Internal Completeness of Categories of Domains

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Abstract

One of the objectives of category theory is to provide a foundation for itself in particular and mathematics in general which is independent of the traditional use of set theory. A major question in this programme is how to formulate the fact that **Set** is "complete", *i.e.* it has all "small" (*i.e.* set- rather than class-indexed) limits (and colimits). The answer to this depends upon first being able to express the notion of a "family" of sets, and indexed category theory was developed for this purpose.

This paper sets out some of the basic ideas of indexed category theory, motivated in the first instance by this problem. Our aim, however, is the application of these techniques to two categories of "domains" for data types in the semantics of programming languages. These are $\mathbf{Retr}(\Lambda)$, whose objects are the retracts of a combinatory model of the λ-calculus, and $\mathbf{bcCont}_\omega$, which consists of countably-based boundedly-complete continuous posets. They are (approximately) related by Scott's [1976] $\mathrm{P}\omega$ model.

In the case of **Set** we would like to be able to define an indexed family of sets as a function from the indexing set to the "set" of all sets. Of course Russell showed long ago that we cannot have this. However there is a trick with disjoint unions and pullbacks which enables us to perform an equivalent construction called a *fibration*.

$\mathbf{Retr}(\Lambda)$ and $\mathbf{bcCont}_\omega$ do not have all pullbacks. This of course means that they're not strictly speaking complete however we formulate smallness: what we aim to show is that they have all "small" *products*. More importantly, this pullback trick is on the face of it not available to us. They do, on the other hand, have a notion of "universal" set (of which any other is a retract), and indeed of a "set" of sets, though space forbids discussion of this. These we use in stead to provide the indexation.

Having constructed the indexed form of $\mathbf{Retr}(\Lambda)$ we discover that it *does* after all have enough pullbacks to present it as a fibration in the same way as **Set**. However whereas in **Set** *any* map may occur as a display map, in this case we have only a restricted class of them. We then identify this class for $\mathbf{bcCont}_\omega$ and find that it consists of the *projections* (continuous surjections with left adjoint) already known to be of importance in the solution of recursive domain equations.

We formulate the abstract notion of a class of display maps and define *relative cartesian closure* with respect to it. The maximal case of this (as applies in **Set**, where all maps are displays, is known as *local* cartesian closure. The minimal case (where only product projections are displays) is known in computer science as (ordinary) cartesian closure, though in category theory it is now more common to use this term only when all pullbacks exist, though not necessarily as displays.

This work will be substantially amplified (including discussion of the "type of types") in [Taylor 1986?].

1. Indexed Families

We begin by formalising the notion of an indexed family of objects. For basic category theory see [Arbib and Manes 1975], [Mitchell 1965] or [Mac Lane 1971]. This account of indexed categories is very loosely based on [Johnstone *et al.* 1978] and [Johnstone 1983]. [Kock and Reyes 1977] and [Lawvere 1969] provide excellent introductions to some of the underlying ideas of categorical logic.

Let S be some category, whose objects we are thinking of as sets — in the first instance $S = \mathbf{Set}$. In order to talk about products, coproducts and so on in S we need some notion of an A-`indexed family of objects` of S, for each $A \in \mathrm{ob}S$. A naive notation for such a thing might be $(X_a : a \in A)$. Between these there are A-indexed functions, $(f_a : X_a \to Y_a : a \in A)$, so that we have a category S^A. In the basic example this category is simply the A-fold power of the category $S = \mathbf{Set}$.

As well as A-indexed families, we have *substitution* or *relabelling* functors. If $\alpha : B \to A$ is any S-map and $X_a : a \in A)$ is an A-indexed family, we have a B-indexed family $(X_{\alpha b} : b \in B)$. The same applies to morphisms, so this is in fact a functor $S^\alpha : S^A \to S^B$. The assignment $\alpha \mapsto S^\alpha$ is itself (pseudo) (contra) functorial, in that $S^{\mathrm{id}} \cong \mathrm{id}$ and $S^{\alpha\beta} \cong S^\beta S^\alpha$. These natural isomorphisms will have to satisfy some coherence conditions, but we shall not pay too much attention to them.

$(X_a : a \in A)$ appears to be a function from A to the class of all sets, which is a very troublesome notion. As has been remarked, we shall be able to think in this way in $\mathbf{Retr}(\Lambda)$, but not in $\mathbf{Set}$. The trick in $\mathbf{Set}$ is to code this up using the disjoint union, making use of our *a priori* knowledge of the structure of the category but quietly subsuming the *Axiom of Replacement*. The indexed set $(X_a : a \in A)$ is represented by its disjoint union together with the *display map* which identifies the index:

$$\begin{array}{ccc} X = \coprod_{a \in A} X_a & \qquad & x \in X_a \\ \downarrow & & \downarrow \\ A & & a \end{array}$$

An object of S^A is therefore just a function, or S-morphism; X_a is picked out as the inverse image of a, *i.e.* the pullback of the singleton function $a : 1 \to A$ against the display map. In the case $S = \mathbf{Set}$ *any* map may occur as a display map.

The substitution functor $S^\alpha : S^A \to A^B$ over $\alpha : B \to A$ is easily seen to be given by pullback along α and is consequently written $\mathrm{P}\alpha$ or α^*:

$$\begin{array}{ccc} \coprod_{b \in B} X_{\alpha b} \cong X \times_A B & \longrightarrow & X \cong \coprod_{a \in A} X_a \\ \downarrow & & \downarrow \\ B & \xrightarrow{\alpha} & A \end{array}$$

There is an alternative description of this set-up in terms of *fibrations*. The objects over $A \in S$ are the S-maps with codomain A, but besides forming categories over each A (called *fibres*) all the objects together form a category called $S^{\mathbf{2}}$ because it is the category of functors from $\mathbf{2} = (\bullet \to \bullet)$ to S; the morphisms are just the commutative squares in S. There is then a functor $\mathrm{cod} : S^{\mathbf{2}} \to S$ with the property that $X \in S^A$ iff $\mathrm{cod}(X) = A$ and the maps $X \to Y$ over $\alpha : B \to A$ (*i.e.* the squares whose lower side is α) correspond bijectively to the maps $X \to \mathrm{P}\alpha Y$ in S^B. The maps within a single fibre (*i.e.* the squares with an identity along the bottom) are (for obvious reasons) called *vertical* whilst those which give pullback squares are called *horizontal* or *cartesian*.

The fibred (as opposed to indexed) approach was pioneered by Bénabou [1975] in recognition of the fact that substitution is in practice defined only up to isomorphism.

The fibre over A is in this case may be seen as either the A-fold power of S or as the *slice category* S/A whose objects are the S-morphisms with codomain A and whose morphisms are the S-morphisms making the triangles commute. In the relative case the objects of the slice will be display maps but the morphisms will still be arbitrary maps.

There is, however, nothing in the definition of a general fibration to require the fibre over an arbitrary A to be the A-fold power of that over the terminal object of S. Indeed the difference is quite crucial to this and many other applications. We shall denote the fibre over A by $\mathrm{P}A$ and the substitution functor over $\alpha : B \to A$ by $\mathrm{P}\alpha$.

Let us conclude this introduction by considering the form of (binary) products in the fibre categories. Let X and Y be objects over A, presented either as A-indexed things or as displays (S-morphisms with codomain A), and write $X \times_A Y$ for their product according to either interpretation. Naively this is $(X_a \times Y_a : a \in A)$, but as a display it turns out to be precisely the pullback, hence the alternative name *fibre product* for the latter. In English we may therefore say "fibre products are pullbacks", although the (Gaullist) French can't make the *a priori* distinction! Fibred equalisers can also be described quite straightforwardly.

2. Indexed Products

In this section we shall look at products in **Set** from the "indexed family" point of view and justify the Lawvere [1969] dictum that *quantification is adjoint to substitution.* This gives us a notion of "internal product" applicable to $\mathbf{bcCont}_\omega$ and $\mathbf{Retr}(\Lambda)$.

Given an B-indexed family of sets, $(X_b : b \in B)$, their product has elements the indexed families $(x_b : b \in B)$ where $(\forall b)(x_b \in X_b)$. This is a B-indexed family of choices of elements, which is the same as specifying a B-indexed family of maps from the (constant) singleton to the X_b, *i.e.* a map $1_B \to X$ from the terminal object in the fibre over B. Now the display map of the *terminal* object over B is precisely (as an S-morphism) the *identity* on B (which to some extent excuses the ambiguous notation 1_B) so this is just a *section* or *splitting* of the display map.

Write $\prod_B X$ for the product set, and think of it as an object of the fibre over the terminal object (*i.e.* of the category of single sets). It has elements $1_1 \to \prod_B X$, and these are to correspond to the maps $1_B \to X$ over B. Now 1_B is the pullback of 1_1 against the terminal projection $\alpha = !_B : B \to 1$, *i.e.* its image under the substitution functor $\mathrm{P}\alpha$. Thus $\prod_B$ is the right adjoint of $\mathrm{P}\alpha$:

$$\begin{array}{ccccccc} 1_B & \cong & \mathrm{P}\alpha 1_1 & \to & X & \text{over} & B \\ & & 1_1 & \to & \prod_B X & \text{over} & 1 \end{array}$$

Now let us do this *indexedly*. So given $((X_b : b \in B_a) : a \in A)$, an A-indexed family of B_a-indexed families of objects, we need to show how to construct the ath product, $(\prod_{b\in B_a} : a \in A)$, and present it as a member of an A-indexed family.

To do this we begin by displaying the B_a's over A, *i.e.* we construct a morphism $\alpha : B \to A$, where $B = \coprod B_a$; then we present the objects $((X_b : b \in B_a) : a \in A)$ as a B-indexed family $(X_b : b \in B)$. The elements of the member $(\Pi\alpha X)_a = \prod_{b\in B_a} X_b$ of the product are maps to $1 \to (\Pi\alpha X)_a$ which are to correspond to indexed families $(1 \to X_b : b \in B_a)$ and so the maps $1_B \to \Pi\alpha X$ over A correspond to those $1_B \to X$ over B. In other words $\Pi\alpha$ (or α_* is the right adjoint to the substitution or pullback $\mathrm{P}\alpha$ (or α^*).

$$\begin{array}{llll} 1_B \cong \mathrm{P}\alpha 1_A \longrightarrow X & & & \text{over} B \\ 1_A \longrightarrow \Pi\alpha X \cong & & (\prod_{b\in B_a} X_b : a \in A) & \text{over} A \end{array}$$

Definition An internal *product* in an indexed category is a right adjoint to substitution over a display map.

The are no conceptual difficulties in doing this for **Set**: collections of maps may be understood naively, and *any* morphism in the base category S occurs as a display map. This is not so in $\mathbf{bcCont}_\omega$ or $\mathbf{Retr}(\Lambda)$: we have to make our indexing "continuous", and not every map occurs as a display map (though we have yet to define them).

Likewise the collections of maps (cones) in the definition of product have to be "continuously varying". Whilst clearly $\mathbf{Retr}(\Lambda)$ and $\mathbf{bcCont}_\omega$, being (very) small, do not have all products externally, they still "think" they have them in this sense. Where the "continuously varying" is taken to mean computable or definable we have the appropriate restriction on the definitions to make them appropriate to programming or intuitionistic type theory.

Let us consider the corresponding notion to product for Natural Deduction, which is *universal quantification.* In order to prove $(\forall b)Y(b)$ from X, where the bound variable is of type B and there are no free variables, it is necessary and sufficient to prove $Y(b)$ from the same hypotheses (in which the variable b does not occur freely). This rule corresponds formally to the natural bijection

$$\begin{array}{ccccc} X & \Rightarrow & Y(b) & \text{over} & B \\ \\ X & \Rightarrow & (\forall b)(Y(b)) & \text{over} & 1 \end{array}$$

(in which, as is usual, we have written in invisible ink above the line the substitution functor which gives X an invisible free variable) which says that $(\forall b)$ is right adjoint to the substitution.

If we had allowed Y to have other free variables besides b, we should have been performing the same argument indexedly over the type(s) of these other free variables, and the relevant substitution functors would have been over product projections in the base category. Now a product projection is the display of a *constant* family; in the case of a general display map $\alpha : B \to A$ the quantifier becomes (at a) $(\forall b)(\alpha b = a \Rightarrow ...)$.

There is a mild technicality in this called the *Beck condition.* We want to be sure that substitution and quantification interact properly, in the sense that if $\mathrm{P}\alpha$ is the introduction of a variable b and $\Pi\beta$ the quantification over c (where $\alpha : B \times A \to A$ and $\beta : C \times A \to A$) we have $\mathrm{P}\alpha(\Pi\beta Y) = \Pi\beta(\mathrm{P}\alpha Y)$. More generally, if the left-hand figure is a pullback in the base category then the right-hand square must commute (at least up to isomorphism):

$$\begin{array}{ccc} C \times_A B & \xrightarrow{\gamma} & C \\ {\scriptstyle\delta}\downarrow & & \downarrow{\scriptstyle\beta} \\ B & \xrightarrow{\alpha} & A \end{array} \qquad\qquad \begin{array}{ccc} \mathrm{P}(C \times_A B) & \xrightarrow{\Pi\gamma} & \mathrm{P}C \\ {\scriptstyle\mathrm{P}\delta}\uparrow & & \uparrow{\scriptstyle\mathrm{P}\beta} \\ \mathrm{P}B & \xrightarrow{\Pi\alpha} & \mathrm{P}A \end{array}$$

Seely [1983] has given a detailed discussion of the meaning of the Beck condition. Since our interest is in certain concrete examples (where it holds automatically) rather than the abstract formulation, we shall pay little attention to it.

Now let us consider this adjoint to substitution in the context of the fibration cod : $S^{\mathbf{2}} \to S$, in which the substitution functors are pullbacks. Take first the case of the terminal projection $\alpha : B \to 1$; pulling $X \to 1$ back along this yields simply the product $X \times B$, so $\mathrm{P}\alpha = (-) \times B$. The right adjoint to this is quite familiar and is written $(-)^B$:

$$\begin{array}{cccc} X \times B & \to & Y & \text{over} \quad B \\ \\ X & \to & Y^B & \text{over} \quad 1 \end{array}$$

Thus S (*qua* indexed category cod : $S^{\mathbf{2}} \to S$) has global products iff it (*qua* category) has exponentials. We have deliberately avoided the discussion in terms of fibrations, but more generally,

Proposition The fibration cod : $S^{\mathbf{2}} \to S$ is complete (has all indexed limits) iff S is locally cartesian closed. □

This at first struck me as somewhat remarkable, but of course it is because of the idea (which there is a tendency to push to the back of one's mind as too childish) that powers are iterated products.

In fact the above result is really a definition of local cartesian closure since we haven't yet formally given one.

3. Indexed Coproducts

The case for sums is very similar, although we are not allowed to argue in terms of elements any more. The corresponding deduction rule for the existential quantifier is formally the same as the definition of left adjoint to substitution For the cod fibration the following easily-overlooked triviality is appropriate:

Proposition For an S-morphism $\alpha : B \to A$, the pullback functor $\mathrm{P}\alpha : S/A \to S/B$ along α has a left adjoint $\Sigma\alpha$ (or $\alpha_!$) given by postcomposition with α. □

Definition An internal *coproduct* or *sum* in an indexed category is a left adjoint to substitution over a display map.

Proposition The fibration $\mathrm{cod} : S^{\mathbf{2}} \to S$ is cocomplete iff S has all *finite* (limits and) colimits. □

Starting from the indexed approach we now have a direct route to the display map: recall that this was originally given as a disjoint union. Let X be an A-indexed family (object of the fibre over A). It has a terminal projection $X \to 1_A$ in this fibre, and the image of this under the sum functor is of course $\sum_A X \to \sum_A 1_A$ over 1; but $\sum_A 1_A$ is (isomorphic to) A (in the canonical identification of S with the fibre over 1).

Proposition In the fibration $\mathrm{cod} : S^{\mathbf{2}} \to S$, the fibre over the terminal object is equivalent to S and the display $X \to A$ corresponds to the map $\sum_A X \to \sum_A 1_A$ in this fibre. □

This is the method by which we shall identify displays of retracts.

The analogue of coproducts or sums in natural deduction is existential quantification; the reader is invited to demonstrate that this is indeed left adjoint to substitution. The ordinary case $(\exists c)(\phi)$ arises as before from a product projection $C \times A \to A$; for a general morphism $\alpha : B \to A$ in the base category we have the idiom $(\exists b)(\alpha b = a \wedge \phi)$. We also need a Beck condition, but those for Σ and Π (or $\exists$ and $\forall$) are equivalent (so long as they both exist) because a diagram of left adjoints commutes up to isomorphism iff the corresponding diagram of right adjoints does so.

4. Local Cartesian Closure

So far we have been using the term "locally cartesian closed" to mean having right adjoints to pullback functors, without giving any explanation of what it has to do with "ordinary" cartesian closure in the sense of having exponentials. In this section we shall rectify this omission.

Recall that for an object Y in the fibre over A in $\mathrm{cod} : S^{\mathbf{2}} \to S$, the product with Y in this fibre, and substitution from the fibre over A to that over Y, are both given by pullback along the display map $Y \to A$.

Lemma The object Y in the fibre over A of $\mathrm{cod} : S^{\mathbf{2}} \to S$ is exponentiable iff pullback along $Y \to A$ has a right adjoint; moreover in this case exponentiation by Y is preserved by any pullback.

Proof Let $\beta : Y \to A$ be the display of Y and $\Pi\beta$ be the right adjoint to the pullback $\mathrm{P}\beta$; then

$$\begin{array}{ccccccccc} & & X \times_A Y & \to & Z & & & \text{over} & A \\ X \times_A Y & \cong & \mathrm{P}\beta X & \to & \mathrm{P}\beta Z & \cong & Y \times_A Z & \text{over} & Y \\ & & X & \to & \Pi\beta(\mathrm{P}\beta Z) & \cong & Z_A^Y & \text{over} & A \end{array}$$

Conversely suppose $(-)_A^Y$ is right adjoint to $- \times_A Y$ in the fibre over A and let W be over Y. Then

$$\begin{array}{ccccccccc} X \times_A Y & \cong & \mathrm{P}\beta X & \to & W & & & \text{over} & Y \\ & & X \times_A Y & \to & W \times_A Y & & & \text{over} & A \\ & & X & \to & (W \times_A Y)_A^Y & \cong & \Pi\beta W & \text{over} & A \end{array}$$

By *preservation* we mean that $\mathrm{P}\alpha(Z_A^Y) \cong (\mathrm{P}\alpha Z)_B^{(\mathrm{P}\alpha Y)}$ for any map $\alpha : B \to A$.

$$\begin{array}{ccccccc}
 & & U & \to & \mathrm{P}\alpha(Z_A^Y) \cong Z_A^Y \times_A B & \text{over} & B \\
 & & U & \to & Z_A^Y & \text{over} & A \\
 & & U \times_A Y & \to & Z & \text{over} & A \\
U \times_A Y & \cong & \mathrm{P}\alpha Y \times_B U & \to & \mathrm{P}\alpha Z & \text{over} & B \\
 & & U & \to & (\mathrm{P}\alpha Z)_B^{(\mathrm{P}\alpha Y)} & \text{over} & B
\end{array}$$

□

The particularly alert reader will have noticed in following this an implicit use of the Beck condition, which is essentially equivalent to the preservation of exponentials, but is of course a theorem in this case.

Proposition The fibres of cod : $S^{\mathbf{2}} \to S$ are cartesian closed and substitution preserves arbitrary limits and exponentials iff S is locally cartesian closed. In this case substitution also preserves any colimits which exist. □

It is essential here to include the condition that exponentials be *preserved*, since otherwise we have a strictly weaker notion.

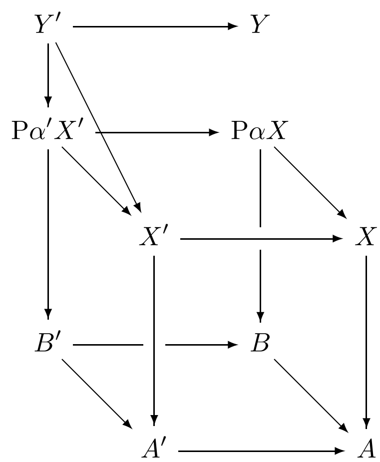
5. Local Smallness and other Internal Notions

The word "local" is used in the context of (indexed) category theory with reference to the fibres or slices S/A. This is a generalisation of the fact that for the open set lattice of a topological space (considered as a poset and hence a category), the slice over (*i.e.* open subsets of) an open set gives (the open set lattice of) the corresponding open subspace. A *local* notion in category theory is therefore one which is preserved by pullbacks, so that it happens in the fibres (slices) and is preserved by substitution.

We can use these methods to formulate definitions and constructions internally (say in a locally cartesian closed category). This usually takes the form of finding a *generic* construction, of which any other is obtained by substitution (pullback), preferably uniquely.

We shall illustrate this by formulating the idea of a category *having small hom-sets* or being *locally small*. Since cartesian closure is concerned with exponentials, *i.e. sets* of functions, it will not come as a surprise that these are equivalent. We can formulate local smallness as having a generic morphism, *i.e.* one from which any other may be obtained by substitution (pullback).

Thus if $Y \in \mathrm{P}B$ and $X \in \mathrm{P}A$ with $\alpha : B \to A$, by a *generic morphism* from Y to X over α we mean a diagram of the form



in which the squares are pullbacks, which is generic in the sense that any other diagram of the same shape (but with $''$ for $'$) is obtained by pulling back this one by a unique $A'' \to A'$.

Proposition S is locally small iff it is locally cartesian closed.

Proof Suppose Y is exponentiable in its fibre. Put $B' = (\mathrm{P}\alpha X)^Y_B$ and $A' = A \times_B B'$. Then

$$
\begin{array}{ccccccc}
 & & Y'' & \to & X'' & & \text{over} & \alpha'' \\
Y \times_B B'' & \cong & Y'' & \to & \mathrm{P}\alpha'' X'' & \cong \ \mathrm{P}\alpha X \times_B B'' & \text{over} & B'' \\
 & & B'' \times_B Y & \to & \mathrm{P}\alpha X & & \text{over} & B \\
 & & B'' & \to & (\mathrm{P}\alpha X)^Y_B & =_{\text{def}} \ B' & \text{over} & B
\end{array}
$$

and it's not difficult to see that the correspondence is obtained by pullback.

Conversely if B' is generic then

$$
\begin{array}{ccccc}
B'' \times_B Y & \to & X & \text{over} & B \\
B'' \times_B Y & \to & B'' \times_B X & \text{over} & B'' \\
B'' & \to & B' =_{\text{def}} X^Y_B & \text{over} & B
\end{array}
$$

The same approach gives a notion of a generic subobject. A category with finite limits and generic subobjects is called an *elementary topos*. Unfortunately, although $\mathbf{bcCont}_\omega$ does have something which might serve as an object-of-subobjects, the display of a generic subobject is (not surprisingly) a mono, whilst the displays in $\mathbf{Retr}(\Lambda)$ and **bcCont** are all epi because of global habitation. Consequently we shall not discuss these ideas.

Finally we might ask for a generic family of objects, *i.e.* a display map of which any other is a pullback. The codomain of this would be a "type of types", each type occurring as the inverse image of an element of it. From any family of objects (display map $G \to V$) we may construct the full subcategory whose objects are in the family; its object set is V and its morphism set $G^G_{V\times V}$ over $V \times V$. The inverse image of $\langle X, Y\rangle \in V \times V$ is just $Y^X = \hom_S(X, Y)$.

However it may be shown that if a *locally* cartesian closed category has a generic family then it has both a generic subobject, $\{true\} \subset \Omega$, (making it a topos) and a "universal set", G, (of which any other, in particular its own powerset Ω^G, is a subobject). Cantor's theorem shows that this is impossible. But if we drop the requirement for equalisers we find the paradox disappears, and indeed more or less any category of domains has this property.

6. The Category of Retracts of a Combinatory Model

We now introduce the first of our two categories of domains, which may be constructed from any combinatory model of the λ-calculus, Λ. For a comprehensive account see [Barendregt 1981] or [Curry and Feys 1958].

A *combinatory model* is a set Λ with a binary operation (application) and constants K, S satisfying $\mathrm{K}ab = a$, $\mathrm{S}abc = ac(bc)$ and five other equations originally formulated by Curry. We adopt the usual convention for omitting brackets, so abc means $(ab)c$. This enables us to interpret λ-terms in Λ by the scheme

$$\begin{array}{rcll} [\lambda x.x] & = & \mathrm{SKK} & \\ {[ab]} & = & [a][b] & \\ {[\lambda x.a]} & = & \mathrm{K}[a] & \text{if x is not free in a} \\ {[\lambda x.ab]} & = & \mathrm{S}[\lambda x.a][\lambda x.b] & \text{otherwise} \end{array}$$

Given a combinatory model Λ, we can define a cartesian closed category $\mathbf{Retr}(\Lambda)$ in which any term is typable, called the *category of retracts*. I shall now use the term cartesian closed in the weak sense of having products and exponentials, not necessarily all finite limits.

The combinators $\mathrm{I} = \lambda x.x = \mathrm{SKK}$ and $\mathrm{P} = \lambda xyz.y(xz)$ define a monoid $\mathrm{M} \subset \Lambda$ as follows. $\mathrm{M} = \{f : \mathrm{PI}(\mathrm{P}f\mathrm{I}) = f\}$, the composition is $f \cdot g = \mathrm{P}gf$ and the identity is I.

An *object* of $\mathbf{Retr}(\Lambda)$ is an idempotent of M, *i.e.* an element $A \in \Lambda$ satisfying $\mathrm{P}AA = A$. Unfortunately *this* operation is not itself idempotent: indeed in general there is no retract of Λ whose image is $\mathrm{ob}\mathbf{Retr}(\Lambda) \subset \Lambda$. The *morphisms* $\alpha : A \to B$ are the elements $\alpha \in \Lambda$ with $\alpha = \mathrm{P}A\alpha = \mathrm{P}\alpha B$. The *identity* on A is A itself, whilst the *composite* of $\alpha : A \to B$ and $\beta : B \to C$ is $\mathrm{P}P\alpha\beta : A \to C$.

Notice that we have dropped the category-theoretic convention that the various hom-sets be disjoint, so we do not have functions *dom* and *cod*; however it is a straightforward but unenlightening exercise to code these things in if they are required.

Idempotents split in $\mathbf{Retr}(\Lambda)$, so if $\alpha : A \to A$ satisfies $\alpha^2 = \alpha$, *i.e.* $\mathrm{P}\alpha\alpha = \alpha$, then there is an object B and a pair of maps $B \rightleftarrows A$ such that $B \to A \to B$ is the identity and $A \to B \to A$ is α; in fact of course both B and the two maps are represented by α. B is then both the equaliser and the coequaliser of α with the identity. Idempotents split in a category iff it has all finite filtered colimits (or limits).

$\mathbf{Retr}(\Lambda)$ is in fact the universal idempotent splitting category containing the monoid M (considered as a category with one object, identified with $\mathrm{I} \in \mathbf{Retr}(\Lambda)$). So if $\mathrm{M} \to \mathbf{D}$ is a functor to a category in which idempotents split then there is a unique (up to unique isomorphism) functor $\mathbf{Retr}(\Lambda) \to \mathbf{D}$ making the triangle commute. This means that if M is the (external) endomorphism monoid of an object in a category in which idempotents split (as they will if the category has all finite limits) then $\mathbf{Retr}(\Lambda)$ is embedded "concretely" as the category of retracts of the object.

$\mathbf{Retr}(\Lambda)$ has a *terminal object* $\mathrm{T} = \mathrm{K}\bot$, where we choose $\bot = (\lambda x.xx)(\lambda x.xx) = (\mathrm{SII})(\mathrm{SII})$, and this also denotes the terminal projection, the unique map $A \to \mathrm{T}$. If Λ is a *model* (Koymans [1984]) then T is in fact a *generator*, *i.e.* if $\alpha, \beta : A \rightrightarrows B$ are two maps whose composites with all maps $\mathrm{T} \to A$ are equal then $\alpha = \beta$.

The type A can be interpreted as the set $\|A\| = \{a : a = Aa\} \subset \Lambda$ and the map $\alpha : A \to B$ as the function $a \to \alpha a$. The type T has a unique set-theoretic element, viz. $\bot \in \|\mathrm{T}\|$, so we may think of T as the *one-element set*. An arrow $\mathrm{T} \to A$ is called a *global (category-theoretic) element* of A. The set-theoretic and category-theoretic elements of A now correspond using K and $\bot$ ("dropping a variable"): given $a \in \|A\|$ we have $\mathrm{K}a\mathrm{T} \to A$, and given $\alpha : \mathrm{T} \to A$ we have $\alpha\bot \in \|A\|$.

That T is a generator for $\mathbf{Retr}(\Lambda)$ means exactly that functions are *extensional*, *i.e.* they are equal iff they have the same effect on elements. Finally every type A has at least one element, $A\bot \in \|A\|$. This is the property of *global habitation* which is an important feature of these models. The representation of A by $\|A\|$ makes $\mathbf{Retr}(\Lambda)$ into a concrete category, *i.e.* $\| - \| : \mathbf{Retr}(\Lambda)) \to \mathbf{Set}$ is faithful.

For an arbitrary $a \in \Lambda$, we call Aa *a reduced to* A; many constructions are of the form of general or untyped constructions reduced to appropriate types.

We shall construct finite products and exponentials in $\mathbf{Retr}(\Lambda)$, and show that its objects have internal fixpoints. It is a consequence of this that $\mathbf{Retr}(\Lambda)$ cannot have binary coproducts or all finite limits.

We now have to choose pairing and unpairing combinators $\langle\rangle = \lambda xyz.zxy$, $\mathrm{0} = \lambda xy.x = \mathrm{K}$ and $\mathrm{1} = \lambda xy.y = \mathrm{KI}$, so that $\langle\rangle ab\mathrm{0} = a$ and $\langle\rangle ab\mathrm{1} = b$ for all $a, b \in \Lambda$. Write $\langle a, b\rangle$, c_0 and c_1 for $\langle\rangle ab$, $(c\mathrm{0})$, $(c\mathrm{1})$ respectively, noting carefully the positions of the digits.

The *product* $A \times B$ of A and B in $\mathbf{Retr}(\Lambda)$ is $\lambda c.\langle Ac_0, Bc_1\rangle$, which is our abbreviation for $\lambda c.\langle\rangle(A(c\mathrm{0}))(B(c\mathrm{1}))$, with projections $\pi_0 = \lambda c.Ac_0$, $\pi_1 = \lambda c.Bc_1$, *i.e.* $\mathrm{0}$ and $\mathrm{1}$ reduced to (domain $A\times B$ and) codomain A or B. Given $\alpha : D \to A$ and $\beta : D \to B$, $\langle \alpha, \beta\rangle = \lambda d.\langle \alpha d, \beta d\rangle$ is the unique map (*pair*) $D \to A \times B$ making the two triangles commute. There is a combinator $\times = \lambda AB.A \times B$ which, when restricted to $\mathrm{ob}\mathbf{Retr}(\Lambda) \times \mathrm{ob}\mathbf{Retr}(\Lambda) \to \Lambda$, yields (idempotents representing) products. The forgetful functor $\| - \| : \mathbf{Retr}(\Lambda) \to \mathbf{Set}$ creates finite products and preserves all limits which exist in $\mathbf{Retr}(\Lambda)$.

$\mathbf{Retr}(\Lambda)$ also has *function spaces*, because $\lambda f.\mathrm{P}Af$ and $\lambda f.\mathrm{P}fB$ are commuting idempotents (assuming $A = \mathrm{P}AA$ and $B = \mathrm{P}BB$) so that their composite, $B^A = \lambda f.\mathrm{P}A(\mathrm{P}fB) = \lambda f.\mathrm{P}(\mathrm{P}Af)B$, is (idempotent, *i.e.*) a type. Given $\alpha : C \times A \to B$ we have the exponential transpose $\tilde{\alpha} = \lambda ca.\alpha\langle c, a\rangle : C \to B^A$ and conversely $\alpha = \lambda d.\tilde{\alpha}d_0d_1$; this is the ancient trick of *Currying*. The *evaluation map* $\mathrm{ev} : B^A \times A \to B$ is given by $\lambda d.C(d_0(Bd_1)$. Again there are obvious combinators doing these things. B^Af is called f *reduced to domain* A *and codomain* B, but one should beware that this reduction is *not* functorial (it does not preserve identity and composition).

Each type $A \in \mathrm{ob}\mathbf{Retr}(\Lambda)$ has internal fixpoints: put

$$\mathrm{Y}_A = \lambda f.(\lambda x.xx)(\lambda x.A(f(A(xx))))$$

Then $\mathrm{Y}_A : A^A \to A$ makes a certain diagram commute, which says $f(\mathrm{Y}_Af) = \mathrm{Y}_Af$ for all $f \in \|A^A\|$. This is just the reduction of Y to $A^A \to A$. Observe that the canonical fixpoint of the identity is the "bottom" element, $A\bot$, of the type; this is a deliberate and crucial choice.

7. Indexed Category of Retracts

The method used in the opening sections for making **Set**into an indexed category over itself requires the existence of all pullbacks, which are not available in the categories which interest us. On the other hand, this pullback trick was required for **Set** because of the size problem with **Cat**, in other words we have no universal set. In $\mathbf{Retr}(\Lambda)$, the category of retracts of a combinatory model of the λ-calculus, we *do* have a kind of universal set, namely the model itself. In the *large* category **bcCont** there is no "global" universal set, but there is a sense in which it has "local" ones. In this section we shall construct the indexed category of retracts, then in section 9 the display maps will be identified (along with some of the indexed sums and products).

For $A \in \mathbf{Retr}(\Lambda)$, an *A-indexed type* is a ("continuous") function $X : A \to \Lambda$ taking type values, *i.e.* $X = \mathrm{P}AX = \mathrm{P}X\Lambda$ (although, since $\Lambda = \mathrm{I}$, $\mathrm{P}X\Lambda = X$ is tautologous) such that $\mathrm{P}(Xa)(Xa) = Xa$ for all $a \in \|A\|$. We may rewrite this as $X = \mathrm{P}AX = \mathrm{Q}XX$ where $\mathrm{Q} = \lambda wxyz.xy(wyz)$. Thus $\mathrm{ob}\mathbf{P}(A = \{X : \mathrm{P}AX = X = \mathrm{Q}XX\}$.

The structure of $\mathbf{P}(A)$ is given in the same fashion as that of $\mathbf{Retr}(\Lambda)$, except that the combinators take an extra argument (this is only a notational complication, but it provides an ample supply of pitfalls). Again, as with our presentation of $\mathbf{Retr}(\Lambda)$, there is no information coded in to define domain, codomain and fibration, but these may be recoded as before. As before, note that Q has *four* variables and that $\mathrm{Q}(\mathrm{Q})fg)h = \mathrm{Q}f(\mathrm{Q}gh)$ and $\mathrm{Q}(\mathrm{P}\alpha f)(\mathrm{P}\alpha g) = \mathrm{P}\alpha(\mathrm{Q}fg)$.

The *objects* of $\mathbf{P}(A)$, for $A \in \mathrm{ob}\mathbf{Retr}(\Lambda)$, are those $X \in \Lambda$ with $X = \mathrm{P}AX = \mathrm{Q}XX$. The *morphisms* $X \to Y$ are those $f \in \Lambda$ with $f = \mathrm{P}Af = \mathrm{Q}Xf = \mathrm{Q}fY$; these conditions say respectively that f is fibred over A and that in each fibre $a \in A$ it has domain Xa and codomain Ya. The *identity* on X is X itself, and the *composite* of $f : X \to Y$ and $g : Y \to Z$ is $\mathrm{Q}fg : X \to Z$. Note that $\mathrm{P}A$ preserves the fixed points of $\lambda x.\mathrm{Q}xx$ and that $\lambda f.\mathrm{P}Af, \lambda f.\mathrm{Q}Xf$ and $\lambda f.\mathrm{Q}fY$ are commuting idempotents for $A \in \mathrm{ob}\mathbf{Retr}(\Lambda)$ and $X, Y \in \mathrm{ob}\mathbf{P}(A)$.

$\mathbf{P}(A)$ has *terminal object* $\mathrm{U} = \mathrm{K}(\mathrm{K}\bot)$, and this also represents the *terminal projection* $X \to \mathrm{U}$. As before we may consider (A-indexed) elements: $x \in_A X$ is a function $A \to \Lambda$ such that $xa \in Xa$ for each $a \in A$; the type of all such indexed elements of X is the (global) *product*, $\prod X = \lambda pa.Xa(p(Aa))$.

The *product* $X \times_A Y$ of X and Y over A is given by $\lambda az.\langle Xaz_0, Yaz_1\rangle$; the *projection maps* are $\pi_0 = \lambda az.Xaz_0$ and $\pi_1 = \lambda az.Yaz_1$, and if $f : Z \to X$, $g : Z \to Y$ are two maps in $\mathrm{P}A$ then the *pair* is $\langle f, g\rangle = \lambda az.\langle faz, gaz\rangle : Z \to X \times_A Y$.

Because of the observation about commuting idempotents, we have fibred *exponential* types.

$$Y_A^X = \lambda af.\mathrm{P}(Xa)(\mathrm{P}(fa)(Ya)) = \lambda af.\mathrm{Q}(\mathrm{Q}Yf)Za = \lambda af.\mathrm{Q}Y(\mathrm{Q}fZ)a = \lambda afx.Ya(fa(Xax))$$

This is obtained by an interchange of variables from the reduction of f to a solution of $f = \mathrm{P}Af = \mathrm{Q}Xf = \mathrm{Q}fY$; the latter is a "global section" of the former. The *evaluation map* $\mathrm{ev} : Y_A^X \to Y$ is given by $\lambda ap.Ya(p_0(Xap_1))$ and the *transpose* identifies $f : W \times_A X \to Y$ with $g : W \to Y_A^X$ by $g = \lambda awx.fa\langle w, x\rangle$ and $f = \lambda ap.gap_0p_1$.

Now let $\alpha : B \to A$ be any map in the base category $\mathbf{Retr}(\Lambda)$; what is the corresponding *substitution functor* $\mathbf{P}(\alpha) : \mathbf{P}(B) \to \mathbf{P}(A)$, and does it have adjoints? The first question has an easy answer, which gives a pleasing consonance of notation: $\mathbf{P}(\alpha) = \mathrm{P}\alpha$. In the same way as the naive indexing of **Set** over itself performed substitution by composition, so does this.

Thus $\mathbf{P}(\alpha)(X)$ is simply $\mathrm{P}\alpha X$ and $\mathbf{P}(\alpha)(f) = \mathrm{P}\alpha f$; moreover $\mathrm{P}\alpha U = U$, $\mathrm{P}\alpha X \times_A Y = (\mathrm{P}\alpha X) \times_B (\mathrm{P}\alpha Y)$ and $\mathrm{P}\alpha(Y_A^X) = (\mathrm{P}\alpha Y)_B^{(\mathrm{P}\alpha X)}$ *exactly*. The product projections, pairings, evaluation maps and transposes are also preserved exactly. Because of this notational coincidence (which I hope justifies the switch of variables in the combinator P to the even the most uncompromising users of left-handed notation), $\mathbf{P}(A)$ and $\mathbf{P}(\alpha)$ will in future be written $\mathrm{P}A$ and $\mathrm{P}\alpha$ respectively.

The fibration $\mathrm{cod} : \mathbf{Set}^{\mathbf{2}} \to \mathbf{Set}$ of the category of sets over itself had the property that the fibre PT over the singleton (terminal object) T was equivalent to **Set** itself, and the fibre $\mathrm{P}A$ over A was its A-indexed power. The former remains the case for indexed retracts (by dropping a variable, *i.e.* $A \in \mathbf{Retr}(\Lambda)$ corresponds to $\mathrm{K}A \in \mathrm{PT}$ and $X \in \mathrm{PT}$ to $X\bot \in \mathbf{Retr}(\Lambda)$; but not the latter.

In fact $\mathrm{P}A$ should be regarded as the category of *continuously* A-indexed types, where *continuously* may in suitable circumstances be interpreted as *definably* or *computably*. If A is in fact some kind of "type of types" the way is open for the interpretation of polymorphic languages, or of type expressions (*not* functors) of which fixpoints might be sought, *i.e.* the solution of recursive domain equations.

To sum up, we have a fibration $p : \mathbf{P} \to \mathbf{Retr}(\Lambda)$ (or indexed category $\mathrm{P} : \mathbf{Retr}(\Lambda)^{\mathrm{op}} \to \mathbf{Cat}$) over a cartesian closed category, such that each fibre is itself cartesian closed and this structure is preserved *exactly* by the substitution functor. The fibre PT over the terminal object is isomorphic to $\mathbf{Retr}(\Lambda)$ itself.

8. Relatively Cartesian Closed Categories

We have now seen a category without all pullbacks ostensibly indexed over itself, and so not by the cod : $S^{\mathbf{2}} \to S$ fibration we used for **Set**. We now introduce the notion of *relative* cartesian closure, which enables us to unify these constructions. In the next section we identify the display maps in $\mathbf{Retr}(\Lambda)$.

In the sections 2 and 4 the (well-known) connection between internal completeness and local cartesian closure was described. Throughout the account, however, it has been hinted that there is a more general construction in which not all maps map occur as display maps, and that this applies to our $\mathbf{Retr}(\Lambda)$ and **bcCont**. In this section this more general version will be formulated.

Recall that the *slice category* $\mathbf{C}/A$ has objects the $\mathbf{C}$-morphisms with codomain A and morphisms the $\mathbf{C}$-morphisms making the triangle commute. If $\alpha : B \to A$ is a $\mathbf{C}$-morphism there is a functor $\alpha_! : \mathbf{C}/B \to \mathbf{C}/A$ given by postcomposition with α. The right adjoint to $\alpha_!$, if it exists, is called $\mathrm{P}\alpha$ and is given by pullback along α. If $\mathrm{P}\alpha$ itself has a right adjoint, written $\Pi\alpha$, then α is said to be *exponentiable*.

More generally let $\mathbf{D}$ be a class of $\mathbf{C}$-maps the *relative slice* $\mathbf{C}/_{\mathbf{D}}A$ has objects the $\mathbf{D}$-maps with codomain A but still *all* $\mathbf{C}$-maps as morphisms (so long as the triangle still commutes in $\mathbf{C}$). Thus $\mathbf{C}/_{\mathbf{D}}A$ is a *full* subcategory of $\mathbf{C}/A$.

Definition A category $\mathbf{C}$ is said to be *cartesian closed relative to* a class of (*display*) maps $\mathbf{D} \subset \mathbf{C}$ if

(i) The pullback of any $\mathbf{D}$-map against any $\mathbf{C}$-map exists and is in $\mathbf{D}$,

(ii) The composite of any two $\mathbf{D}$-maps is in $\mathbf{D}$,

(iii) $\mathbf{C}$ has a terminal object and any terminal projection is in $\mathbf{D}$, and

(iv) For $\alpha : B \to A$ in $\mathbf{D}$, pullback $\mathrm{P}\alpha : \mathbf{C}/_{\mathbf{D}}A \to \mathbf{C}/_{\mathbf{D}}B$ has a right adjoint $\Pi\alpha$.

Examples

(i) **Set**, or any locally cartesian closed category, is cartesian closed relative to all maps.

(ii) Any cartesian closed category is cartesian closed relative to the class of all product projections.

From these data we can construct a fibred category $p : \mathbf{P} \to \mathbf{C}$, where the *objects* over $A \in \mathbf{C}$ are the display maps $X \to A$ with codomain A, and the *morphisms* over $\alpha : B \to A$ from $Y \to B$ to $X \to A$ are the commutative squares of which three sides have already been given. The fibre $\mathrm{P}A$ over A is therefore $\mathbf{C}/_{\mathbf{D}}A$.

Lemma If $(\mathbf{C}, \mathbf{D})$ satisfy axiom (i) above, then this is a fibration; the fibres are relative slices and the horizontal maps are the pullback squares. □

This coincides with the standard construction in the case of **Set**; for the minimal class of display maps the families are all "constant" ones, so this isnt very interesting.

Axiom (ii) serves a dual role: it performs mundane categorical bookkeeping, but also provides indexed sums. The purpose of axiom (iii) is that we should be able to speak of the fibration as actually being the indexed form of the original category. Henceforward $(\mathbf{C}, \mathbf{D})$ are assumed to satisfy axioms (i) to (iii).

Lemma $\mathbf{C}$ is canonically identified with the fibre over its terminal object. □

Lemma $\mathbf{C}$ has finite products, and their projections (including all isomorphisms) are in $\mathbf{D}$ □

Lemma The fibres have finite products (given by pullback) and these are preserved (up to isomorphism) by arbitrary pullback functors. □

Lemma Pullback along display maps has a left adjoint (namely postcomposition with the display map). The display map in the sense constructed at the end of section 2 coincides with that defining the indexed type. □

Finally axiom (iv) deals with products and exponentials, as was proved in section 4.

Lemma The fibres are cartesian closed (and this structure is preserved (up to isomorphism) by pullback functors) iff axiom (iv) holds. □

Theorem A relatively cartesian closed category gives rise to an indexed category whose fibres are cartesian closed and whose substitution functors preserve this structure. It is complete and cocomplete in the sense that substitution along display maps has adjoints on both sides which satisfy the Beck condition.

Proof It remains only to show that the Beck condition holds, but this is yet another application of the definitions of pullbacks. □

Having now set up the theoretical machinery for talking about internal products, sums and function spaces in categories we devote the remainder of the paper to the identification of the display maps in $\mathbf{Retr}(\Lambda)$ and **bcCont**, showing that these categories are relatively cartesian closed.

9. Displays of Retracts

The next objective is to identify and construct the display maps in $\mathbf{Retr}(\Lambda)$ and hence show that this has indexed products and sums. For this it will be enough to know (in the first instance) about *global* sums, although in fact we shall need to do the local (indexed) case implicitly in the course of the proof of the final lemma of this section; we have already done the work for the indexed products in our study of exponentials.

Recall that the substitution functor over $\alpha : B \to A$ is written $\mathrm{P}\alpha$ and is given by precomposition with α; its adjoints (where they exist) are called $\Sigma\alpha$ and $\Pi\alpha$ for reasons which were discussed in sections 2 and 3. These take values in the fibre over A. The "global" sum and product functors, which take values in $\mathbf{C}$ itself (but recall that this is canonically identified with PT, the fibre over the terminal object) will be written simply $\sum$ and $\prod$, so (under this identification), $\prod = \Pi\mathrm{T} = \Pi!_B$ in the various notations, and likewise with Σ.

Let $B \in \mathbf{C}$ and $Y \in \mathrm{P}B$. The basic idea of $\prod Y$ is the set

$$\{p : B \to \Lambda \mid (\forall b)(pb \in Yb)\}$$

which is the set of solutions of $p = \mathrm{P}Bp = \mathrm{S}Yp$; these are commuting idempotents (given that $B = \mathrm{P}BB$ and $Y = \mathrm{P}BY = \mathrm{Q}YY$), so their composite gives the required type. Likewise $\sum Y$ is based on

$$\{\langle b, y\rangle \mid b \in B \wedge yb \in Yb\}$$

so the corresponding retract is $\lambda p.\langle Bp_0, Yp_0p_1\rangle$.

Instead of tryingto specify those adjoints which exist in terms of combinators, we rely on the general theorem of the previous section. Unfortunately we have lost the preservation of structure "on the nose" because of the need to choose representations for these functors; in particular the global sum and product for the terminal object (singleton family) are not identities because they include redundant coding which is residual from the structure of the indexing set. The remainder of this section concerns the properties of display maps.

Recall that the indexing of **Set** over itself used disjoint unions together with their indicator maps. As a result of section 3 we can describe this in terms of the global sum. If $X = (X_a : a \in A)$ is an A-indexed set, the display map occurs as both the left and top sides of the following commutative square:

$$\begin{array}{ccc} X = \coprod_{a\in A} X_a & \longrightarrow & A \\ \downarrow & & \downarrow \\ A & \longrightarrow & A \end{array}$$

The significance of this banal observation is that this square represents a morphism of $S^{\mathbf{2}}$, specifically one over A, and the right-hand map is the terminal object of A. Composing below with the terminal projections from A, we get a morphism over the terminal object 1, which is the global sum applied to $X \to_A 1_A$.

The same can of course be done in our case, so

Definition A *display map* in $\mathbf{Retr}(\Lambda)$ is the composite of an invertible followed by a map of the form $\pi_0 : \sum Y \to A$ where $Y \in \mathrm{P}A$. π_0 is in fact $\lambda p.Bp_0$.

We shall take as read a number of trivial properties of pullbacks, including the fact that this definition allows invertibles to be "passed through" display maps.

Lemma The pullback of a display map exists and is a display map.

Proof Let $\alpha : B \to A$ in $\mathbf{C}$ and $X \in \mathrm{P}A$. Put $Y = \mathrm{P}\alpha X$; then the following is a pullback square:

$$\begin{array}{ccc}
\sum Y = \lambda q.\langle Bq_0, Yq_0q_1\rangle & \xrightarrow{\lambda q.\langle \alpha q_0, Yq_0q_1\rangle} & \sum X = \lambda p.\langle Ap_0, Xp_0p_1\rangle \\
\downarrow \pi_0 & & \downarrow \pi_0 \\
B & \xrightarrow{\alpha} & A
\end{array}$$

Given any other $\beta : C \to B$ and $\gamma : C \to \sum X$ making the square commute, the pair is $\lambda c.\langle \beta c, (\gamma c)_1\rangle$. □

Lemma Any terminal projection is a display map.

Proof Given $B \to \mathrm{T}$, put $X = \mathrm{K}B$ and $A = \mathrm{T}$. Then $X \in \mathrm{P}A$ and $\sum X = \lambda p.\langle \bot, Bp_1\rangle \cong B$. □

Lemma Any composite of display maps is a display map.

Proof Given $X \in \mathrm{P}A$ and $Y \in \mathrm{P}B$ where $B = \sum X$ we want to construct $Z \in \mathrm{P}A$ with $\sum Y \cong \sum Z$ over A. Put $Z = \lambda ap.\langle Xap_0, Y\langle a, p_0\rangle p_1\rangle$; then $i : \sum Z \to \sum Y$ and $j : \sum Y \to \sum Z$ are mutually inverse where

$$i = \lambda u.\langle\langle Au_0, Xu_0u_{10}\rangle, Y\langle u_0, u_{10}\rangle u_{11}\rangle$$

$$j = \lambda v.\langle Av_{00}, \langle Xv_{00}v_{01}, Yv_0v_1\rangle\rangle$$

□

Lemma Pullback against a display map, considered as a functor between relative slices, has a right adjoint.

Proof This is equivalent by section 4 to the fact (which we have already proved) that the fibres have exponentials which are preserved by substitution. □

Theorem $\mathbf{Retr}(\Lambda)$ is cartesian closed relative to the class of display maps identified above; consequently it has internal sums, products and function spaces. □

The class of display maps constructed is the largest possible in the following sense. Suppose some map $\alpha : B \to A$ in $\mathbf{C}$ has a pullback against any map with the same codomain, *and that this can be done internally*. By this we must mean (restricting to the case of maps from the terminal object, *i.e.* elements of A) that there is a continuous function assigning to each $a \in A$ a type X_a and a pair of maps forming a pullback square. Then X would itself be an A-indexed type and $B \cong \sum X$.

10. Continuous Lattices

Continuous lattices are a generalisation of algebraic lattices (which occur as lattices of subobjects in finitary algebraic theories such as groups and modules), making the notion of finiteness or compactness a relative one. They provide the answers to a number of questions in general topology relating to injectivity and exponentiability as well as "nice" behaviour in the theory of topological (semi)lattices. For an authoritative discussion see Gierz *et al.* [1980]. Johnstone and Joyal [1981] have given a generalisation to categories which answers the corresponding questions for toposes.

Let A be a poset with directed sups. We say a is *well below* b (notation $a \ll b$) if whenever $b \leq \bigvee^{\uparrow} U$ for some (directed, which is what the arrow means) set U, there is some $c \in U$ with $a \leq c$. In the case of the lattice of open sets of the real line, $\mathbf{R}$, this means that there is a compact set lying between a and b. Then A is called a *continuous poset* if $b = \bigvee^{\uparrow}\{a : a \ll b\}$ for all $b \in A$. Scott [1972] gave an argument that continuous posets are the appropriate notion of approximate computation, although most of his followers have since retreated to the algebraic condition.

There is a topology, the *Scott topology*, which is appropriate for A, in which the basic open sets are those of the form $\Uparrow a = \{b : a \ll b\}$ (likewise we write $\Downarrow b = \{a : a \ll b\}$). Conveniently, a function $f : A \to B$ between two continuous posets is continuous w.r.t. this topology iff it preserves directed sups, whilst separate and joint continuity coincide for functions of two variables.

The previous remark gives a topology from an order: there is also a converse operation called the *specialisation order* on a topological space. Let $x \leq y$ if y lies within any open set which contains x; this relation is antisymmetric iff the space is T_0 and discrete iff it is T_1. The specialisation order on a sober space has directed sups, but the converse is false [Johnstone 1982].

We say that a topological space I is *injective* if given any subspace inclusion $A \subset B$ and a continuous map $f : A \to I$, there is some (not necessarily unique) continuous $g : B \to I$ making the triangle commute. Likewise I is *densely injective* if this holds for dense subspace inclusions. An easy (but important) example of an injective space is the *Sierpinski space*, which has two points exactly one of which is open.

Proposition The following are equivalent for an ordered T_0 space I:

(i) I is injective

(ii) I is a continuous lattice with the Scott topology

(iii) I is a retract of a (Tychonov) power of the Sierpinski space

(iv) I is has arbitrary infs ($\bigwedge$) which distribute over directed sups ($\bigvee^{\uparrow}$)

(v) I is an algebra for the filter monad. □

$\mathrm{P}\omega$ is the first infinite example of part (iii); it carries a well-known combinatory algebra structure [Scott 1976] and $\mathbf{Retr}(\mathrm{P}\omega) \simeq \mathbf{ContLat}_{\omega}$, the category of *countably based* continuous lattices and Scott continuous maps.

We shall have occasion to make extensive use of these characterisations. In particular, part (iv) suggests that there is another class of maps of importance between continuous lattices. These are the *homomorphisms* of the $(\bigvee^{\uparrow}, \bigwedge)$ structure, *i.e.* functions preserving these operations. Of course these are just continuous functions with left adjoint, so a surjective homomorphism is the same as a projection. Write **CL** for the *algebraic* category of continuous lattices and homomorphisms. Part (v), due to Day [1975], identifies the free functor **Set** → **CL**, left adjoint to the forgetful functor.

We shall need a few fragments of universal algebra for the proofs in the next section (see, for example, [Cohn 1965] or [Manes 1975]). In particular the forgetful functor **CL** → **Set** creates arbitrary limits (*i.e.* we calculate them at the level of **Set** and impose the obvious algebra structure); this means we can talk about pullbacks of continuous lattices *and homomorphisms*.

Secondly, a *congruence* on an algebra A is a reflexive, symmetric and transitive subalgebra of $A \times A$, *i.e.* a subset R containing the diagonal and closed under the operations such that $(a, b) \in R$ iff $(b, a) \in R$ and if $(a, b), (b, c) \in R$ then

$(a, c) \in R$. In **Set**, **Gp**, $K-$**Vect** and **Rng** these are usually presented as equivalence relations, normal subgroups, subspaces and ideals, respectively. Given a congruence, we may construct the *quotient*, A/R, whose elements are the classes $[a] = \{b : (a, b) \in R\}$, together with the function $A \to A/R$ by $a \mapsto [a]$. A/R carries a unique algebra structure making this a homomorphism.

Slightly more generally, a *partial congruence* is the same thing but without reflexivity, so the union of the classes may be a *proper* subalgebra. We can still construct A/R, but now it is a *subquotient*, *i.e.* a quotient of a subalgebra.

Finally, a poset is *boundedly-complete* if any *bounded* (but possibly empty) set has a least upper bound; equivalently any *nonempty* set has a greatest lower bound. We shall assume the posets to be inhabited (nonempty) and so have a least element, although this conflicts with the definition naturally provided by universal algebra. An easy generalisation of the characterisation of continuous lattices, as it turns out more appropriate to our studies, is

Proposition The following are equivalent for an inhabited T_0 space I:

(i) I is densely injective

(ii) I is a boundedly complete continuous poset (with the Scott topology)

(iii) I is a closed subset of some continuous lattice. □

11. bcCont is Relatively Cartesian Closed

In this section we shall show that **bcCont**, the category of inhabited boundedly complete continuous posets and Scott-continuous maps, has a nontrivial relatively cartesian closed structure, in which **D** is the class of *projections*, *i.e.* surjective maps with left adjoint.

In order to avoid developing a separate theory for boundedly complete continuous posets we shall add top elements where convenient (denoted by $X^\top$) and make extensive use of the algebraic and topological characterisations of continuous lattices. We shall work in the category **IPO** of posets with directed sups and least element and Scott-continuous maps.

Lemma Suppose $\alpha : B \to A$ in **bcCont** has a pullback against any map $f : X \to A$. Then α is a projection.

Proof Consider the special case of the inclusion of the element $a \in A$. By hypothesis the pullback

$$\begin{array}{ccc} \alpha^{-1}(a) & \hookrightarrow & B \\ \downarrow & & \downarrow \alpha \\ 1 & \xrightarrow{a} & A \end{array}$$

exists in **bcCont**, so in particular $\alpha^{-1}(a)$ has a least element. Write $\beta(a)$ for the corresponding element of B. Then $\beta(\alpha(a)) = a$ and $\alpha(\beta(b)) \leq b$ so β is left adjoint to α (and so preserves *all* sups, in particular directed ones) and α is surjective. □

If we were to insist on working with (total) continuous lattices, we should have to have right as well as left adjoints in order to preserve top. Intuition seems, however, to suggest on the one hand that top is a red herring and on the other that the projections are the important class of maps.

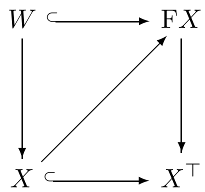
Write $--\triangleright$ for projections. We shall show that $\mathbf{C} = \mathbf{bcCont}$ is cartesian closed relative to the class **D** of projections. Conditions (ii) and (iii) are trivial.

Proposition The forgetful functor $\mathbf{CL} \to \mathbf{IPO}$ has a left adjoint F.

Proof Recall that **CL**, the category of continuous lattices and *homomorphisms*, is algebraic and in particular there is a free algebra functor $\mathbf{Set} \to \mathbf{CL}$ and **CL** has all limits. If $X \in \mathbf{IPO}$ then we may take the continuous lattice generated by the elements of X subject to the equations that directed sups in X remain so in the continuous lattice. $\square$

Lemma Let $X \in \mathbf{C}$. Then X is a (Scott-continuous) retract of a (Scott-) closed subset of $\mathrm{F}X$.

Proof Consider the map $X \to X^\top$; then $X^\top \in \mathbf{CL}$ so by the universal property of $\mathrm{F}X$ there is a unique homomorphism $\mathrm{F}X \to X^\top$ making the triangle commute. Let $W \sqsubseteq \mathrm{F}X$ be the inverse image under this of the closed subset $X \sqsubseteq X^\top$.



Clearly $X \to \mathrm{F}X$ factors through this so X is a retract of W. $\square$

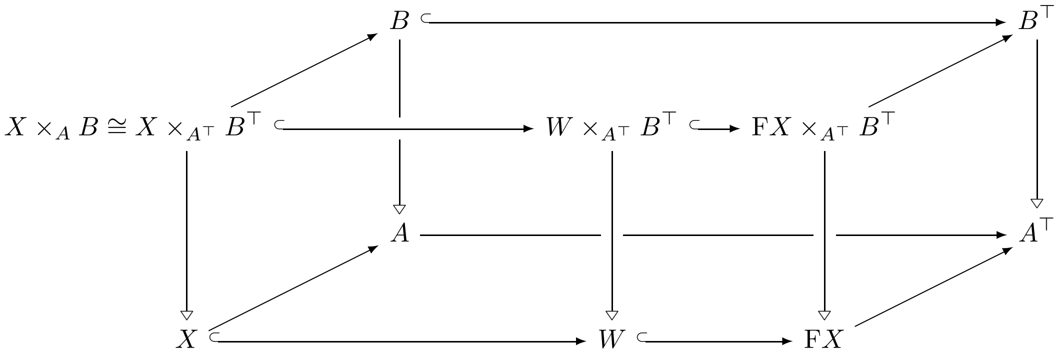
Lemma A closed subset or a retract of a boundedly complete continuous poset is another such. $\square$

We now have the machinery to prove axiom (i) for relative cartesian closure.

Proposition Pullbacks of $\mathbf{D}$ maps against $\mathbf{C}$ maps exist and are in $\mathbf{D}$.

Proof Let $\alpha : B \to A$ in $\mathbf{D}$ and $f : X \to A$ in $\mathbf{C}$; let $\beta : A \to B$ be the left adjoint to α. There is no difficulty in constructing the pullback $X \times_A B$ in **IPO** ; it consists of the pairs $(x, b) \in X \times B$ with $fx = \alpha b$ and by the continuity of f and α this equation respects directed sups. Moreover it has a least element $(\perp_X, \beta(f\perp_X))$ and the left adjoint to the projection onto X takes x to $(x, \beta(fx))$. The problem is to show that $X \times_A B$ is continuous.

By the universal property of $\mathrm{F}X$ and the fact that $A^\top \in \mathbf{CL}$, there is a unique homomorphism $\mathrm{F}X \to A^\top$ making the base of the following pentagonal prism commute; the top of the prism is given by pulling back along $B^\top \to A^\top$. The aim is to show successively that $\mathrm{F}X \times_{A^\top} B^\top$, $W \times_{A^\top} B^\top$ and $X \times_{A^\top} B^\top \cong X \times_A B$ are continuous.



The right-hand square forms a pullback in **CL** because the bottom and right-hand maps are homomorphisms. Next we have the inverse image of the closed set $W \sqsubseteq \mathrm{F}X$ under a continuous map, so by the lemma this is continuous. Finally the required pullback is the fixed-point set of the image under the pullback functor of the retract of W which gives X. $\square$

Finally we prove axiom (iv).

Lemma Let A and B be continuous lattices and $[A \to B]$ the poset of Scott-continuous functions from A to B with the pointwise order (arising from that on B). Then $[A \to B]$ is a continuous lattice in which $\bigvee^\uparrow$ and $\bigwedge$ are evaluated pointwise. $\square$

Proposition Let $\alpha : B \to A$ in $\mathbf{D}$. The pullback functor $\alpha^* : \mathbf{C_D}A \to \mathbf{C}/_\mathbf{D}B$ has a right adjoint, $(-)^B_A$.

Proof Let $\xi : X \to B$ in $\mathbf{D}$. Working in **Set** and then in **IPO** , X^B_A has to be

$$\{(a, f : \alpha^{-1}(a) \to X) \mid (\forall b.\alpha b = a)(\xi(fb) = b)\}$$

Indeed for $\upsilon : Y \to A$ in $\mathbf{D}$ we have a bijection

$$\begin{array}{ccccc} Y \times_A B & \to_B & X & (y,b) & \mapsto & g(y,b) \\ \hline Y & \to_A & X_A^B & y & \mapsto & (\upsilon y, g(y,-)) \end{array}$$

which preserves directed sups and is natural in X and Y. The bottom element of X_A^B is $(\bot_A, \zeta|_{\alpha^{-1}(\bot)})$ where $\zeta : B \to X$ is the left adjoint to ξ. Once again, the problem is in showing that this is a continuous poset.

Again it is convenient to move to continuous lattices, but this time in order to make use of the injectivity. Then any continuous map $\alpha^{-1}(a) \to X$ can be extended to one $B^\top \to X^\top$, and we can recover the information we want by identifying functions which agree on $\alpha^{-1}(a)$.

$$\begin{array}{ccc} \alpha^{-1}(a) & \hookrightarrow & B^\top \\ \downarrow & & \\ X & \hookrightarrow & X^\top \end{array}$$

Let U be the continuous lattice $A^\top \times [B^\top \to X^\top]$ and $R \subset U \times U$ be the subset

$$\{(a_1, f_1, a_2, f_2) : a_1 = a_2 \neq \top \wedge \ (\forall b. \alpha b = a_1)(f_1 b = a_2 b \wedge \xi(f_1 b) = b)\}$$

This is symmetric, transitive and closed under directed sup and nonempty inf, *i.e.* it is a partial congruence of $(\bigvee^\uparrow, \bigwedge^{\neq\emptyset}$-algebras. Thus the subquotient U/R which consists of the equivalence classes under R (which do not exhaust U) carries a $(\bigvee^\uparrow, \bigwedge^{\neq\emptyset}$-algebra, *i.e.* boundedly-complete continuous poset, structure.

But this is isomorphic as a poset to the required X_A^B which is therefore in $\mathbf{C}$ as required. □

Theorem bcCont is cartesian closed relative to the projections, and is therefore fibred over itself so that each fibre is cartesian closed, this structure is preserved up to isomorphism by pullback against arbitrary maps, and pullback against projections has adjoints on both sides. □

This theorem is true in rather greater generality. We have implicitly proved it for **IPO** already (just delete the "difficult" bits), but inclusion of categories is not the same as specialisation of proof. To demonstrate it for, say, retracts of Plotkin's "SFP" objects, requires more conceptual technology.

The intuition behind this result is that a display of domains is given by a "patchwork" of components (each of which has a least element). In order for the component posets to be "continuously indexed" it is sufficient that the composite be itself continuous (*qua* domain). In fact these display maps are themselves fibrations of domains, as will be shown in [Taylor 1986?].

By way of a corollary, "raised sums" of domains (in which we take the disjoint union of two domains and add a new bottom element) are seen as a special case of indexed sums in which the indexing domain is $\bigvee$-shaped and the $\bot$-component is the singleton.

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