U$ define happens to be a *rational* number $q \in \mathbb{Q}$ then $q$ belongs to *neither* $D$ nor $U$, so our notion of cut is a slight modification of Richard Dedekind's original one [Ded72]. But the treatment of rationals is not the important issue, which is instead that the preceding example defines an *upper* real $U$ that has no partner to form a Dedekind cut.

**Lemma 2.9** For both the upper and Euclidean reals,

$$\mathsf{inf}\,\{d(x,a) \mid a \in A\} < r \iff \exists a \in A.\, d(x,a) < r.$$

**Proof** Since an upper real is a upwards-closed set of rational numbers, infima are computed as unions. If the numbers are Euclidean then they have lower cuts that respect this, but testing the equalities in this statement only requires the upper cut. □

**Lemma 2.10** If a function $d : X \to \mathbb{R}$ is both upper and lower semicontinuous then it is continuous.

**Proof** Any open subspace is a union of open intervals, whilst any open interval is an intersection $(a, b) = (-\infty, b) \cap (a, +\infty)$, so it suffices that inverse images of lower and upper rounded subsets be open. □

However, it is better to think of this situation as a *pair* of functions, one lower and the other upper semicontinuous, that co-operate to define a Dedekind cut for each point of $X$. Given that analysts already use semicontinuous *functions*, they should be willing to consider upper and lower *reals*. This is because many phenomena that are often regarded as pathological are simply cases where the upper and lower cuts come adrift, *i.e.* where the locatedness condition above fails.

Where it does exist, the lower partner of the upper $d$ that we have just defined will be described in Section 4.

If you are skeptical of treating the existential quantifier as a topological notion, or of dealing with logical operators on open subspaces, then maybe you will feel more comfortable with this metric-like function. However, we still have to check that, if we *define* $\Diamond$ from an arbitrary upper semicontinuous function $d$ satisfying the properties in Lemma 2.3, then it preserves unions and we may recover $d$. We shall do that in the next section, where we investigate *from what* it measures a distance. Section 4 provides the lower cut for $d(x)$.

## 3 Accumulation points

Now we will show that the two structures $\Diamond$ and $d$ give rise to a notion of accumulation point, for which we need Dependent Choice and Cauchy Completeness.

**Lemma 3.1** For any point $x_0$ with $d(x_0) < r_0$, there are sequences $(x_n)$ and $(r_n)$ such that

$$d(x_n, x_{n+k}), \quad d(x_n) \quad < r_n < 2^{-n} r_0,$$

so $(x_n)$ is a Cauchy sequence.

**Proof** From roundedness and converegence of $d$ (Lemma 2.3),

$$\begin{aligned} d(x) < r &\implies \exists r'.\ d(x) < r - r' < r \ \land\ 0 < r' < \tfrac{1}{2} r \\ &\implies \exists r' x'.\ d(x, x') < r - r' \ \land\ d(x') < r' < \tfrac{1}{2} r. \end{aligned}$$

Hence by iteration and Dependent Choice, there are sequences $(x_n)$ and $(r_n)$ such that

$$d(x_n, x_{n+1}) < r_n - r_{n+1} \qquad \text{and} \qquad d(x_n) < r_n < 2^{-n} r_0,$$

so

$$B_{r_0}(x_0) \ \supset\ \cdots\ \supset\ B_{r_n}(x_n)\ \supset\ B_{r_{n+1}}(x_{n+1})\ \supset\ \cdots$$

By the triangle law for the metric and induction, we also have $d(x_n, x_{n+k}) < r_n$, so $(x_n)$ is a Cauchy sequence. $\square$

For simplicity, we now assume that the metric space $X$ in which we are working is complete, *i.e.* that any Cauchy sequence has a unique limit. However, we shall return to this assumption at the end of this section, where we will see that techniques similar to the one that we are developing can be used to *construct* the Cauchy completion.

**Corollary 3.2** Any ball $B_r(x)$ that satisfies $\Diamond B_r(x)$ has some $a \in B_r(x)$ with $d(a) = 0$.
This means $\forall \epsilon > 0.\, d(a) < \epsilon$ and is equivalent to $\forall V.\ a \in V \implies \Diamond V$.
We call such an $a$ an ***accumulation point*** of $d$ or $\Diamond$.

**Proof** For the last part: [$\Rightarrow$] If $a \in V$ then $a \in B_r(a) \subset V$ for some $r$, but $d(a) < r$ so $\Diamond B_r(a)$ and $\Diamond V$ hold, by Lemma 2.3. [$\Leftarrow$] Consider $V \equiv B_r(a)$ for each $r > 0$. $\square$

**Theorem 3.3** ("Tangency") If an overt subspace defined by $\Diamond$ or $d$ ***touches*** an open subspace $U$, *i.e.*

$$\Diamond U, \qquad \text{or equivalently} \qquad \exists x r.\, d(x) < r \ \land\ B_r(x) \subset U,$$

holds, then $U$ contains an accumulation point of $\Diamond$ or $d$. $\square$

**Definition 3.4** The ***extent*** of $\Diamond$ or $d$ is the subspace of *all* accumulation points,

$$A \ \equiv\ \{a \mid d(a) = 0\} \ \equiv\ \bigcap_r \{a \in X \mid d(a) < r\} \subset X.$$

This is a $G_\delta$***-set***, *i.e.* the intersection of a countable family of open subspaces.

**Remark 3.5** Constructively, the subspace $A$ need not be closed in the sense of being the complement of an open subspace (note that $0 \equiv \{r \in \mathbb{Q} \mid 0 < r\}$ is not a closed point of $\overline{\mathbb{R}}$: Remark 2.7). We shall return to this in the next section.

However, it *is* closed in various weaker senses.

It is ***sequentially closed***: any convergent sequence $a_n \to x$ with $a_n \in A$ also has $x \in A$.

It is ***rest-closed*** [Vic07, Vic06] it contains all of its closure points. We call $x \in X$ a ***closure point*** of $A$ if every neighbourhood of $x$ intersects $A$, which is $d_A(x) = 0$ in our notation. This is because it suffices to consider open balls, so $x$ is a closure point iff $\forall r > 0.\, \exists a \in A.\, d(x, a) < r$.

**Corollary 3.6** We may express the Tangency Theorem above as

$$d(x) < r \iff \exists a \in A.\, d(x, a) < r \iff d(x, A) \equiv \mathsf{inf}\, \{d(x, a) \mid a \in A\} < r. \qquad \Box$$

This explains why we have re-used the letter $d$: it says how far away the nearest member of $A$ is from $x$.

**Corollary 3.7** In terms of $\Diamond$, the Tangency Theorem is

$$\Diamond U \iff \exists a \in A.\, a \in U,$$

so $\Diamond$ expresses ***existential quantification*** over $A$. $\Box$

When we need to give a *name* to $\Diamond$ or $d$, we sometimes call it after its extent $A$ and write $\langle A \rangle$ for $\Diamond$ and $d(x, A)$ for $d$. However, we shall see that $\Diamond$ and $d$ are much more important than the subset $A$.

**Definition 3.8** An overt subspace $\Diamond$ for which $\Diamond X$ is true is called ***inhabited***. Then by the basis expansion, $\exists xr.\, d(x) < r$. However, by the rules for $d$,

$$d(x) < r' < r \implies d(x) < r \implies \exists x'.\, d(x, x') < r \wedge d(x') < r',$$

whence $\exists x.\, d(x) < r \iff \exists x'.\, d(x') < r'$ for *any* $r$ and $r'$. That is, this statement is independent of $r$, so we just write $\exists x.\, d(x) < \infty$ in this case. Then, by the Tangency Theorem, $\Diamond$ has an accumulation point.

On the other hand, the ***empty subspace*** is also overt. Both $\langle \emptyset \rangle U$ and $d(x, \emptyset) < r$ are everywhere false, so we write $d(x, \emptyset) \equiv \infty$.

Hence any overt subspace is inhabited iff it has an accumulation point. This is not a tautology but a consequence of the Tangency Theorem.

**Remark 3.9** The ***union*** of two (or more generally an overt family of) overt subspaces is overt, with

$$\Diamond U \;\equiv\; \exists i.\, \langle i \rangle U, \qquad d(x) \;\equiv\; \mathsf{inf}\, d_i(x).$$

However, the ***intersection*** of two overt subspaces need not be overt; see Section J 16 for counterexamples for this and other situations.

The whole of any metric space $X$ is overt, with

$$\langle X \rangle U \;\equiv\; \exists x.\, x \in U \qquad \text{and} \qquad d(x, X) \;\equiv\; 0,$$

but since $X$ may have non-overt subspaces, not every space is overt.

In the previous section we showed how $d$ is defined from $\Diamond$ and that $\Diamond$ is recovered from this. We will now use the Tangency Theorem as a short cut to prove the bijection between $d$ and $\Diamond$, although ideally this should be done without invoking the extent. (Actually it's valid in a locally compact metric space.)

**Lemma 3.10** Let $d : X \to \overline{\mathbb{R}}$ be an upper semicontinuous function that satifies the properties in Lemma 2.3 and define

$$\Diamond U \;\equiv\; \exists xr.\; d(x) < r \;\wedge\; B_r(x) \subset U.$$

Then the operator $\Diamond$ preserves unions.

**Proof** We use the Tangency Theorem for $d$ to find $a$ in the second line:

$$\begin{aligned}
\Diamond \bigcup U_i &\equiv \exists xr.\, d(x) < r \;\wedge\; B_r(x) \subset \bigcup U_i \\
&\Rightarrow \exists xra.\, d(a) = 0 \;\wedge\; d(x,a) < r \;\wedge\; B_r(x) \subset \bigcup U_i \\
&\Rightarrow \exists a.\, d(a) = 0 \;\wedge\; a \in \bigcup U_i \\
&\Rightarrow \exists ai\epsilon.\, d(a) < \epsilon \;\wedge\; B_\epsilon(a) \subset U_i \\
&\equiv \exists i.\, \Diamond U_i.
\end{aligned}$$

The reverse implication follows from the fact that if $U \subset V$ then $\Diamond U \Longrightarrow \Diamond V$. $\square$

**Lemma 3.11** The function $d$ is recovered from from $\Diamond$.

**Proof** We easily have

$$d(x) < r \Longrightarrow \exists ys.\, d(y) < s \;\wedge\; B_s(y) \subset B_r(x) \;\equiv\; \Diamond B_r(x) \;\equiv\; d'(x) < r.$$

Conversely, since the $\Diamond$ derived from $d$ preserves unions by the previous result, the $d'$ derived from it satisfies Lemma 2.3. Hence

$$\begin{aligned}
d'(x) < r &\Rightarrow \exists r'.\, d'(x) < r' < r \quad \equiv \quad \exists r'ys.\, d(y) < s \;\wedge\; B_s(y) \subset B_{r'}(x) \;\wedge\; r' < r \\
&\Rightarrow \exists r'ys.\, \exists z.\, d(y,z) < s \;\wedge\; d(z) < r - r' \;\wedge\; d(x,z) < r' < r \\
&\Rightarrow d(x) < r.
\end{aligned}$$

$\square$

**Proposition 3.12** In a complete metric space $X$, the formulae

$$d(x) < r \;\equiv\; \Diamond B_r(x) \quad \text{and} \quad \Diamond U \;\equiv\; \exists xr.\; d(x) < r \;\wedge\; B_r(x) \subset U$$

define a bijective correspondence between
(a) an operator $\Diamond$ on open subspaces that satisfies $\Diamond \bigcup U_i \iff \exists i.\, \Diamond U_i$ and
(b) an upper semicontinuous function $d : X \to \overline{\mathbb{R}}$ satisfying the conditions in Lemma 2.3.
Moreover, either of these defines a subspace $A$. However, there may be many such subspaces that correspond to a given $d$ or $\Diamond$. $\square$

**Proposition 3.13** Any ***singleton*** $\{x_0\} \subset X$ is overt, by which we mean that it is the extent of

$$\Diamond U \;\equiv\; \langle x_0 \rangle U \;\equiv\; (x_0 \in U) \qquad \text{or of} \qquad d(x) \;\equiv\; d(x, \{x_0\}) \;\equiv\; d(x, x_0).$$

In this case, $\Diamond$ preserves finite meets as well as joins,

$$\Diamond X \Leftrightarrow \top \qquad \text{and} \qquad \Diamond U \;\wedge\; \Diamond V \;\Longrightarrow\; \Diamond(U \cap V),$$

whilst the distance function has the property that

$$d(y,z) \;\leq\; d(y) \;+\; d(z) \;<\; \infty,$$

in which $\leq$ means $\qquad d(y,z) < r + s \;\Longleftarrow\; d(y) < r \;\wedge\; d(z) < s.$

Conversely, if $\Diamond$ or $d$ satisfies these properties in a complete metric space then its extent is a singleton.

**Proof** The statements about a given singleton are easy properties of topological and metric spaces and we derive

$$d(x) < r \;\equiv\; \Diamond B_r(x) \;\equiv\; \big(x_0 \in B_r(x)\big) \;\equiv\; d(x, x_0) < r.$$

Conversely, if $\Diamond$ has these properties then it is inhabited. So, by Tangency, there is *some* point in its extent $A$. By the triangle law, tangency and symmetry,

$$\begin{aligned} d(x) < r \ \wedge\ d(y) < s \quad &\equiv \quad \Diamond B_r(x) \ \wedge\ \Diamond B_s(y) \quad \Longrightarrow \quad \Diamond\big(B_r(x) \cap B_s(y)\big) \\ &\Rightarrow \quad \exists a \in A.\, d(x,a) < r \ \wedge\ d(x,a) < s \quad \Longrightarrow \quad d(x,y) < r+s. \end{aligned}$$

The point is unique because if $x, y \in A$, so $d(x) = d(y) = 0$, then $d(x,y) = 0$ and $x = y$. $\square$

Any operator $\Diamond$ that perserves finite meets as well as arbitrary joins defines a ***completely prime filter*** $\mathcal{F} \equiv \{U \mid \Diamond U\}$ and the space is called ***sober*** if every such $\mathcal{F}$ corresponds to a unique point [Joh82].

*Starting* with the singleton makes this look rather trivial, but that is an example of our more general message that an overt subspace is *defined* by $\Diamond$ or $d$, from which the subspace $A$ is *derived*.

The standard way of specifying a point in a metric space is as a limit of a Cauchy sequence:

**Lemma 3.14** Any Cauchy sequence $(x_n)$ with $d(x_n, x_{n+k}) < 2^{-n}$ defines

$$d(x) < r \ \equiv\ \exists n.\, d(x_n, x) + 2^{-n+1} < r$$

and hence a singleton, which is the limit.

**Proof** It is easy to show that the relation $d(x) < r$ is rounded upper and satisfies the triangle law. For convergence, first observe that the $n$ in the definition of $d$ may be increased arbitrarily, because

$$\begin{aligned} d(x_{n+k}, x) + 2^{-n-k+1} \quad &\leq \quad d(x_n, x_{n+k}) + d(x_n, x) + 2^{-n-k+1} \\ &< \quad 2^{-n} + d(x_n, x) + 2^{-n-k+1} \quad < \quad d(x_n, x) + 2^{-n+1}. \end{aligned}$$

Then given $d(x) < r$ and $0 < \epsilon$, if $2^{-n+1} < \epsilon$ then $y \equiv x_n$ satisfies

$$d(x_n, y) + 2^{-n+1} < \epsilon \qquad \text{and} \qquad d(x,y) < d(x, x_n) + 2^{-n+1} < r.$$

The overt subspace defined by $d$ is inhabited because $\exists xr.\, d(x) < r$. To show that it is a singleton, suppose that

$$d(x) < r \ \equiv\ \exists n.\, d(x_n, x) + 2^{-n+1} < r \qquad \text{and} \qquad d(y) < s \ \equiv\ \exists m.\, d(x_m, y) + 2^{-m+1} < s$$

with $n \leq m$, so $d(x_n, x_m) < r_n$. Then by symmetry and the transitive law,

$$\begin{aligned} d(x,y) \quad &\leq \quad d(x_n, x_m) + s(x_n, x) + d(x_m, y) \\ &< \quad r_n + s(x_n, x) + d(x_m, y) \ < \ r + s - r_m \ < \ r + s. \end{aligned}$$

Finally, the unique accumulation point is the limit because

$$\begin{aligned} d(x) = 0 \quad &\Longleftrightarrow \quad \forall \epsilon.\, \exists n.\, d(x_n, y) + r_n < \epsilon \\ &\Longleftrightarrow \quad \forall \epsilon.\, \exists n.\, d(x_n, y), r_n < \tfrac{1}{2}\epsilon, \end{aligned}$$

where we have $\forall \epsilon.\, \exists n.\, r_n < \frac{1}{2}\epsilon$ because we were given a Cauchy sequence, so this says that $x$ is its limit. $\square$

**Lemma 3.15** Let $(y_n)$ and $(z_n)$ be Cauchy sequences with $y_n \to y$ and $z_n \to z$. If $p(x)$ and $q(x)$ are the corresponding (upper semicontinuous) distance functions then

$$d(y,z) < t \iff \exists x.\, p(x) < r \ \wedge\ q(x) < s \ \wedge\ r + s < t. \qquad \square$$

**Remark 3.16** In assuming Cauchy completeness in this section we have rather begged some important questions. The alternative is to postulate *formal* balls (as in Formal Topology) with rational centres and radii, and *construct* the completion. Writing $a(x)$ and $b(x)$ instead of $d(x)$ for distance-like functions obtained as above from Cauchy sequences, one may show that

$$d(a,b) \;\equiv\; \mathsf{inf}\,\{r+s \mid \exists x.\, a(x) < r \wedge b(x) < s\}$$

defines a metric. This has been studied for locale theory by Steven Vickers [Vic98, Vic05] and for constructive analysis by Fred Richman [Ric00, Section 5].

Now, instead of assuming completion, we may incorporate the Vickers–Richman technique into ours by defining or encoding an accumulation point $a \in A$ in our sense as the distance-like function $a(x)$ of a Cauchy sequence that "leads into" the overt subspace in the manner of Lemma 3.1. Then, with a more conventional use of $\exists$, we have

$$\Diamond U \iff \exists a \in A.\, \langle a \rangle U \qquad \text{and} \qquad d(x) < r \iff \exists a \in A.\, a(x) < r,$$

where $A$ is now a set of distance-like functions $a(x)$ instead of points, and $\langle a \rangle$ is obtained as above.

This section is by no means *verbatim* the Bolzano–Weierstrass theorem that every bounded sequence has a convergent subsequence, but it should be understood in the same spirit. Our theme is that $\Diamond$ or $d$ encapsulates a soluble problem and the Tangency Theorem is the fundamental one that provides its solution.

# 4 Closed and compact overt subspaces

In our treatment so far, we have taken some trouble *not* to assume that $d$ takes Euclidean real values (Definition 2.8) or that the extent $A$ is closed. We shall now see that these two natural additional conditions are equivalent, and survey their consequences.

Although we have said that one should think of an overt subspace as defined by $\Diamond$ or $d$ and not $A$, let's see what happens when we do that:

**Definition 4.1** A subspace $A \subset X$ of a metric space is called ***located*** if

$$d(x) \quad \equiv \quad \mathsf{inf}\,\{d(x,a) \mid a \in A\}$$

is well defined as a (Euclidean) real number or Dedekind cut.

Locatedness is one of the notions that the Bishop school uses extensively but which we regard as a composite. For them, it means that the infimum takes Euclidean values, but Bas Spitters observed [Spi10] that it is just the *upper* cut that carries the constructive force. This is what we too found in Section 2, but we shall now study both cuts and show that $d$ is a continuous function.

**Lemma 4.2** The locatedness function $d$ obeys the same properties as in Lemma 2.3:

$$\begin{array}{llll}
d(x) < r' < r & \Longrightarrow & d(x) < r & \text{monotonicity} \\
d(x) < r & \Longrightarrow & \exists r'.\, d(x) < r' < r & \text{roundedness} \\
d(x) < \epsilon \;\wedge\; d(x,y) < r & \Longrightarrow & d(y) < r + \epsilon & \text{triangle law} \\
d(x) < r & \Longrightarrow & \forall \epsilon.\, \exists y.\, d(x,y) < r \;\wedge\; d(y) < \epsilon. & \text{convergence}
\end{array}$$

In particular, $\{x \mid d(x) < r\} \subset X$ is open for each $r$ and $d$ is upper semicontinuous. Finally, if $a \in A$ then $d(a) = 0$.

**Proof**

$$\begin{array}{lll}
d(x) < r & \Longrightarrow & \exists a \in A.\, d(x,a) < r \\
& \Longrightarrow & \exists a r'.\, d(x,a) < r' < r \quad \Longrightarrow \quad \exists r'.\, d(x) < r' < r \\
d(x) < r \;\wedge\; d(y,x) < s & \Longrightarrow & \exists a \in A.\, d(x,a) < r \;\wedge\; d(y,x) < s \\
& \Rightarrow & \exists a \in A.\, d(y,a) < r + s \quad \Longrightarrow \quad d(y) < r + s, \\
d(x) < r & \equiv & \exists a.\, d(a) = 0 < \epsilon \;\wedge\; d(x,a) < r.
\end{array}$$

If $a \in A$ then $d(a) = 0$ directly from the formula. $\square$

By Proposition 3.12, any located subspace is therefore overt — by which we mean simply that the function $d$ defined by locatedness is of the kind that we discussed in the previous two sections. The given subspace $A$ is *contained* in the extent of $d$ or $\lozenge$ as we defined it before, but it could be a *proper* subspace. The obvious way to make them agree would be to require $A$ to be closed.

**Corollary 4.3** Any closed located subspace is overt, where these properties come from the same distance function $d$, and this is upper semicontinuous. [Spi10]. $\square$

For the converse we still need to show that if the extent of an overt subspace is closed then the distance function has a lower cut and is lower semicontinuous. First we look at the situation using topology instead of the metric.

**Definition 4.4** A ***closed overt*** subspace is defined by a join-preserving operator $\lozenge$ together with an open subspace $W$ that satisfy

$$\neg \lozenge W \qquad \text{and} \qquad x \in U \implies \lozenge U \ \vee \ x \in W.$$

The second property is called the *relative instantiation rule* in Section J 11. In this case the extent of $\lozenge$ is the complementary closed subspace to $W$, the idea being that, using Corollary 3.7,

$$\lozenge U \iff \exists a \in A.\, a \in U \iff U \not\subset W.$$

Returning to the setting of a metric space, we have seen that the upper cut for the distance function $d$ is derived from the operator $\lozenge$ for overtness. What is the relationship between the lower cut and the open complement $W$ of the subspace? By the Tangency Theorem,

$$d(x) < r \iff \exists a.\, a \in B_r(x) \ \wedge \ a \in A,$$

so classically we would expect something like

$$r \le d(x) \iff B_r(x) \ \subset \ W.$$

However, this doesn't yield a *rounded* lower cut, so we force the result to be rounded:

**Notation 4.5** We may say that a point $x$ lies in an arbitrary open subspace $W$ (not necessarily the complement of an overt one) to ***depth*** (more than) $q$ if it satisfies

$$q < d(x), \qquad \text{defined as} \qquad \exists r.\, q < r \ \wedge \ B_r(x) \subset W.$$

Alternatively, $x$ is ***bounded away*** from the complementary closed subspace by this amount.

**Lemma 4.6** For any open subspace $W \subset X$ of a metric space, the depth defines a lower semicontinuous function $d : X \to \underline{\mathbb{R}}$ by

$$d(x) \ \equiv \ \mathsf{sup}\,\{q \mid q < d(x)\}$$

that satisfies $\ x \in W \iff 0 < d(x)$.

We need to put another condition on $d$ in order to recover it from $W$, but there seems to be no better way of stating this than that $d$ arises as the depth function for some $W$.

**Proof** The notation with an inequality is justified because

$$\begin{aligned} q < d(x) \quad &\equiv \quad \exists r.\, q < r \ \wedge \ B_r(x) \subset W \\ &\Leftrightarrow \quad \exists q' r.\, q < q' < r \ \wedge \ B_r(x) \subset W \quad \equiv \quad \exists q'.\, q < q' < d(x). \end{aligned}$$

The subspace $\{x \mid q < d(x)\}$ is open because

$$\begin{aligned}
q < d(x) \quad &\equiv \quad \exists \epsilon.\, B_{q+2\epsilon}(x) \subset W \\
&\Rightarrow \quad \exists \epsilon.\, \forall yz.\, d(x,y) < \epsilon \ \wedge\ d(y,z) < q + \epsilon \Rightarrow z \in W \\
&\equiv \quad \exists \epsilon.\, \forall y \in B_\epsilon(x). \quad B_{q+\epsilon}(y) \subset W \\
&\equiv \quad \exists \epsilon.\, B_\epsilon(x) \subset \{y \mid q < d(y)\},
\end{aligned}$$

using the triangle law, so the function $d : X \to \underline{\mathbb{R}}$ is lower semicontinuous. Since $W$ was open, we recover it from $d$ because

$$x \in W \iff \exists \epsilon.\, 0 < \epsilon \ \wedge\ B_\epsilon(x) \subset W \ \equiv\ 0 < d(x). \qquad \square$$

**Example 4.7** In order to find an example of a *single* open subspace $W$ for which the depth $d$ is only lower semicontinuous and not continuous, we have to think in an explicitly constructive way.

However, we can find counterexamples in classical topology if we look for them *parametrically.* For any continuous function $f : X \to Y$ between metric spaces, each $W_y \equiv \{x \in X \mid fx \neq y\}$ is open and we may define $d_y(x)$ as above. This is *lower* semicontinuous in $x$ and $y$ (jointly), but only (upper semi)continuous when $f$ is an open map, as we shall see in the next section. If $f : \mathbb{R} \to \mathbb{R}$ is instead the squaring function that we used in paragraph 1.2 then $W_y = \{x \mid x^2 \neq y\}$ and $d_y(x)$ is lower but not upper semicontinuous.

Whilst the fibre $f^{-1}(y)$ in this example is overt for *each real number $y$*, it is not overt when $y$ is a lower real for which it is not decidable whether $y = 0$ or $y < 0$. For some counterexamples, see Section J 16.

Finally, we put the upper and lower cuts for $d$ together.

**Lemma 4.8** Let $\lozenge$ and $W$ define an overt closed subspace of a locally compact metric space. Then

$$q < d(x) \ \equiv\ \exists q'.\, q < q' \ \wedge\ B_{q'}(x) \subset W \qquad \text{and} \qquad d(x) < r \ \equiv\ \lozenge\, B_r(x)$$

define a continuous function $d : X \to \mathbb{R}$.

**Proof** We know that the two cuts are rounded and respectively lower and upper. For disjointness,

$$r < d(x) < r \ \equiv\ \exists r'.\, r < r' \ \wedge\ B_{r'}(x) \subset W \ \wedge\ \lozenge\, B_r(x) \implies \lozenge\, W \implies \bot.$$

For locatedness, if $q < r$ then $q < q' < r$ for some $q'$. Then

$$d(x,y) < q' \implies y \in U \equiv B_r(x) \implies y \in W \ \vee\ \lozenge\, U$$

by relative instantiation, so

$$\forall y \in B_{q'}(x). \quad y \in W \quad \vee \quad \lozenge\, U.$$

The second disjunct is independent of the bound variable $y$, so *classically* we may move it outside the quantifier:

$$\big(\forall y \in B_{q'}(x).\ y \in W\big) \quad \vee \quad \lozenge\, U,$$

whence

$$q < r \quad \implies \quad \exists q'.\, q < q' < r \ \wedge\ B_{q'}(x) \subset W \ \vee\ \lozenge\, B_r(x) \quad \equiv \quad q < d(x) \ \vee\ d(x) < r.$$

When the metric space is locally compact, the closed ball $\overline{B}_{q'}(x)$ is compact and $q' < r \implies \overline{B}_{q'}(x) \subset B_r(x)$. Moving the constant disjunct outside the universal quantifier is a constructive property of compact subspaces that is dual to the Frobenius law for open ones mentioned in paragraph 1.9. Then the above argument becomes constructively valid if we replace the open ball $B_{q'}(x)$ by the compact one $\overline{B}_{q'}(x)$. Hence

$$\lozenge\, B_r(x) \quad \vee \quad q < q' \ \wedge\ B_{q'}(x) \subset \overline{B}_{q'}(x) \subset W, \qquad \text{and so} \qquad d(x) < r \ \vee\ q < d(x).$$

Therefore we have a Dedekind cut (Definition 2.8). Moreover, $d : X \to \mathbb{R}$ is both upper and lower semicontinuous, so it is continuous. $\square$

**Theorem 4.9** In a locally compact complete metric space, there is a bijective correspondence amongst

(a) a join-preserving operator $\lozenge$ together with an open subspace $W$ such that

$$\neg \lozenge W \quad \text{and} \quad x \in U \Longrightarrow \lozenge U \vee x \in W;$$

(b) a closed located subspace $A$; and

(c) a continuous function $d : X \to \mathbb{R}$ satisfying the properties in Lemma 2.3,

where

$$\begin{array}{rcl@{\qquad\qquad}rcl} q < d(x) & \Longleftrightarrow & \exists q'.\, q < q' \wedge B_q(x) \subset W & d(x) < r & \Longleftrightarrow & \lozenge B_r(x) \\ d(x) & = & \inf\{d(x,a) \mid a \in A\} & \lozenge U & \Longleftrightarrow & \exists a \in A.\, a \in U \\ W & = & \{w \mid 0 < d(w)\} & A & = & \{a \mid 0 = d(a)\}. \end{array}$$ $\square$

**Remark 4.10** There are further interesting results in the case where the extent of an overt subspace is **compact**. Just as we may represent an overt subspace $A$ by an operator $\lozenge$ so that $\lozenge U \iff \exists a \in A.\, a \in A$ or $A$ *touches* $U$, it has long been recognised that, for a compact subspace $K$, its *open neighbourhoods* (those open $U$ for which $K \subset U$) are more important than its points, so we may write

$$\square U \;\equiv\; (K \subset U) \;\equiv\; \forall x \in K.\, x \in U.$$

If a subspace is both overt and compact, these operators are related by laws that are familiar in ***modal logic***,

$$\lozenge U \;\wedge\; \square V \;\Longrightarrow\; \lozenge(U \wedge V) \qquad \text{and} \qquad \lozenge U \;\vee\; \square V \;\Longleftarrow\; \square(U \vee V).$$

This situation is studied in the context of an ambient Hausdorff space in Section J 12.

The definition that Errett Bishop gives for a compact subspace in constructive analysis actually requires it to be overt too. However, it is stated using a notion called *totally bounded* that we will consider in Definition 7.9.

# 5 Open maps

Now we can prove the characterisation of open maps between locally compact metric spaces that we stated in the Introduction.

Recall that an open map $f : X \to Y$ is one for which the direct image $f_! U \subset Y$ of any open subspace $U \subset X$ is again open. The operator $f_!$ then preserves unions. Hence, if $\blacklozenge$ is a join-preserving operator on the open subspaces of $Y$ then $\lozenge U \equiv \blacklozenge f_! U$ defines one on those of $X$. In particular, since any singleton $\{y\} \subset Y$ is overt by Proposition 3.13, with $\langle y \rangle V \equiv (y \in V)$, we expect its fibre or inverse image $f^{-1}(y) \subset X$ under an open map $f : X \to Y$ to be overt too.

**Proposition 5.1** For any open map $f : X \to Y$ between metric spaces,

$$\big\langle f^{-1}(y) \big\rangle U \;\equiv\; (y \in f_! U) \qquad \text{and} \qquad d\big(x, f^{-1}(y)\big) < r \;\equiv\; \big(y \in f_! B_r(x)\big)$$

satisfy the properties in Lemma 2.3 for each $y \in Y$, whilst $\{y \mid d\big(x, f^{-1}(y)\big) < r\} \subset Y$ is open for each $x \in X$, and $d\big(x, f^{-1}(y)\big) = 0 \iff y = fx$.

**Proof** We have just proved the first part. The subspace is open because it is $f_! B_r(x)$:

$$\begin{array}{rcl} d\big(x, f^{-1}(y)\big) < r & \equiv & \big(y \in f_! B_r(x)\big) \iff \exists \epsilon.\, B_\epsilon(y) \subset f_! B_r(x) \\ & \equiv & \exists \epsilon.\, \forall y'.\, \big(e(y,y') < \epsilon \;\Rightarrow\; d\big(x, f^{-1}(y')\big) < r\big). \end{array}$$

This statement is $\forall\delta\exists\epsilon$ since it is stating continuity of $f^{-1}$ rather than of $f$. For the last part,

$$\begin{aligned} d\big(x, f^{-1}(y)\big) = 0 &\equiv \forall\delta.\, d\big(x, f^{-1}(y)\big) < \delta \\ &\equiv \forall\delta.\, y \in f_! B_\delta(x) \\ &\equiv \forall\delta.\, \exists x'.\, fx' = y \ \wedge\ d(x, x') < \delta, \end{aligned}$$

so $fx = y \Longrightarrow d\big(x, f^{-1}(y)\big) = 0$. Conversely, we combine the last form with $\epsilon$–$\delta$-continuity for $f$ to get

$$\forall\epsilon.\, \exists\delta x'.\, fx' = y \ \wedge\ e(fx, fx') < \epsilon,$$

which is $fx = y$. $\square$

**Definition 5.2** A ***parametric overt subspace*** of $X$ dependent on $y \in Y$ is given by a ternary relation $d\big(x, A(y)\big) < r$ that satisfies the properties in Lemma 2.3 for each $y \in Y$ and has $\{y \mid d\big(x, A(y)\big) < r\} \subset Y$ open for each $x \in X$.

In this, the letter $A$ is just a cipher, but we shall construct an open map $g$ for which $A(y) = g^{-1}(y)$. Indeed, from the first condition alone, we may write

$$A(y) \ \equiv\ \{x \in X \mid d\big(x, A(y)\big) = 0\}$$

*for each $y$ on its own*, as in Definition 3.4 for the subspace of accumulation points.

We only put a very weak requirement for continuity on the expression because we can derive everything else that we need:

**Lemma 5.3** For any parametric overt subspace of a locally compact metric space, the expression $d(x, A(y))$ is a function $X \times Y \to \overline{\mathbb{R}}$ that is upper semicontinuous in $x$ and $y$ jointly.

**Proof** Proposition 2.4 gives semicontinuity in $x$ for each $y$ and we can derive the joint property,

$$\begin{aligned} & d\big(x, A(y)\big) < r \\ \Rightarrow\ & \exists\delta.\, \forall x'.\, d(x, x') \le \delta \Rightarrow d\big(x', A(y)\big) < r && \text{upper semicontinuous in } x \\ \Rightarrow\ & \exists\delta.\, \forall x'.\, d(x, x') \le \delta \Rightarrow \exists\epsilon.\, \forall y'.\, \big(e(y, y') \le \epsilon \ \Rightarrow\ d\big(x', A(y')\big) < r\big) && \textit{ditto} \text{ in } y \\ \Rightarrow\ & \exists\delta\epsilon.\, \forall x'y'.\, d(x, x') \le \delta \ \wedge\ e(y, y') \le \epsilon \ \Rightarrow\ d\big(x', A(y')\big) < r, \end{aligned}$$

from compactness of $\overline{B}_\delta(x)$. $\square$

**Lemma 5.4** The following composite is an open map:

$$g : A \ \equiv\ \{(x, y) \mid d\big(x', A(y)\big) = 0\} \rightarrowtail X \times Y \xrightarrow{\pi_1} Y.$$

**Proof** For each $r > 0$, we also write

$$A_r \ \equiv\ \{(x, y) \mid d\big(x, A(y)\big) < r\} \ \subset\ X \times Y, \qquad \text{so that} \qquad A \ =\ \bigcap_r A_r.$$

By the remarks in Definition 3.8, its image

$$\pi_1 A_r \ \equiv\ \{y \in Y \mid \exists x.\, d\big(x, A(y)\big) < r\} \ \equiv\ \{y \in Y \mid \exists x.\, d\big(x, A(y)\big) < \infty\} \ \subset Y$$

is independent of $r$ and says which $y$ name inhabited subspaces $A(y)$. Moreover, this is an open subspace of $Y$ by upper semicontinuity of $d\big(x, A(-)\big)$ in $y$.

Then, for a typical basic open subspace $W \equiv A \cap \big(B_\delta(x) \times B_\epsilon(y)\big)$ of $A$,

$$\begin{aligned} g_! W &= \{y' \mid \exists x'.\, d\big(x', A(y')\big) = 0 \ \wedge\ d(x, x') < \delta \ \wedge\ e(y, y') < \epsilon\} \\ &= \{y' \in B_\epsilon(y) \mid d\big(x, A(y')\big) < \delta\} \end{aligned}$$

by Corollary 3.6. This is an open subspace of $Y$, so $g$ is an open map. $\square$

**Proposition 5.5** A continuous function $f : X \to Y$ between metric spaces is open iff there is a parametric overt subspace such that $d\big(x, A(y)\big) = 0 \iff fx = y$.

**Proof** In this case, $\pi_0 : X \times Y \to X$ and $(\mathsf{id}, f) : X \to X \times Y$ restrict to a homeomorphism $X \cong A \subset X \times Y$ with $f = (\mathsf{id}, f)\, ; g : X \to Y$ in the previous construction. $\square$

**Lemma 5.6** The direct image of an open subspace $U \subset X$ is

$$f_! U \;\equiv\; \{y \mid \exists x\delta.\; d\big(x, f^{-1}(y)\big) < \delta \;\wedge\; B_\delta(x) \subset U\}.$$

**Proof** By similar arguments to those in Lemmas 3.10 and 3.11, $f_!$ preserves unions and

$$f_!\, B_\delta(x) \;=\; \{y \mid d\big(x, f^{-1}(y)\big) < \delta\}.$$

This and $f_! U \subset Y$ are open by upper semicontinuity of $d\big(x, f^{-1}(y)\big)$ in $y$.

Since the conditions for an open map respect unions, it suffices to consider $U \equiv B_\delta(x)$ and $V \equiv B_\epsilon(y)$ in them. Then, by the triangle law,

$$f^{-1} f_! B_\delta(x) \;=\; \{x' \mid d\big(x', f^{-1}(fx)\big) < \delta\} \;\supset\; B_\delta(x).$$

Since $d\big(x, A(y)\big) < \delta \Longrightarrow e(fx, y) < \epsilon$ when $\epsilon$ and $\delta$ are related by the continuity axiom,

$$f_! f^{-1} B_\epsilon(y) \;=\; \{y' \mid \exists x\delta.\, d\big(x', f^{-1}(y')\big) < \delta \;\wedge\; B_\delta(x) \subset f^{-1} B_\epsilon(y)\} \;\subset\; B_\epsilon(y).$$

We only need to prove one direction of the Frobenius law, so let

$$y' \in f_! B_\delta(x) \cap B_\epsilon(y) \qquad i.e. \qquad d\big(x, f^{-1}(y')\big) < \delta \;\wedge\; e(y, y') < \epsilon.$$

By the Tangency Theorem there is some $x' \in X$ with

$$fx' = y' \qquad \text{and} \qquad d(x, x') < \delta$$

and by continuity of $f$ there is some $\delta'$ with

$$0 < \delta' < \delta - d(x, x') \qquad \text{and} \qquad \forall x''.\, d(x', x'') \le \delta' \Longrightarrow e(fx', fx'') < \epsilon.$$

Then $$d\big(x', f^{-1}(y')\big) = 0 < \delta' \qquad \text{and} \qquad \overline{B}_{\delta'}(x') \;\subset\; B_\delta(x) \;\cap\; f^{-1} B_\epsilon(y).$$

Hence
$$\begin{aligned} y' \;&\in\; \{y' \mid \exists x'\delta'.\, d\big(x', f^{-1}(y')\big) < \delta' \;\wedge\; \overline{B}_{\delta'}(x') \subset B_\delta(x) \cap f^{-1} B_\epsilon(y)\} \\ &\equiv\; f_!\big(B_\delta(x) \cap f^{-1} B_\epsilon(y)\big), \end{aligned}$$

so $f : X \to Y$ is an open map with direct and inverse images $f^{-1}$ and $f_!$ as claimed. $\square$

**Corollary 5.7** In terms of the distance functions, the inverse image of an overt subspace of $Y$ defined by $e$ is

$$d(x) < \delta \;\equiv\; \exists y\epsilon.\; e(y) < \epsilon \;\wedge\; \big(\forall y'.\, e(y, y') < \epsilon \Rightarrow d\big(x, f^{-1}(y)\big) < \delta\big). \qquad \square$$

We have only used upper semicontinuity, but when the target of an open map is a metric or Hausdorff space, the subspaces $\{y\} \subset Y$ and $f^{-1}(y) \subset X$ are closed.

**Lemma 5.8** Let $f : X \to Y$ be an open map between locally compact metric spaces. Then the distance $d\big(-, f^{-1}(-)\big)$ takes Euclidean real values.

**Proof** Consider the depth function for $W \equiv f^{-1}(Y \setminus \{y\})$ from Notation 4.5,

$$q < d\big(x, f^{-1}(y)\big) \quad\equiv\quad \exists q'.\, q < q' \;\wedge\; \forall x'.\, \big(d(x, x') < q' \Rightarrow f(x') \neq y\big).$$

In Lemma 4.6, we showed that this is rounded in $q$, defines an open subspace of $X$ (so it is lower semicontinuous in $x$). Lemma 4.8 showed that its upper partner $d\big(x, f^{-1}(y)\big) < r$ is the function that we have just been studying. $\square$

**Lemma 5.9** This distance function $d\big(-, f^{-1}(-)\big) : X \times Y \to \mathbb{R}$ is jointly lower semicontinuous.

**Proof** Using compactness of $\overline{B}_{q+\delta}(x)$:

$$\begin{array}{rcll}
q < d\big(x, f^{-1}(y)\big) & \Rightarrow & \exists \delta.\ q + \delta < d\big(x, f^{-1}(y)\big) & \text{roundedness} \\
& \equiv & \exists \delta.\ \forall x''.\ d(x, x'') \leq q + \delta \Rightarrow (f x'' \neq y) & \text{definition} \\
& \Leftrightarrow & \exists \delta.\ \forall x''.\ d(x, x'') \leq q + \delta \Rightarrow \exists \epsilon.\ \epsilon < e(f x'', y) & \text{metric} \\
& \Rightarrow & \exists \delta \epsilon.\ \forall x''.\ d(x, x'') \leq q + \delta \Rightarrow \epsilon < e(f x'', y) & \text{compactness} \\
& \Rightarrow & \exists \delta \epsilon.\ \forall x' y'.\ d(x, x') \leq \delta \ \wedge\ e(y, y') \leq \epsilon \ \Rightarrow & \text{triangle laws} \\
& & \qquad \big(\forall x''.\ d(x', x'') \leq q \Rightarrow f x'' \neq y'\big) & \text{for metrics} \\
& \equiv & \exists \delta \epsilon.\ \forall x' y'.\ d(x, x') \leq \delta \ \wedge\ e(y, y') \leq \epsilon \ \Rightarrow\ q < d\big(x', f^{-1}(y')\big). & \square
\end{array}$$

This lower function identifies solutions of the equation by the same condition as for the upper one in Proposition 5.1. This is that $d\big(x, f^{-1}(y)\big) = 0$, which happens iff $q < d\big(x, f^{-1}(y)\big)$ fails for all $q$, which is equivalent to $f(x) = y$.

**Theorem 5.10** The following are equivalent for any continuous function $f : X \to Y$ between complete locally compact metric spaces:

(a) the map $f$ is open, *i.e.* the direct image $f_! U$ is open in $Y$ for any open subspace $U \subset X$;

(b) the expression

$$q < d\big(x, f^{-1}(y)\big) \ \equiv\ \exists q'.\ q < q' \ \wedge\ \forall x'.\ \big(d(x, x') < q \Rightarrow f(x') \neq y\big),$$

which always defines a lower semicontinuous function, actually gives a continuous one $X \times Y \to \mathbb{R}$; and

(c) there is a continuous function $d\big(-, f^{-1}(-))\big) : X \times Y \to \mathbb{R}$ such that

$$d\big(x, f^{-1}(y)\big) < r \implies \exists x'.\ d\big(x', f^{-1}(y)\big) < \delta \ \wedge\ d(x, x') < r \implies d\big(x, f^{-1}(y)\big) < r + \delta,$$

and
$$d\big(x, f^{-1}(y)\big) = 0 \iff f x = y.$$

In fact, upper semicontinuity in each variable is enough. $\square$

# 6 Algorithms

We claimed in the Introduction that algorithms for solving equations yield distance functions like those that we have now defined. However, this is an idealisation.

**Remark 6.1** In Corollary 3.6, we showed that our $d(x)$ gives the *actual* distance to the *nearest* solution (accumulation point). It was defined in that way so that it would satisfy the triangle law $d(x') \leq d(x) + d(x, x')$ and correspond uniquely to the operator $\Diamond$.

The estimate $\Delta(x)$ of the proximity of a solution that underlies an algorithm such as Newton–Raphson does not satisfy all of these properties. The *local* information that it provides at a particular test point $x_n$ tells us *roughly* how far away *some* solution is, but the actual distance may be different, because of changes (higher derivatives) along the way. Also, there may be some other solution lurking nearby.

Therefore, we expect $\Delta$ to satisfy convergence and roundedness but not necessarily the triangle law.

On the other hand, an algorithm gives a *particular* next approximant, $x_{n+1}$, not just an existential formula $\exists x'$. So the algorithm has a more definite convergence property like

$$\Delta(x_n) < r \implies d(x_{n+1}, x_n) < \tfrac{1}{2}r \ \wedge\ \Delta(x_{n+1}) < \tfrac{1}{2}r.$$

In fact, the Newton–Raphson algorithm is known to converge *much* faster than this (after a certain stage, it *doubles* the number of valid bits at each iteration), but it is the *fact* of convergence that interests us here, not its rate.

Unfortunately, such an algorithm by no means does always converge. Sometimes it may behave *chaotically*, wildly diverging from the test point, maybe even when there happens to be a solution nearby. In this case, we let $\Delta(x) = \infty$ instead. That is, we define $\Delta(x)$ to have a finite value, not by blind application of a *formula*, but just in those cases were we can *prove* that it has the required property. Hence $\Delta$ must combine this logical property with the arithmetical information that is provided by the derivative.

**Remark 6.2** Recall that the derivative of a function $f : \mathbb{R}^{\mathbf{n}} \to \mathbb{R}^{\mathbf{n}}$ has matrices as values,

$$\dot{f} : \mathbb{R}^{\mathbf{n}} \to \mathbb{R}^{\mathbf{n}} \otimes \mathbb{R}^{\mathbf{n}},$$

which we want to have inverse matrices $\big(f(x)\big)^{-1}$ at each point and be continuous. Then the Newton–Raphson algorithm defines a sequence $(x_n)$ by

$$x_{n+1} \ \equiv\ x_n \ +\ g(x_n) \qquad \text{where} \qquad g(x) \ \equiv\ \big(\dot{f}(x)\big)^{-1} \cdot \big(y - f(x)\big)$$

and we are looking for conditions to ensure that $x_n \to a$ with $f(a) = y$.

**Notation 6.3** Let $\Delta(x) < r$ be the conjunction of the three conditions

$$\dot{f}(x) \text{ invertible}, \qquad\qquad |\, g(x)\,| < \tfrac{1}{2}r,$$

and
$$\forall x'x'' \in \overline{B}_r(x).\quad \Big|\, \dot{f}(x)^{-1} \cdot \big(\dot{f}(x') - \dot{f}(x'')\big) \Big| \ \le\ |\, x' - x''\,|/r.$$

Comparing this predicate with Lemma 2.3, it is rounded ($r$ may be reduced slightly), because of the $< \frac{1}{2}r$ and $/r$ in the second and third conditions, whilst it is semicontinous in $y$ because $g(x)$ is continuous in it.

**Lemma 6.4** $\Delta$ satisfies the convergence property,

$$\Delta(x_0) < r \quad\implies\quad |\, x_1 - x_0\,| < \tfrac{1}{2}r \quad\wedge\quad \Delta(x_1) < \tfrac{1}{2}r.$$

**Proof** We adapt Theorem 5 of [CM12]. The second part of $\Delta(x_0)$ is

$$|\, x_1 - x_0\,| \ \equiv\ \Big|\, \dot{f}(x_0)^{-1} \cdot (y - f(x_0)) \Big| \ <\ \tfrac{1}{2}r,$$

so we may put $x \equiv x'' \equiv x_0$ and $x' \equiv x_1$ in the third part to give

$$\Big|\, \dot{f}(x_0)^{-1} \cdot \dot{f}(x_1) - \mathsf{id} \Big| \ =\ \Big|\, \dot{f}(x_0)^{-1} \cdot \big(\dot{f}(x_1) - \dot{f}(x_0)\big) \Big| \ \le\ \frac{|\, x_1 - x_0\,|}{r} \ <\ \tfrac{1}{2}.$$

This justifies finding the inverse of $\dot{f}(x)$ as the power series $(\mathsf{id} - M)^{-1} = \mathsf{id} + M + M^2 + M^3 + \cdots$ (*op. cit.*, §2.1) and then

$$\Big|\, \dot{f}(x_1)^{-1} \cdot \dot{f}(x_0) \Big| \ \le\ \frac{1}{1 - |\, x_1 - x_0\,|/r} \ <\ 2.$$

We use this to change the denominator from $\dot{f}(x_0)$ to $\dot{f}(x_1)$ in the third part of $\Delta$,

$$\Big|\, \dot{f}(x_1)^{-1} \cdot (\dot{f}(x') - \dot{f}(x'')) \Big| \ \le\ \Big|\, \dot{f}(x_1)^{-1} \cdot \dot{f}(x_0) \Big| \cdot \Big|\, \dot{f}(x_0)^{-1} \cdot (\dot{f}(x') - \dot{f}(x'')) \Big| \ <\ \frac{2|\, x' - x''\,|}{r}.$$

Next, by the mean value theorem (*op. cit.*, §2.4),

$$\left| \dot{f}(x_0)^{-1} \cdot \big( f(x') - f(x'') - \dot{f}(x') \cdot (x' - x'') \big) \right| \;\leq\; \frac{|\, x' - x'' \,|^2}{2r}.$$

Using the definition of $x_1$ and since $|\, x_1 - x_0 \,| < \frac{1}{2}r$,

$$\begin{aligned} \left| \dot{f}(x_0)^{-1} \cdot \big( y - f(x_1) \big) \right| &= \left| \dot{f}(x_0)^{-1} \cdot \big( f(x_1) - f(x_0) - \dot{f}(x_0) \cdot (x_1 - x_0) \big) \right| \\ &\leq |\, x_1 - x_0 \,|^2 / 2r \;<\; r/8. \end{aligned}$$

Changing the denominator of this from $\dot{f}(x_0)$ to $\dot{f}(x_1)$ as before,

$$\left| \dot{f}(x_1)^{-1} \cdot \big( y - f(x_1) \big) \right| \;<\; \tfrac{1}{4}r$$

which is the second part of $\Delta(x_1)$. $\square$

**Lemma 6.5** If $\Delta(x_0) < r$ then the Newton–Raphson sequence converges to some $a \in B_r(x)$ with $f(a) = y$. Moreover this is unique.

**Proof** We may find the limit either directly or using Lemma 3.1. For uniqueness, suppose that $b \in B_r(x)$ also satisfies $f(b) = y$. Then

$$\begin{aligned} x_1 - b &= x_0 - b + \dot{f}(x_0)^{-1} \cdot \big( y - f(x_0) \big) \\ &= \dot{f}(x_0)^{-1} \cdot \big( f(b) - f(x_0) - \dot{f}(x_0) \cdot (b - x_0) \big) \end{aligned}$$

so, by the mean value theorem again,

$$|\, x_1 - b \,| \;\leq\; |\, x_0 - b \,|^2 / 2r \;\leq\; \tfrac{1}{2}r.$$

Hence $x_n \to b$, so $a = b$ since limits of Cauchy sequences are unique. $\square$

**Remark 6.6** The function $\Delta(x)$ has *similar* properties to the $d(x)$ that is related to an overt subspace, except that $\Delta(x)$ fails the triangle law. However, from an estimate $\Delta(x)$ with the "raw" properties in Remark 6.1 we can define

$$d(x) < r \quad \equiv \quad \exists x'\delta.\, \Delta(x') < \delta \quad \wedge \quad d(x, x') < r - \delta,$$

and this does obey the missing triangle law. We deduce the convergence property of $d(x)$ by iterating the analogous one for $\Delta$ enough $(m)$ times to make $\epsilon < 2^{-m}r$.

In general, $d(x) \leq \Delta(x)$, that is, $\Delta(x) < r \Longrightarrow d(x) < r$. However, $\Delta(x)$ may over-estimate the distance of $x$ from the nearest zero, as we have explained, whilst $d(x)$ says *exactly* how far away it is (Corollary 3.6). The latter is the *ideal*, appropriate in a *pure* mathematical context, but the former is what we expect in practical computation. In the Newton–Raphson case, $\Delta(x)$ is given by *local* properties of the function near $x$, so it cannot know what will happen between the current point $x$ and a zero elsewhere.

**Remark 6.7** In fact, we may use $\Delta(x)$ directly for many of the purposes of $d(x)$. In particular, they have the same zeroes (accumulation points).

Since $d(x) \leq \Delta(x)$, if $\Delta(x) = 0$ then $x$ is a solution. More generally $\Delta(x) < r$ ensures that there is one *within* $r$,

$$\Delta(x) < r \implies d(x) < r \iff \exists x'.\, d(x, x') < r \wedge d(x') = 0.$$

Conversely, suppose that $\Delta(x)$ is *uniformly* semicontinuous in a suitable (compact) domain,

$$\Delta(x) < 2^{-k} \iff \exists \epsilon_k.\, \forall x''.\, d(x, x') \leq \epsilon_k \Longrightarrow \Delta(x'') < 2^{-k},$$

whilst $d(x) = 0$, so
$$\forall r > 0.\, \exists x'\delta.\, \Delta(x') < \delta \;\wedge\; d(x, x') < r - \delta.$$
Let $0 < r_k < \mathsf{min}(2^{-k}, \epsilon_k)$, so since $d(x) < r_k$, there are $x'$ and $\delta$ with
$$\Delta(x') < \delta < 2^{-k} \qquad \text{and} \qquad d(x, x') < r_k - \delta < \epsilon_k,$$
so $\Delta(x) < 2^{-k}$. $\square$

**Remark 6.8** Some of the other properties in Section 3 also transfer from $d$ to $\Delta$.

The statement $\exists x.\, \Delta(x) < \infty$ is enough to say that a solution exists *somewhere*, because
$$d(x) < \infty \iff \exists x'.\, \Delta(x') < \infty \;\wedge\; d(x, x') < \infty.$$
The condition for uniqueness (Proposition 3.13) is also the same, because
$$\forall xy.\, d(x, y) \le d(x) + d(y) \qquad \text{iff} \qquad \forall x'y'.\, d(x', y') \le \Delta(x') + \Delta(y'),$$
where $\Rightarrow$ is easy and $\Rightarrow$ follows from the formula in Remark 6.6 and the triangle law for the metric.

We can therefore use $\Delta$ instead of $d$ to characterise local homeomorphisms.

**Theorem 6.9** (Open Mapping Theorem for Differentiable Functions) Any function $f : \mathbb{R}^{\mathbf{n}} \to \mathbb{R}^{\mathbf{n}}$ that has a continuous invertible derivative is an open map and is locally invertible. Moreover this inverse may be found using the Newton–Raphson algorithm.

**Proof** Re-introducing the parameter $y \in Y$ to the definition of $\Delta$,
$$d(x, f^{-1}(y)) = 0 \iff \Delta(x, f^{-1}(y)) = 0 \iff fx = y,$$
so $f$ is an open map by Proposition 5.5. Also, $\Delta$ and $d$ satisfy the condition to make the fibres discrete, so $f$ is a local homeomorphism. $\square$

**Remark 6.10** We have used an algorithm of venerable utility to prove that maps in a certain class have an important abstract property, *openness*. This theorem is well known, so where is the novelty?

Recall that the results of Sections 2 and 3 linked three aspects of overt subspaces (and therefore also of open maps), namely the distance $d(x)$, the union-preserving operator $\lozenge$ and the subspace $A$ of accumulation points.

In the result that we have just proved, the issues that are peculiar to differentiability are confined to a short calculation with "old fashioned" analysis (Lemma 6.4) that yields our distance function more or less directly.

After that, its use does not depend on the differential structure but is meaningful in any metric space.

Conversely, if we have a function like this for some other problem in analysis, along with a proof of its convergence property that is *explicit* enough to yield a value for the existential quantifier, then that proof already contains an algorithm for finding solutions of the problem.

**Remark 6.11** Notice that the *set* $A$ of all solutions is pretty much irrelevant in this. That we can even *form* this set is a conceit of *twentieth century* mathematics: Earlier mathematicians obtained efficient numerical solutions for such problems without pretending that they could comprehend their totality. Even if it is meaningful to form the set, this is of very little use: Neither traditional methods nor the ones that are described in this paper can be applied to *arbitrary* subsets or to *all* solutions of a problem, in particular to tangential roots of polynomials.

On the other hand, General Topology was a *conceptual* advance in mathematics that allows us to discuss the existence and *behaviour* of solutions without getting lost in a jungle of $\epsilon$s and $\delta$s. Whilst our distance function gets us away from manipulating derivatives, it still relies on having a metric available, whereas the $\lozenge$ operator only depends on open subspaces.

# 7 Direct images and nets

We have seen by way of an example that overtness is related to the existence of a practical algorithm for solving a problem. In this section we start from a fundamentalist view of what a computation of a solution *has* to be, and show that this must correspond to a join-preserving operator. Conversely, any such operator arises in this way. Topologically, this argument is concerned with *direct* images of overt subspaces under arbitrary continuous maps.

**Definition 7.1** By a ***net*** in a space $X$ we mean a function $f : M \to X$ from a set $M$, where we require $M$ to be "computationally representable" and we give it the discrete topology.

Any computation is a net, for the following seemingly trite reason: *Whatever* program we have, if its output is to be understood as a point of the space $X$ then it computes a function $f : M \to X$. Whatever its inputs $M$ are (parameters to the problem or a seed approximation for the solution), they have to be encoded in a discrete way. Whether this program *correctly* solves some problem is a separate question that amounts to the relationship between nets and subspaces that we are about to investigate.

Nets are also familiar in analysis. If $M \equiv \mathbb{N}$ then we have a *sequence*, and our notion of accumulation point could be developed into one of a *limit*. *Countable dense subsets* are commonly used in functional analysis, although unfortunately our setting is not that general.

However, the idea of a net rather loses its value in either computation or analysis if we allow $M$ to be an arbitrary set such as the "underlying set" of points in a classical topological space. We shall therefore return later to the question of what we might mean by "computationally representable".

**Notation 7.2** For any net $f : M \to X$ and open subspace $U \subset X$, we write

$$\Diamond U \;\equiv\; \exists m \in M.\, fm \in U,$$

which satisfies $$\Diamond \bigcup U_i \iff \exists mi.\, fm \in U_i \iff \exists i.\, \Diamond U_i.$$

Conversely, if the space $X$ and operator $\Diamond$ are *definable* in some formal system then we will show how to use these *definitions* to construct a net $f : M \to X$ such that $\Diamond U \iff \exists m \in M.\, fm \in U$.

**Notation 7.3** We need to distinguish between points $x \in X$ that *have* rational coordinates and those coordinates themselves. We therefore write $p : P$ for the latter and $i : P \to X$ for the map that takes the coordinates to the point. (This is an example of a net that is dense in the *whole* space, *i.e.* of which *every* point of $X$ is an accumulation point.)

**Notation 7.4** Now we define the set

$$M \;\equiv\; \{(p, r) \mid p : P,\ r : \mathbb{Q},\ d(ip) < r > 0\}$$

of *names* $m \equiv (p, r)$ of open balls $B_r(ip) \subset X$ that $\Diamond$ touches.

We will turn $M$ into a net by reformulating the Tangency Theorem.

**Lemma 7.5** There is a net $g : M \to X$ such that, given the *name* of some ball $(p, r)$ that the overt subspace touches, $g(p, r)$ is an accumulation point that lies inside that ball.

**Proof** In the construction of Lemma 3.1, we have

$$d(ip) < r \quad\Longrightarrow\quad \exists p'r'.\, d(ip, ip') < r - r' \quad\wedge\quad d(ip') < r' < \tfrac{1}{2}r,$$

so that there is a surjective binary relation $(p, r) \rightharpoondown (p', r')$ on $M$.

We need a *function* $e : M \to M$ such that each $(p, r) \in M$ is linked in this relation to $e(p, r)$. The existence of such a function is called the ***Choice Principle***. An *algorithm* like Newton–Raphson provides it.

More abstractly, if the relation is expressed in a suitable formal language then we obtain a function by some method such as Gentzen's Existence Property (paragraph 1.17), Logic Programming or some new technique that is specifically designed for this purpose.

As before, we may iterate this endofunction from a given starting point $(p_0, r_0)$ to yield a sequence $(p_n, r_n)$ in $M$.

Then we either invoke Cauchy completeness as we did in Corollary 3.2 to obtain the limit, or define it using a distance-like function as in the Vickers–Richman approach in Remark 3.16.

Finally, all of this is parametric in the given name $m \equiv (p, r) \in M$, so we have defined a function $f : M \to X$ such that $f(p, r) \in B_r(ip)$ and $d\big(f(p, r)\big) = 0$. □

**Corollary 7.6** $\lozenge U \iff \exists m.\, f(m) \in U$.

**Proof** For basic $U \equiv B_r(ip)$,

$$\lozenge U \implies (p, r) \in M \implies f(p, r) \in U \ \wedge\ d(f(p, r)) = 0 \implies \lozenge U$$

by the definition of $M$ and the construction of $f : M \to X$. This extends to general open subspaces because both sides preserve unions. □

This yields the Representation Theorem for overt subspaces as nets:

**Theorem 7.7** For any definable locally compact complete metric space $X$ there is a correspondence amongst

(a) a definable operator $\lozenge$ such that $\lozenge \bigcup U_i \iff \exists i.\ \lozenge U_i$;

(b) a definable upper semicontinuous function $d : X \to \overline{\mathbb{R}}$ with the properties in Lemma 2.3; and

(c) a net $f : M \to X$,

defined by $$d(x) < r \iff \lozenge B_r(x) \iff \exists m.\, d(x, fm) < r$$

and $$\lozenge U \iff \exists xr.\, d(x) < r \ \wedge\ B_r(x) \subset U \iff \exists m.\, fm \in U.$$

The net $M$ is not uniquely determined by $\lozenge$ or $d$, but may be obtained from their syntactic definition in whatever axiomatisation of topology is being used. □

**Remark 7.8** When the subspace $A$ of accumulation points is **compact**, we can say more about the net $f : M \to X$. Since it is dense, its $\epsilon$-balls cover $A$:

$$\text{for any } \epsilon > 0, \quad A \subset \bigcup_{m \in M} B_\epsilon(fm),$$

but *finitely many* of these will do, by compactness.

Conversely, if $f : M \to X$ with $M$ finite and $\epsilon > 0$ such that

$$A \subset \bigcup_{m \in M} B_\epsilon(fm) \subset \bigcup_{m \in M} \overline{B}_\epsilon(fm)$$

then $A$ is compact so long as it is closed, by local compactness.

**Definition 7.9** If there is a finite $\epsilon$-net for *every* $\epsilon > 0$ then the subspace is called ***totally bounded***.

This stronger version of the property is needed for metric spaces (such as Banach spaces) that need not be locally compact. Bishop defines a subspace to be compact if it is closed and totally bounded. So this is equivalent to being compact and *overt* in our usage.

For us, the interval $[d, u] \subset \mathbb{R}$ is compact for *any* bounded $d$ and $u$, even when $d$ is just an upper and $u$ a lower real. This interval is compact *overt* just when $d$ and $u$ are Euclidean real numbers [J].

**Remark 7.10** Behind the coding of spaces using nets in this section lies the topological notion of the ***direct image***. Recall that the direct image $fK$ of a compact subspace $K \subset Y$ along any continuous map $f : Y \to X$ is again compact. We have the same result for overt subspaces.

Let $\blacklozenge$ be a join-preserving operator on $Y$ and $f : Y \to X$ a continuous function. Then

$$\lozenge U \;\equiv\; \blacklozenge(f^{-1}U),$$

also preserves unions, since both $f^{-1}$ and $\blacklozenge$ do.

If $y \in Y$ is an accumulation point of $\blacklozenge$ then

$$fy \in U \implies y \in f^{-1}U \implies \blacklozenge(f^{-1}U) \implies \lozenge U$$

so $fy \in X$ is an accumulation point of $\lozenge$.

For example, when we have a net $f : M \to X$, the operator $\blacklozenge$ is given by $\blacklozenge V \equiv \exists m.\, m \in V$ and then

$$\lozenge U \;=\; \exists m \in M.\, m \in f^{-1}U \;=\; \exists m \in M.\, fm \in U$$

as before. Also, any $m \in M$ satisfies $fm \in U \Rightarrow \lozenge U$.

**Corollary 7.11** It is enough to define an overt subspace as a join-preserving operator, without any Frobenius condition.

**Proof** Certainly we want $\mathbb{N}$ to be overt, along with any computationally representable object $M$. The direct image of any net $f : M \to X$ is therefore overt too, but we have shown that any join-preserving operator is of this form. $\square$

**Remark 7.12** The direct image is more interesting and familiar when expressed in terms of the distance function, so let $Y$ and $X$ have metrics $e$ and $d$ respectively. Then

$$\begin{aligned}
d(x) < \delta \quad &\equiv \quad \lozenge B_\delta(x) \;\equiv\; \blacklozenge f^{-1}B_\delta(x) \\
&\equiv \quad \exists y\epsilon.\, e(y) < \epsilon \;\wedge\; \overline{B}_\epsilon(y) \subset f^{-1}B_\delta(x) \\
&\equiv \quad \exists y\epsilon.\, e(y) < \epsilon \;\wedge\; \big(\forall y'.\, e(y,y') \le \epsilon \Longrightarrow d(x,fy') < \delta\big) \\
&\Rightarrow \quad \forall \epsilon'.\, \exists yy'\epsilon.\, e(y') < \epsilon' \;\wedge\; \exists y'.\, e(y,y') < \epsilon \;\wedge \\
&\qquad\quad \wedge\; \forall y'.\, e(y,y') \le \epsilon \Longrightarrow d(x,fy') < \delta \\
&\Rightarrow \quad \forall \epsilon'.\, \exists y'.\, e(y') < \epsilon' \;\wedge\; d(x,fy') < \delta
\end{aligned}$$

in which we see the metrical $\epsilon$–$\delta$ definitions of continuity and density.

In the case of a net where $M$ has decidable equality, $M$ may be given a metric with $e(m,m') = 0$ if $m = m'$ and 1 otherwise. Then the formula above reduces to

$$d(x) < \delta \;\equiv\; \exists m \in M.\, d(x, fm) < \delta.$$

# 8 Conclusion: the meaning of overtness

We have left meaning of the phrase "computationally representable" open because different formulations of topology have different logical strengths, whilst the results that we shall develop are valid for each of them.

At one extreme, in classical topology $M$ can simply be the underlying set (of points) of the space $X$, but in this case everything is vacuous.

The set theory without excluded middle or the axiom of choice (topos logic), in which locale theory is most naturally formulated, and Martin-Löf Type Theory (the usual setting for formal topology) impose stricter restictions on $M$.

At the other extreme, we may insist that the set $M$ actually have some computable encoding. Abstract Stone Duality [J] is a reformulation of general topology in which any function that can be defined is automatically both continuous and computable.

This means that we can reason *mathematically* in an (almost) familiar topological way but then (in principle) extract a *program* from the proof.

The formalisation of the theory allows existential quantification $\exists m \in M$ over sets of this kind. So when we have proved that this net gives rise to $\Diamond$ we will settle our earlier doubt about whether $\exists a \in A$ is meaningful.

In order to formulate this, the ***existential quantifiers*** $\exists m \in M$ and $\exists i$ must be meaningful (and commute). Being equipped with such a quantifier is the one of the definitions of an overt *space*. Indeed, in the kind of topology that we need to use, only families of open subsets that are indexed by an *overt* set have unions, not arbitrary ones.

Such questions do not arise in traditional topology, where "all" unions and quantifiers exist. However, it is a natural part of computation, where unions of *recursively enumerable* subsets, say of $\mathbb{N}$, may be formed only for recursively enumerable families.

What, then, *is* an overt subspace? From the discussion of accumulation points, we see that $\Diamond$ and $d$ remain the same if we add limits of subsequences to a net, so many different subspaces or nets may give rise to the same $\Diamond$ operator.

The message that we want to give in this paper is that $\Diamond$ and $d$ capture a problem such as an equation to be solved. It is they that define the overt "subspace", rather than any of the many nets that generate them, or even the (canonical) set of solutions as accumulation points. Overtness is not about these solutions themselves but the way in which $\Diamond$ and $d$, as measurements of the *environment*, give *evidence* of their existence and therefore a means of finding them.

The simplest example of this is the Intermediate Value Theorem, for which [J] made a detailed study of the distinction between *zeroes*, where a function $f : \mathbb{R} \to \mathbb{R}$ *happens* to have $f(x) = 0$ (but maybe as a tangent), and *stable zeroes*, where it *crosses* the axis.

The phenomenon that larger or smaller sets of points may give rise to the same $\Diamond$ operator is the dual of the situation for a compact subspace of a non-Hausdorff space. There [HM81]demonstrated a correspondence between compact *saturated* subspaces and their Scott-open filters of open neighbourhoods.

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