Well Founded Coalgebras and Recursion

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Abstract

We define well founded coalgebras and prove the recursion theorem for them: that there is a unique coalgebra-to-algebra homomorphism to any algebra for the same functor. The functor must preserve monos, whereas earlier work also required it to preserve their pullbacks. The argument is based on von Neumann's recursion theorem for ordinals. Extensional well founded coalgebras are seen as initial segments of the free algebra, even when that does not exist. We have a categorical form of Mostowski's theorem that imposes extensionality.

The assumptions about the underlying category, originally sets, are examined thoroughly, with a view to ambitious generalisation. In particular, the "monos" used for predicates and extensionality are replaced by a factorisation system.

These proofs exploit Pataraia's fixed point theorem for dcpos, which Section 2 advocates (independently of the rest of the paper) for much wider deployment as a much prettier (as well as constructive) replacement for the use of the ordinals, the Bourbaki–Witt theorem and Zorn's Lemma.

Contents:

Categorical set theory is the study of *ideas* from Set Theory *as ordinary mathematical objects, without their foundational pretensions.* The first application of our subject was to show that the logic of an elementary topos with natural numbers is more or less the same as Zermelo Set Theory [Mik76, Osi74].

In this paper we study well-foundedness, extensionality and the arguments behind von Neumann's recursion theorem and the Mostowski extensional quotient. We strip them of everything else from their set- (or even topos-) theoretic origins and then identify what is needed of some completely different setting for them to be re-deployed there.

The value of these particular ideas is that they provide ways of expressing very strong principles of *induction* and *recursion.* These may be used in Proof Theory to prove the *consistency* of other logical systems and in Process Algebra to investigate termination or persistence of processes and ask whether one process is "the same" as another.

The role of category theory is that it is a good tool for capturing the essential features of a mathematical argument, whilst demanding almost nothing by way of foundational beliefs.

In the first section we give the traditional ideas from Set Theory and universal algebra that we are seeking to capture. Chief among these is the theorem of John von Neumann that defines functions by *recursion* over well-founded relations, *i.e.* those for which we have *induction* for predicates.

Section 2 introduces a novel order-theoretic fixed point theorem that we consider deserves a place in the mathematical canon beyond its application in this paper. We need it here because, whereas the original form of the recursion theorem relies on the fact that a poset with *all* joins has a greatest element, our generalisation need not have binary joins at a key point. We demonstrate how it maybe used for induction and recursion in familiar relational and process algebra.

Section 3 begins our categorical treatment by showing how coalgebras for a functor put these properties of Set Theory and term algebras in a common abstract setting, summarising earlier work.

Section 4, which you should omit on first reading, examines our precise requirements of the category and its notion of "mono". In future work this will enable a considerable generalisation of similar previous results from **Set** to other categories.

Section 5 shows how well founded coalgebras are generated and Section 6 proves our central result, the recursion theorem.

Section 7 introduces extensional well founded coalgebras and shows how they behave like the von Neumann hierarchy in Set Theory.

Section 8 shows how to *impose* well-foundedness and extensionality on a coalgebra, giving adjoints to the inclusions of categories, on the additional assumption of image factorisation.

Section 9 re-introduces the requirement that the functor preserves pullbacks and proved the (relatively few) results that depend on this. Finally, Section 10 considers binary joins, in particular the "overlapping" union in Set Theory.

1 Background

We are going to study the axioms of *foundation* and *extensionality*.

The axiom of foundation and the notion of a well *founded* relation are the (to us, natural) generalisation of the well-*orderings* or *ordinals* $(X, \prec)$ that Georg Cantor introduced [Can95, Can97]. He stated their defining property in two ways:

(a) every non-empty subset $\emptyset \neq U \subset X$ has a $\prec$-least element; or

(b) there is no infinite descending sequence $\cdots \prec d \prec c \prec b \prec a$.

In fact Euclid had invoked the second of these principles for the natural numbers long beforehand, in *Elements* VII 31 [Fow94, p 262].

It took some time to recognise the weaker notion and how to use it to show that Zermelo's Set Theory and infinitary proofs are not vulnerable to circular arguments like Russell's Paradox. Dimitry Mirimanoff seems to have been the first to do this [Mir17a, Mir17b, Mir19], introducing ideas such as *rank* that we will see later. His style is in the same spirit as our own, treating membership like any ordinary relation.

John von Neumann proposed a meta-axiom, that the system of Set Theory be the minimal one [vN25]. Ernst Zermelo asserted the two properties above for $\in$ as his axiom of foundation [Zer30]. He then introduced the general notion of a well founded relation and applied it to proof theory [Zer35].

If we state either of the properties (a,b) as the definition, we have to make frequent use of *excluded middle* or *dependent choice*, respectively. For the intuitionistic definition, we identify what we actually want to *do* with the notion. It is difficult to say who first did this, because of constructivists' habit of retaining classical definitions *verbatim* and then arguing at length about their faults, but one early formal intuitionistic account of induction is [HK66].

Definition 1.1 A binary relation $\prec$ on a carrier A is ***well founded*** if it obeys the ***induction scheme***

$$\frac{\forall a\!:\!A.\ \ (\forall b\!:\!A.\ \ b \prec a \Rightarrow \phi b) \Longrightarrow \phi a}{\forall a\!:\!A.\ \ \phi a}$$

for any predicate ϕ on A.

It will be convenient to dissect this triply nested implication. The innermost one,

$$\forall b\!:\!A.\ b \prec a \Longrightarrow \phi b$$

is standardly called the ***induction hypothesis*** (for ϕ at a).

When ordinary mathematicians *use* induction to prove something, their effort goes in to justifying the next implication, the one above the line. However, our focus is on the *validity* of induction, *i.e.* the outermost implication (written *as* the line), and therefore we call the whole of the top line the ***induction premise*** (for ϕ).

We recognise that the middle implication is typically not two-way (*â priori*, but of course it always *becomes* two-way *after* we have invoked induction), but in the case where it is we call it ***tight***.

Remark 1.2 This still leaves the variable ϕ free. For simplicity we will usually speak of well-foundedness as if it were *quantified over all* ϕ. However, the word *scheme* in the name indicates that we may restrict attention to individual predicates or to a class of them of a certain logical complexity, such as those with at most a particular number of alternations of quantifiers. Our categorical structure will be able to accommodate this generalisation (Assumption 4.17) and we will indicate for what predicates we are using induction. This consideration is particularly relevant in proof theory, but we shall not get involved in that subject in this paper.

Example 1.3 With the successor relation $n \prec n+1$ on the natural numbers, the induction scheme is known as Peano induction:

$$\frac{\phi 0 \qquad \forall n{:}\mathbb{N}.\ \ \phi n \Longrightarrow \phi(n+1)}{\forall n{:}\mathbb{N}.\ \ \phi n}$$

although this idiom predates Giuseppe Peano [Pea89] by at least three centuries.

Whilst the general notion of well-foundedness is natural and long-established, many mathematicians seem to be reluctant to use it. Instead they say that they are doing induction or recursion on the *length* of a string, the *height* of a tree, its *depth* in computer science, or some other such numerical measure. This is also the way in which iterative or recursive programs are shown to terminate.

The general result that lies behind such usage is this:

Proposition 1.4 If $(A, \prec)$ is well founded and $f : (B, <) \to (A, \prec)$ is ***strictly monotone*** in the sense that

$$\forall b_1 b_2{:}B. \quad b_1 < b_2 \implies f b_1 \prec f b_2$$

then $(B, <)$ is also well founded.

Proof If B has an infinite descending sequence then so does A, which is forbidden. Alternatively, if $\emptyset \neq V \subset B$ then $\emptyset \neq fV \subset A$, so there is a minimal $a \in fV$, where $a = fb$ for some $b \in V$ and this is minimal there. The more difficult intuitionistic proof will be given in Proposition 9.2. $\square$

From the ability to *prove a predicate by induction* we may derive that of *defining a function by recursion.* The principal goal of this paper is to see how far we can generalise the setting of this ***recursion theorem***. John von Neumann proved it for the ordinals in his reformulation of their theory that became the classic one [vN28, § III] and the following is the (mild) adaptation of his argument to intuitionistic well founded relations.

This result appears in most Set Theory textbooks, usually without attribution, but Paul Bernays [Ber58, p 100] credits von Neumann; this book also has a detailed historical introduction by Abraham Fraenkel, who probably knew the developments personally.

Theorem 1.5 Let $(A, \prec)$ be a carrier with a well founded binary relation and Θ another carrier with a function $\theta : \mathcal{P}\Theta \to \Theta$ that takes an arbitrary subset of Θ as its argument and returns a single element. Then there is a unique function $f : A \to \Theta$ such that

$$\forall a{:}A. \quad fa \ = \ \theta\left(\{fb \mid b \prec a\}\right).$$

We call this equation the ***recursion scheme***, because we do not quantify over (Θ, θ): we only ever consider a *particular* target structure.

Proof An ***initial segment*** of A is a subset $B \subset A$ such that

$$\forall bc{:}\, A.\;\; c \prec b \in B \Longrightarrow c \in B$$

and an ***attempt*** is a partial function $f : A \rightharpoonup \Theta$ whose ***support*** (domain of definition) $B \subset A$ is an initial segment and

$$\forall b{:}\, A.\;\; b \in B \;\Longrightarrow\; fb \;=\; \theta(\{fc \mid c \prec b\}).$$

(a) There is a unique attempt with empty support.

(b) The union of any directed family of initial segments or attempts is another such.

(c) The restriction of $\prec$ to any initial segment is well founded.

(d) Any two attempts f, g with the same support B are equal, which we prove by induction over $(B, \prec)$ for the predicate

$$\phi b \;\equiv\; (fb = gb).$$

(e) Hence any two attempts with supports B_1 and B_2 agree on $B_1 \cap B_2$ and so may be amalgamated into an attempt with support $B_1 \cup B_2$.

(f) Given any attempt f with support B, there is a successor attempt g with support

$$C \;\equiv\; sB \;\equiv\; \{c : A \mid \forall b{:}\, A.\;\; b \prec c \Longrightarrow b \in B\} \qquad \text{given by} \qquad gc \;\equiv\; \theta\{b : A \mid b \prec c\},$$

whilst any attempt with support C restricts to B and these constructions are inverse.

(g) In this construction, $C = B$ iff $B = A$, which we prove by induction over $(A, \prec)$ for the predicate

$$\phi a \;\equiv\; (a \in B),$$

indeed "$C = B$ iff $B = A$" is exactly the induction scheme for this predicate.

(h) The required solution to the recursion equation is the union of all of the attempts (a,b,e); this is total because it is fixed by the successor operation (g) and unique by (d). $\square$

It is *essential* to understand the steps of this traditional proof before proceeding with the rest of this paper and we label them because they will each be the subject of lemmas in our categorical proof. However, we shall give our proof in a generality in which Proposition 1.4 *fails* (even though that is plainly an extremely important property of well founded relations). We therefore lose steps (c) and (e) of the proof and so cannot simply form the union of all attempts in the final part.

For these reasons, the next section gives a revised proof of the Theorem, as a guide to the way we subsequently do it categorically.

Remark 1.6 Steps (a) and (f) in the traditional proof provide the initial and next attempts, so by Peano recursion we can define the nth one for all $n : \mathbb{N}$. Can we not then just use step (b) at limit stages to continue this through the ordinals (here and in the fixed point theorems in the next section)?

No:

(a) Ordinals are not "infinite numbers" in which ω follows the finite ones and we continue ever *upwards*: the (classical) definition involves *downward* sequences. We require a proof to justify recursion, namely the result due to von Neumann that we have just stated. Using a theorem to prove itself is begging the question, as is citing a result from later in a textbook to prove an early one; even a journal paper is to be understood as forming some *stage* in a logical development.

(b) Even when ordinal recursion is legitimate, there are two *further* but commonly overlooked components to a valid proof, besides the zero, successor and limit cases. Firstly, the ordinals go on “forever” — Cesare Burali-Forti [BF97] showed early on that they do not form a “set” — so when do we stop? Secondly, there is *more work to be done in the target structure* to deduce that the recursion provides a solution to the problem being considered.

(c) The question of when to stop was answered by Friedrich Hartogs [Har15]: For any set X, let λ be the set of isomorphism classes of well-orderings of subsets of X. Then λ is well ordered and there is no injection $\lambda \rightarrowtail X$.

Hartogs' proof was one of the earliest formal applications of Zermelo's Set Theory [Zer08b] and he set out the prerequisites from that and Cantor's original work [Can97] very clearly. Principal among the latter is that, for any two well ordered sets, one is uniquely isomorphic to an initial segment of the other; we would now deduce that from von Neumann's (later) recursion theorem (*cf.* Proposition 2.9), but Cantor had actually given a valid direct proof of it.

A key feature von Neumann's paper (already present in [Mir17a]) was the use of the global set-theoretic membership relation $\in$ for the order $\prec$ on an *ordinal*, whereas for Cantor, Hartogs and us, the relation $\prec$ on a well ordered or well founded set is *superstructure*.

(d) Now we have some ordinal λ that *does not* embed in the given set, so somewhere it must repeat itself (classically). The second question is why is this point unique and how does it solve the problem? We will see a new result in the next section that does this — intuitionistically and much more simply.

(e) Remark 2.15 shows that there are lightweight alternatives to Hartogs' Lemma for obtaining the ordinal λ. However, instead of vindicating the use of ordinals, they suggest a deep rethinking of how we prove things by recursion.

(f) No proof of the fixed point theorem correctly using ordinal recursion and citing Hartogs seems to have been published prior to these alternatives, *i.e.* in 1928–49.

(g) Making new rules after the game has begun, such as Collection, Inaccessible Cardinals or Universes, is also illegitimate, especially as there are valid proofs according to the original rules (Zermelo Set Theory or an elementary topos).

(h) The traditional theory of the ordinals depends *very heavily* on excluded middle. There are two existing intuitionistic accounts [JM95, Tay96a], which show that there are several different notions. Even so, (the use of) Hartogs' lemma remains irretrievably classical.

(i) The ordinals themselves are significant applications of the generalisations that our categorical approach will offer, but they deserve a treatment of their own [Tay23]. It is no more reasonable to use an old theory to justify its replacement than it would be to power a carbon-neutral vehicle with a steam engine.

Remark 1.7 In order to start generalising these ideas, consider first the recursion scheme: θ is the evaluation operation for some sort of *algebra* Θ. In taking a *set* of arguments instead of a *list*, we are saying that θ is *idempotent* and *commutative* with respect to them, although these conditions are inessential.

Indeed, we can consider any ***free theory***, *i.e.* one with no equations at all, but a (possibly infinite) collection Σ of operation symbols, each r of which has a (possibly infinite) ***arity*** $\mathsf{ar}\,(r)$. Then for any set X (of constants, generators, indeterminates or variables as you please), there is a set

$$TX \quad \equiv \quad \coprod_{r:\Sigma} X^{\mathsf{ar}\,(r)}$$

of ***terms*** of depth 1 built from these generators and operation symbols. With no generators, $T\emptyset$ is the set of constants or nullary operation-symbols. Of course TTX is the set of terms of depth 2 and so on.

An ***algebra*** for these operation *symbols* is a carrier Θ that is equipped with an *operation* $\Theta^{\mathsf{ar}\,(r)} \to \Theta$ for each symbol $r : \Sigma$. These may be combined into a single function on the disjoint union:

$$\theta : T\Theta \longrightarrow \Theta.$$

In particular, at least in the case where all of the arities are finite, there is a ***term-*** or ***free algebra*** that is obtained by forming the union A of all of the iterates of T, applied to the empty set. Since we have already done so exhaustively, applying T again to A yields the same thing, so

$$TA \underset{\text{parse}}{\overset{\text{ev}}{\rightleftarrows}} A, \quad (\cong)$$

where ev and parse are the operations of wrapping and unwrapping the outermost symbol of a term.

Therefore,

$$b \prec a \quad \equiv \quad (r, b) \in \mathsf{parse}\,(a)$$

defines the ***immediate sub-term relation*** on A. Since A only consists of expressions that are formed by repeated application of the operation symbols, this relation clearly satisfies the "descending sequence" definition of well-foundedness.

Whilst pure mathematicians still typically do induction on the *depth* of such an expression (*cf.* Proposition 1.4), it is increasingly common for theoretical computer scientists and logicians to say directly that this is ***structural induction*** or ***structural recursion*** on the expressions or language instead.

Returning to Set Theory, the second idea that we want to develop is the following — at first sight innocent — property of the $\in$-relation:

Definition 1.8 A (well founded) binary relation $\prec$ such that

$$\forall ab\colon A. \quad (\forall c\colon A.\ \ c \prec a \iff c \prec b) \implies a = b$$

is called ***extensional***. The analogous property of sub-terms in a free algebra is that the parse map is one-to-one, because any term is uniquely determined by its sub-terms (and outermost operation-symbol).

Remark 1.9 In this paper we will put the ideas of well-foundedness and extensionality in a more powerful categorical setting. Together they explain many characteristic features of Set Theory, even when stripped of what we might suppose to be its most important ingredients. They are also important properties of term algebras, underlying the algorithm for *unification*, *i.e.* for assigning (sub-)terms to indeterminates in two or more terms so that they match.

In Set Theory, when we form the "union" of two supposedly independent objects, we may find that they already overlap. (Besides being bizarre from the point of view of any other kind of mathematics, this is irritating for those who use set theory as a foundation.) The way that unification "matches up" sub-terms is similar to the overlapping union.

We shall find in Section 7 that the *category* of extensional well founded structures and the appropriate homomorphisms is actually a *pre-order*, *i.e.* there is at most one map between any two objects. When we put two objects together, they (typically) have a non-empty intersection (meet in this order) and therefore an "overlapping" union.

Remark 1.10 Applying universal algebra back to Set Theory, when we take (the functor) T to be the (covariant) powerset $\mathcal{P}$, we see that the terms of successive depth are just sets ($\in$-structures). We usually like to have *free algebras* for structures, which in this case would be the ***universal set***, but this does not exist as a legitimate object.

However, the extensional well founded structures are legitimate *fragments* of the universal set. These are known in Set Theory as ***transitive sets***, by which is meant those X for which

$$y \in x \in X \Longrightarrow y \in X, \qquad \text{but not necessarily} \qquad z \in y \in x \in X \Longrightarrow z \in x.$$

The analogue in algebra is a collection of terms that includes all of their sub-terms This is a familiar situation: a language processor such as a compiler forms just such a collection when it parses a particular program or text.

Such structures are *parts* of the free algebra, whether the latter exists legitimately or not. More precisely, the (possibly illegitimate) *union* (colimit) of the preorder of extensional well founded structures is the free algebra.

Remark 1.11 Continuing with the fiction of the universal set, let's use it as the target Θ of the recursion theorem. Then, for any well founded relation $(A, \prec)$, we may define

$$fa \qquad \text{recursively as the set } (\in\text{-structure}) \qquad \{fb \mid b \prec a\}.$$

Even if $(A, \prec)$ was not extensional, the result is, because Θ is extensional by the axioms of Set Theory.

Therefore, following Andrzej Mostowski [Mos49, Thm 3],

(a) any extensional well founded relation is isomorphic to a unique set ($\in$-structure); and

(b) any well founded relation has an extensional quotient, with a suitable universal property.

The first of these obliges us to subscribe to the belief that a set *is* some particular thing, instead of having a mathematical property that is shared by any isomorphic structure. (There is the same distinction between von Neumann's *ordinals* and Cantor's *well ordered* sets.) Moreover, if we admit that, then we commit ourselves even more deeply, because this $\in$-structure is not defined within Zermelo's original Set Theory [Zer08b], but requires the axiom-scheme of replacement.

Since we are not using Set Theory as our foundations, we do not need to be concerned with that (as yet). On the other hand, the second statement is an ordinary theorem of higher order logic. It's a quotient, so we may construct it using an equivalence relation, albeit one that has a *co-recursive* definition. This is done symbolically in Theorem 2.13 and in a more general categorical form in Theorem 8.9.

Remark 1.12 This discussion of whether Mostowski's theorem requires Replacement or not is a distraction. There undoubtedly are constructions that ordinary mathematicians do, but which are not available in Zermelo Set Theory or its modern substitutes:

(a) It is common to *iterate* constructions, either over $\mathbb{N}$ or an ordinal, the simplest case being $\bigcup \mathcal{P}^n(\mathbb{N})$.

(b) By methods variously known as realisability, gluing or logical relations, one can compare the term model of a logic system with a semantic one to prove consistency or completeness. Since this seems to conflict with Gödel's Incompleteness Theorems, the recursion over the term model must be one that goes beyond what that logic can prove for itself.

To give a categorical account of the axiom-scheme of replacement would go well beyond what we can consider in this paper. We will make a proposal towards it in [Tay23] by demonstrating how our categorical methods can *define* transfinite iteration of functors. Of course we cannot *construct* this: we will simply add a new tool to the categorical lexicon. This will lie alongside, for example, the *definition* of the subobject classifier in a category with finite limits, which defines but does not construct an elementary topos.

Mostowski's theorem is nevertheless the conceptual key to this, because our definition of transfinite iteration will be another example of the extensional quotient. However, this is in a framework where we use categorical tools to generalise the notions of "injective" (and "surjective") functions. Sections 4 and 8 explain how this is done.

Remark 1.13 Finally, since we have gone to the trouble of saying how induction and recursion are *schemes*, we should also state our position *vis à vis* the two traditions in Set Theory: one that employs *completed infinities* (classes, universes, inaccessible cardinals) and another that eschews them, developing *potential infinities* instead.

Completed infinities feature in ordinary mathematics in the form of *free algebras*, as we have seen. André Joyal and Ieke Moerdijk [JM95], approaching the analysis of Set Theory from this point of view, treated the universes of sets and of (three kinds of intuitionistic) ordinals as the free algebras for the powerset functor together with "successor" functions having various properties.

It was their key contribution to model the small/large distinction using ideas that had been developed in topology and sheaf theory to handle *open maps*. Their *algebraic set theory* has been developed further by a number of authors [Awo13] and now gives a categorical account of several highly powerful notions in Set Theory.

Type theories also commonly include (multiple) *universes*, because, when the motivations are symbolic formulae, it is quite natural to internalise the whole system within itself. This is also used to provide results that would otherwise be obtained *impredicatively*.

Our view, on the other hand, is in the tradition of *potential* infinities. We take on board the fact that we cannot solve $X \cong \mathcal{P}(X)$, *i.e.* that there are functors such as the covariant powerset that have no free algebra. In place of this, we characterise and work with *fragments* of what ought to be the free algebra. In the case of the powerset, these fragments are the $\in$-structures or transitive sets of traditional Set Theory.

Working without completed infinities is also important if we want to understand Replacement, because of the way that it can be dismissed as apparently trivial in the context of universes. Somehow Replacement allows us to express *very large* things using *small* specifications, like an architect's plan for a skyscraper, even without an encompassing universe.

Remark 1.14 In this setting we therefore need to explain what we mean when we write **Set** for the ***category of all*** sets (or whatever) and functors between such categories. Categorists commonly and happily talk about these without being clear what they mean.

Plainly, to do *sheaf* theory we would need to consider functors $F : \mathcal{X}^{\mathsf{op}} \to \mathbf{Set}$ as legitimate objects, and also collections of them. These are completed infinities, although they can in fact be re-formulated to avoid this by considering fibrations $\mathcal{F} \to \mathcal{X}$ instead.

But we're not going to do sheaf theory in this paper. For us, **Set** and other "large" categories are not really the completed infinities of *all* objects but just a shorthand for the *scheme* that says *what it is to be* an object or morphism of the relevant kind. Similarly, a functor is a *process* that turns an object or morphism of one kind into one of the other, not the completed infinity that collects all instances of this transformation.

2 A novel fixed point theorem

In order to prove our *categorical* generalisation of the recursion theorem we need to know about *order-theoretic* fixed points. Here we also recall the properties of simulation and bisimulation that both the set-theoretic membership relation and coalgebra homomorphisms obey.

The best known fixed point theorem is for complete (semi)lattices. It was stated tersely by Bronisław Knaster [Kna28] and later elaborated by Alfred Tarski [Tar55], but the key idea was already a commonplace to Zermelo in 1908. In the same year as Knaster, von Neumann used it in his proof of the Recursion Theorem 1.5.

However, there are many systems, especially in algebra, where general (especially binary) joins need not exist, or if they do they are unmanageably complicated. For example, in the proof that we gave of the recursion theorem, step (e) pasted two partial functions together, relying on an inductive argument in step (d) that they agree on their common domain.

We will generalise the recursion theorem to situations where these steps may not be valid, so we need a more subtle result that avoids binary joins. Classically, this is accommodated by using those of *linear orders* or *chains*, but for a constructive result we replace these with a looser notion:

Definition 2.1 Let $(X, \leq)$ be a poset (partially ordered set), so the relation $\leq$ is reflexive, transitive and antisymmetric ($x \leq y \leq x \Rightarrow x = y$). Then

(a) a subset $I \subset X$ is ***directed*** if

$$\exists x{:}X.\ x \in I \qquad \text{and} \qquad \forall xy \in I.\ \exists z \in I.\ x \leq z \geq y;$$

(b) $(X, \leq)$ is a ***dcpo*** (directed-complete poset) if it has joins of all directed subsets, written $\bigvee^{\uparrow}$ or $\bigcup^{\uparrow}$; and

(c) it is a ***ipo*** (inductive poset) if it also has a least element, written $\bot$ or $\emptyset$.

If an endofunction $s : X \to X$ of an ipo *preserves* directed joins then there is another simple fixed point property that is well known in universal algebra and the semantics of programming languages, where it was promoted by Dana Scott. Here, however, we will only require the function s to be ***monotone***, *i.e.* to ***preserve order***,

$$\forall x, y{:}X. \quad x \leq y \quad \Longrightarrow \quad sx \leq sy,$$

and also to be ***inflationary***,

$$\forall x{:}X. \quad x \leq sx,$$

although there are numerous alternative names for these two properties.

Lemma 2.2 Any dcpo $(X, \leq)$ has a greatest inflationary monotone endofunction, $t : X \to X$. This is idempotent ($\forall x.\ t(tx) = tx$) and its fixed points are exactly the points that are fixed by *all* inflationary monotone endofunctions.

Proof Consider the poset Y of all inflationary monotone endofunctions of X, equipped with the pointwise order,

$$r \leq_Y s \quad \equiv \quad \forall x{:}X. \quad rx \leq_X sx.$$

This inherits directed joins from the pointwise values in X. Also, id_X is the least element of Y, so Y is an ipo.

Now, for any $r, s \in Y$, the composites $r\,;s$ and $s\,;r$ both lie above both r and s in the pointwise order on Y, because

$$\forall x. \quad x \quad \leq \quad rx,\ sx \quad \leq \quad r(sx),\ s(rx),$$

using both the inflationary and monotone properties.

Hence the whole of Y is *directed*.

Since Y is also directed-*complete*, it therefore has a greatest element, $t : X \to X$.

Now, for any $s \in Y$, the composites $s\,;t$ and $t\,;s$ are in Y too, so $s\,;t \geq t \leq t\,;s$ by the previous argument, but also $s\,;t \leq t \geq t\,;s$ since t is the greatest element of Y. Hence $s\,;t = t = t\,;s$ and in particular $t = t\,;t$.

Finally, if $x = tx$ then $sx = s(tx) = tx = x$ for any $s \in Y$. In particular $x \equiv ty$ satisfies this for any $y \in X$. $\square$

Our novel fixed point theorem incorporates the idea that recursion doesn't "pause for a breather and then start up again", *cf.* Remark 1.6(d).

Theorem 2.3 Let $s : X \to X$ be an inflationary monotone endofunction of an ipo satisfying the ***special condition*** that

$$\forall xy{:}X. \quad x = sx \ \leq\ y = sy \quad \Longrightarrow \quad x = y.$$

Then

(a) X has a greatest element, which we call $\top$;

(b) $\top$ is the unique fixed point of s;

(c) if $\bot$ satisfies some predicate that is preserved by s and directed joins then this also holds for $\top$.

Proof By the Lemma, let $t : X \to X$ be the greatest inflationary monotone endofunction. Then

$$\forall x{:}X. \quad \bot \ \leq \ x \ \leq \ sx \ \leq \ tx \ = \ s(tx),$$

whence
$$\forall x. \quad t\bot \ = \ s(t\bot) \ \leq \ s(tx) \ = \ tx \ \geq \ x,$$

so the $\leq$ is equality by the special condition and $t\bot$ is the greatest element ($\top$) and unique fixed point.

For the final part, the subset $U \subset X$ defined by the predicate is closed under $\bot$, s and $\bigvee^{\uparrow}$. It therefore satisfies the same properties as X itself, so it contains a fixed point, which must be the same as the one in X. □

The innovation of using functions instead of subsets was made by Dito Pataraia in 1997, but he never wrote it up formally himself and died in 2011 at the untimely age of 48 [Jib11]. His original argument was more complicated but was simplified by Alex Simpson [JS97] and our new "special condition" simplifies it further.

We call the third part of the conclusion ***Pataraia induction***, although it was first exploited in a constructive setting by Martín Escardó [Esc03, Thm 2.2].

It is easy but instructive to find examples where s fails to be monotone or inflationary and there is no top element. We continue the discussion at the end of this section.

To illustrate the use of the Theorem, we now prove the analogous results for relations to those that we later discuss for coalgebras, starting with a proof of the Recursion Theorem 1.5. Some of the intermediate steps are weaker, but note that the use of *uniqueness* is a novelty that will be crucial in the categorical version:

Theorem 2.4 Let $(A, \prec)$ be a carrier with a well founded binary relation and Θ another carrier with a function $\theta : \mathcal{P}\Theta \to \Theta$. Then there is a unique function $f : A \to \Theta$ such that

$$\forall a{:}A. \quad fa \ = \ \theta\left(\{fb \mid b \prec a\}\right).$$

Proof Initial segments, attempts and the successor operations on them are defined as before.

The poset $(\mathsf{Seg}\,, \subset)$ of initial segments of $(A, \prec)$ has least ($\emptyset$) and greatest (A) elements and directed unions; we will not use binary unions in this proof. Part (g) of the earlier version says that the top element (A) is the unique fixed point of the successor, as in the special condition.

We haven't used Pataraia's Theorem so far, because we already have $\top$, but we do use Pataraia *induction* for the attempts:

There is another poset $(\mathsf{Att}\,, \leq)$ of attempts, which also has a least element and directed joins, but it is not obvious that there is a greatest element or that it is total.

There is a "support" function $\mathsf{Att} \to \mathsf{Seg}$, which commutes with $\bot$, s and $\bigvee^{\uparrow}$.

Next consider the successor operation on attempts (part (f)) more carefully: it doesn't just *extend* an attempt from an initial segment to its successor, but defines a *bijection* between attempts with supports B and sB.

Now let $\Phi(B)$ be the predicate on $B \in \mathsf{Seg}$ which says that there is a *unique* attempt with support B. Our remark about bijections means that Φ is preserved by successor.

It also holds for $\emptyset \in \mathsf{Seg}$ and is preserved by directed unions, since these are colimits. We do not need to consider binary unions. We deduce by Pataraia induction that $\Phi(A)$ holds, *i.e.* there is a unique attempt with total support. □

Another form of the first part of the proof does make use of the special condition and the first two conclusions of Pataraia's Theorem:

Lemma 2.5 Any binary relation $(A, <)$ contains a largest well founded initial segment $WA \subset A$.

Proof This time we consider the poset WfSeg of *well founded* initial segments, which again has a least element and directed unions, but no longer *obviously* a greatest element.

The special condition is a *relative* version of part (g) of the original proof: if $B' = sB' \subset B = sB \subset A$ with B well founded then $\forall x \in B.\, x \in B'$ by induction.

Hence WfSeg does have a greatest element by Pataraia's Theorem. $\square$

We now turn to *extensional* well founded relations, which are the basis of Cantor's theory of the ordinals and the $\in$-structures of Set Theory. Since our point of view is that sets are partial $\mathcal{P}$-algebras, we adapt the recursion theorem to allow its target (Θ, θ) to be partial. This makes it rather more complicated, so it is correspondingly less obvious that there is a *greatest* attempt. Pataraia's theorem comes to the rescue.

For this we need the notion, introduced by Mirimanoff as "isomorphisme", that spells out what equality of sets means for their elements, but it is nowadays best known in Process Algebra [San11]:

Definition 2.6 A function $f : B \to A$ between sets with binary relations is called a ***simulation*** if it has the "lifting" property

$$\forall a' {:} A.\ \forall b {:} B. \quad a' \prec_A fb \implies \exists b' {:} B.\ a' = fb' \wedge b' \prec_B b.$$

We may define a similar property for a *relation* $(\sim) : B \rightleftharpoons A$:

$$\forall a a' {:} A.\ \forall b {:} B. \quad a' \prec_A a \ \wedge\ b \sim a \implies \exists b' {:} B.\ b' \sim a' \wedge b' \prec_B b$$

and then $\sim$ is a ***bisimulation*** if it also satisfies the symmetrical property,

$$\forall a {:} A.\ \forall b b' {:} B. \quad b' \prec_A b \sim a \implies \exists a' {:} A.\ b' \sim a' \wedge a' \prec_A a.$$

Since this makes the empty relation a bisimulation, we need to be clear whether we are talking about functions or relations.

Lemma 2.7 The bisimulation relations between any two sets with extensional well founded relations form an ipo under inclusion.

Proof The definition is finitary, so it is closed under directed unions. $\square$

Lemma 2.8 The ***relative successor*** $b \approx a$ of a bisimulation relation $(\sim) : (B, \prec_B) \rightleftharpoons (A, \prec_A)$ is defined by

$$\big(\forall a'.\ a' \prec_A a \Rightarrow \exists b'.\ a' \sim b' \wedge b' \prec_B b\big) \ \wedge\ \big(\forall b'.\ b' \prec_B b \Rightarrow \exists a'.\ a' \sim b' \wedge a' \prec_A a\big).$$

It extends $(\sim)$ and is also a bisimulation. If $(A, \prec_A)$ is extensional and $(\sim) : B \rightharpoonup A$ is functional,

$$(b \sim a_1) \ \wedge\ (b \sim a_2) \implies a_1 = a_2,$$

then the successor $(\approx)$ is functional too.

Proof If $b \approx a_1$, $b \approx a_2$ and $a' \prec a_1$ then $\exists b'.\, b' \sim a' \wedge b' \prec b$, so

$$\exists b' a''.\ b' \sim a' \prec a_1 \ \wedge\ b' \sim a'' \prec a_2 \ \wedge\ b' \prec b,$$

in which $a' = a''$ since $\sim$ is functional, so $a' \prec a_2$. The converse is similar, so $a_1 = a_2$ by extensionality of A. $\square$

Proposition 2.9 Between any two well founded relations $(B, \prec_B)$ and $(A, \prec_A)$ there is a greatest bisimulation relation. If A is also extensional then the bisimulation is functional and if B is also extensional then it is a partial bijection.

Proof To apply Pataraia induction, it remains to verify the special condition, so suppose that

$$\forall ab. \quad a \backsim b \iff a \mathrel{\overset{\backsim}{\backsim}} b \implies a \sim b \iff a \approx b$$

and (for induction)

$$\forall a'b'. \quad a' \prec_A a \ \wedge\ b' \prec_B b \ \wedge\ a' \sim b' \Longrightarrow a' \backsim b'.$$

Then

$$\begin{array}{rcl} a \approx b & \equiv & \big(\forall a'.\, a' \prec_A a \Rightarrow \exists b'.\, a' \sim b' \wedge b' \prec_B b\big) \ \wedge\ \big(\cdots\big) \\ & \Rightarrow & \big(\forall a'.\, a' \prec_A a \Rightarrow \exists b'.\, a' \backsim b' \wedge b' \prec_B b\big) \ \wedge\ \big(\cdots\big) \quad \equiv \quad a \approx b, \end{array}$$

so $a \sim b \iff a \backsim b$. This is true for all $a \in A$ and $b \in B$ by well-foundedness of $\prec_A$ and $\prec_B$.

Hence, using extensionality, the greatest bisimulation is functional by Pataraia induction. When this is so both ways it is a partial bijection. $\square$

Section 7 goes on to show how, with the natural notion, all morphisms between extensional well founded relations are mono, they from a preorder with meets and their joins are like set-theoretic union. For *classical ordinals*, this is also the result that says that one must be an initial segment of the other, in a unique way [Can97, §13 Thms N&E].

Proposition 1.4 said that well-foundedness is reflected by order-preserving functions and in particular is inherited by initial segments, *cf.* Theorem 1.5(c). There is a simpler result about the induction *premise* that will be an important tool (Lemma 5.1) in our categorical construction:

Lemma 2.10 Substitution along simulation functions preserves the induction premise.

Proof Let $f : (B, \prec) \to (A, <)$ be a simulation function and ϕ a predicate on A that satisfies the induction premise (Definition 1.1),

$$\forall a. \quad (\forall a'.\ a' < a \Rightarrow \phi a') \Longrightarrow \phi a.$$

Put $\psi \equiv f^*\phi$, so $\psi b \equiv \phi(fb)$, and suppose that it satisfies the induction hypothesis

$$\forall b'.\ b' \prec b \Longrightarrow \psi b'$$

for $b : B$. Let $a' : A$ be such that $a' < a \equiv fb$. Then, since f is a simulation, there is some lifting $b' : B$ with $a' = fb'$ and $b' \prec b$. By the induction hypothesis for B at b, this satisfies $\psi b'$, which is $\phi(fb')$ or $\phi a'$.

Hence we have proved the induction hypothesis for ϕ on A at $a \equiv fb$. It follows from the induction premise for A that $\phi a \equiv \phi(fb) \equiv \psi b$. Therefore we have proved that

$$\forall b. \quad (\forall b'.\ b' \prec b \Rightarrow \psi b') \Longrightarrow \psi b,$$

which is the induction premise for B. $\square$

Corollary 2.11 Surjective simulation functions preserve well-foundedness.

Proof If $(B, <)$ is well founded, f is surjective and ϕ obeys the induction premise for A in the Lemma then $\forall b.\, \psi b$ and $\forall a.\, \exists b.\, a = fb$, whence $\forall a.\, \phi a$ [Tay96a, Lemma 2.7]. $\square$

Now we have further applications of Pataraia's theorem:

Lemma 2.12 Any simulation function $f : (B, \prec) \to (A, <)$ from an extensional well founded relation to any binary relation whatever is 1–1.

Proof For any initial segment $C \subset B$, let $\Phi(C)$ be the predicate that the composite $C \to B \to A$ is 1–1. This holds for $C \equiv \emptyset$ and is inherited by directed unions.

Suppose $\Phi(C)$ holds and let $b_1, b_2 \in sC$ with $fb_1 = fb_2 \in A$.

Since f is a simulation function, the trivial statement $a < fb_1 \iff a < fb_2$ becomes

$$(\exists b_1'.\, a = fb_1' \ \wedge\ b_1' \prec b_1) \iff (\exists b_2'.\, a = fb_2' \ \wedge\ b_2' \prec b_2),$$

in which we must have $fb'_1 = fb'_2$. By construction of sC, we have $b'_1, b'_2 \in C$, so $b'_1 = b'_2$ by $\Phi(C)$. Hence

$$\forall b'. \quad b' \prec b_1 \iff b' \prec b_2,$$

so $b_1 = b_2$ by extensionality of B. So we have proved $\Phi(sC)$. The special condition is as in Lemma 2.5, so Pataraia induction gives $\Phi(B)$, which is that $f : B \to A$ is 1–1. □

We use these two results to prove our version of Mostowski's theorem (Remark 1.11), which is that any well founded relation may be made extensional by forming the quotient by an equivalence relation. This replaces the *ad hoc* references to co-recursion in [Tay96a, Thm 2.11] with Pataraia induction. Theorem 8.9 is the generalisation to coalgebras.

Theorem 2.13 Let $(X, \prec)$ be a well founded relation,

$$\begin{tikzcd} (E,<) = X/{\sim} \arrow[rr, dotted, "h"] & & (E',<') \\ (X,\prec) \arrow[u, "f"] \arrow[urr, "g"'] & & \end{tikzcd}$$

Then there is an extensional well founded relation $(E, <)$ and a surjective simulation function $f : X \to E$, with the universal property that, for any simulation function $g : X \to E'$, where $(E', <')$ is extensional and well founded, there is a unique simulation function $h : E \to E'$ such that $g = h \circ f$.

Proof First consider the universal property. Extensionality of E' at $g(x)$ and $g(y)$, where $x, y \in X$, says

$$\left[\forall e'.\, e' <' g(x) \Leftrightarrow e' <' g(y)\right] \quad \Longrightarrow \quad g(x) = g(y).$$

Write $x \sim y$ for $g(x) = g(y)$ and use Definition 2.6. By an argument similar to Lemma 2.12,

$$(\forall x' \prec x.\, \exists y' \prec y.\, x' \sim y') \quad \wedge \quad (\forall y' \prec y.\, \exists x' \prec x.\, x' \sim y') \quad \Longrightarrow \quad x \sim y,$$

which is a bisimulation relation. By Proposition 2.9, there is a greatest of these and by Pataraia induction it is reflexive, symmetric and transitive (an equivalence relation).

The order relation on the quotient $X/(\sim)$ is defined by

$$[x] \prec [y] \quad \equiv \quad \exists y'.\, x \sim y' \prec y.$$

Then $X \to X/(\sim)$ preserves $\prec$ and is a surjective simulation function. Hence $X/(\sim)$ is well founded by Corollary 2.11.

Since $\sim$ is fixed by the successor operation, $X/(\sim)$ is extensional. Moreover, for any denser equivalence relation $\approx$, the quotient $X/(\sim) \to X/\approx$ is a simulation function out of an extensional well founded relation, so it is 1–1 by Lemma 2.12 and therefore bijective. □

This is as much as is required as an introduction to the remaining sections of the paper: the rest of this section is a largely historical commentary on fixed point theorems.

Remark 2.14 Theorems like Pataraia's are usually stated without our special condition, concluding instead that s has a least fixed point, but there may be lots of other stuff above and alongside it. We presented the argument as we did in part because it is simpler to *deduce* the least fixed point result from our version than *vice versa.*

But principally, our tool is meant to be a scalpel, not a sledgehammer.

We introduced the special condition because, as the applications have shown, the objective is often to prove that there *is* a top element in some system. Plainly if the fixed point is unique then the special condition must hold, but to *prove* uniqueness we would normally postulate two candidates *without assuming* any *â priori* relation between them. The special condition says that we *may* suppose that they are in the order relation. Compare this with Tarski's version [Tar55], where is a *lattice* of fixed points, so our condition reduces that to just one.

There must have been some kind of "refinement" of the raw setting to make it satisfy our special condition. In our examples this done by assuming well-foundedness in some form.

Remark 2.15 One general way of performing this refinement would be to say that the ipo X to which we apply our version is obtained from a larger one as the subset *generated* by $\bot$, s and joins of either directed subsets or chains.

The classical forerunner of Pataraia's Theorem did exactly that. Ernst Witt [Wit51] and the Bourbaki group [Bou49] showed that this subset is itself a chain (linear order), in fact

$$\forall x, y{:}X. \quad y \leq x \ \vee \ sx \leq y.$$

Hence it has a greatest element and is indeed *well* ordered. It therefore satisfies what we have called Pataraia induction, and Hartogs' lemma is no longer needed.

Once again, the credit really belongs to Ernst Zermelo [Zer08a]: see the last paragraph on page 184 of [vH67], which proves this property, albeit for descending chains of subsets. There $sA \equiv A' \equiv A - \{\phi(A)\}$, where $\phi : \mathcal{P}^+(M) \to M$ is the choice function.

So the historical question is why this result, which should have been in the core of the curriculum, kept being re-discovered, starting with [Hes08, §125], and then re-buried under crude ordinal recursion.

You may perhaps understand this if you try to find your own proof: in contrast to the simplicity of Pataraia's Theorem, the Bourbaki–Witt property is surprisingly awkward, because the natural induction step swaps the two variables and cases. On the other hand, once you find one proof and then study the literature, you will see that there are multiple strategies. Walter Felscher identified a principle of double induction to handle this [Fel62], claiming that it underlay earlier versions. This too was forgotten and re-discovered.

As for constructivity, Todd Wilson has shown that the double induction argument does not use excluded middle [Wil01]. However, the resulting notion of well-ordering just says that every inhabited subset has a least element, as in Cantor's condition, but this is not enough for intuitionistic induction.

Returning to Definition 2.1, Andrej Bauer and Peter LeFanu Lumsdaine have investigated the difference between joins of chains and of directed subsets in the effective topos [Bau09, BL12]. From this, the Bourbaki–Witt property is definitely classical.

Forming the *subset generated* in this way requires second order logic, but there is a first order way of defining a (possibly larger but) suitable subset. This uses the poset translation of the categorical notion of well founded coalgebra that we introduce in the next section.

Proposition 2.16 Let $(X, \leq)$ be an ipo that also has binary meets $(\wedge)$ and $s : X \to X$ a monotone endofunction. We say that $x : X$ is a ***well founded element*** if

$$x \ \leq \ sx \qquad \text{and} \qquad \forall u{:}X. \ \ (su \wedge x \ \leq \ u) \implies x \ \leq \ u.$$

Then there is a greatest well founded element, which is the least fixed point of s and satisfies Pataraia induction.

Proof The subset of well founded elements satisfies the hypotheses of Pataraia's Theorem, including the special condition. □

Proposition 2.17 Let $(A, <)$ be any set with a binary relation and

$$sX \quad \equiv \quad \{a : A \mid \forall b{:}A. \ \ b \prec a \implies b \in X\}$$

the successor operation from Theorem 1.5(f), on the full powerset $\mathcal{P}A$. Then X is a well founded element iff it is an initial segment on which $<$ is a well founded relation.

Proof It is an initial segment iff $X \subset sX$ and the induction premise for U is $sU \cap X \subset U$. $\square$

We have not exploited well founded *elements* in this paper, but doing so might simplify our definitions and proofs further, eliminating the special condition. However, its use in Theorem 8.9 does not apparently follow *directly* from well-foundedness, even though it did in the analogous Lemma 2.12.

Remark 2.18 We have already explained why it is not legitimate to *use* ordinals to prove the Recursion Theorem. Indeed, if you do so and make *full disclosure* of the relevant proofs (by von Neumann and Hartogs) down to your chosen foundations (Zermelo [Zer08b], maybe), you will see that Pataraia's proof is *much simpler*.

It is also intuitionistically valid.

But even if you don't care about using Excluded Middle, maybe you agree with G.H. Hardy's maxim that "there is no permanent place in the world for ugly mathematics" [Har40, §10]. The classical theory of the ordinals is ugly and heavy: Pataraia's proof is *much prettier*.

We have already given a neater complete proof of the validity of ordinal recursion in this section, but (unless you *really* need the value at some random polynomial in ω) its uses in other places throughout mathematics could very probably be reduced to simpler and more natural arguments using Pataraia's Theorem instead.

At a more elementary level, our hypotheses are very similar to those of the famous "lemma" for which Max Zorn denied responsibility [Cam78, Zor35], but which has dominated the literature on fixed points. It is very easy to deduce our Theorem from that assumption. Conversely, in the situation where a maximal element that has been found *à la* Zorn turns out to be unique up to unique isomorphism, it may be possible to adapt the proof of the latter fact to verify our special condition instead, which would eliminate the axiom of choice from the construction.

However passionately we advocate more subtle methods, people will still regress to ordinal recursion and even misreport others' work. Kazimierz Kuratowski [Kur22] argued this a century ago, with worked examples and references from Set Theory, topology and measure theory, saying repeatedly that he was doing *induction*. Sadly, even he, much later in his career, based his textbook on the usual transfinite stuff [Kur61].

For further studies of the history of the fixed point theorem see [Bla14, Cam78, Fel62].

Remark 2.19 Finally, Lemma 2.2 constructs the inflationary monotone function t as the join of *all such*, so it is ***impredicative***. Indeed, the *directed* set Y is much bigger than the X that we required to be directed-*complete*. Possibly related is that there seems to be no categorical or "proof-relevant" version.

This is an issue in some understandings of *constructivity* that we will not address in this paper. If you would like to do so, note that we will not repeat this sin, so it is just that Lemma that you will need to replace. One might imagine obtaining it from the *proof* instead of the *fact* that a dcpo is well formed.

Proof theorists apparently have a different conception of ordinals from the one in the set-theoretic tradition, generated more like infinitary algebra, *cf.* Remark 1.7. We therefore believe that the categorical reformulation that we are about to develop would continue to be of value to do this, in some more predicative category than **Set**.

3 Well founded coalgebras

We now show how the ideas from Set Theory, universal algebra and process algebra in the previous two sections can be expressed in category theory. We build on the work of Christian Mikkelsen and Gerhard Osius. This was done in the years following the introduction of the notion of an elementary topos by Bill Lawvere and Myles Tierney [Law70], when the key issues were to optimise the categorical axioms and show that toposes could do anything that sets could do.

The main part of Mikkelsen's thesis [Mik76] gave an important simplification of the categorical definition of a topos, showing how colimits could be derived from limits (although this was eclipsed by Robert Paré's monadicity result [Par74]), after his supervisor Anders Kock had derived exponentials from powersets [KM74]. As an appendix, he gave the first proof of the recursion theorem in a topos, in a very "diagrammatic" style; apparently he devised the argument himself, not having known von Neumann's Theorem 1.5.

Gerhard Osius was one of several people who demonstrated how to interpret "ordinary" mathematical notation (higher order logic) in a topos. The aspect of this that was not also done by other authors was was to take $\in$-structures seriously as mathematical objects in a categorical setting [Osi74, §§4&6]. He also summarised Mikkelsen's proof of the recursion theorem in more familiar notation [Osi75, §6].

It is a pity that neither of them continued studying categorical logic: Osius became a professor of statistics (and died in 2019) and Mikkelsen a schoolteacher, having been unable to find a permanent university job.

The extension of their theory to any endofunctor T of a topos that preserves inverse images was made in [Tay99, §6.3] and sketched for other categories in [Tay96b]. In this paper we weaken the requirement on T to preservation of mono(morphism)s, but in the next section we show how the latter may be replaced by special notions of inclusion in other categories.

We give precise references to some corresponding results in these earlier works, for historical comparison, but the ones here are often much more general. (Unfortunately, I mis-attributed Mikkelsen's work to Osius in my earlier work.)

We work throughout in the logic of an elementary topos $\mathcal{S}$, remembering to thank Osius and others for allowing us to write this in the vernacular of mathematics. You may therefore treat $\mathcal{S}$ as **Set**, except that we do not use Excluded Middle or the Axiom of Choice, although the key Lemma 2.2 is impredicative.

So far, we have discussed a binary relation on a carrier A. There are many ways of representing a relation in category or type theory, but the one that we choose is as a function (morphism)

$$A \xrightarrow{\alpha} \mathcal{P}A \qquad \text{by} \qquad a \longmapsto \{b \mid b \prec a\} \subset A.$$

This is directly analogous to the parse operation for a free algebra (Remark 1.7), where $\prec$ or $\in$ correspond to the immediate sub-term relation.

We can do the same for any functor T whatever, although we will throughout require it to preserve monos:

Definition 3.1 A ***coalgebra*** for an endofunctor $T : \mathcal{C} \to \mathcal{C}$ of any category is an object A of $\mathcal{C}$ together with a morphism $\alpha : A \longrightarrow TA$. We say (provisionally) that (A, α) is ***extensional*** if α is mono in $\mathcal{C}$, *cf.* Definition 1.8.

$$\begin{tikzcd} TA \arrow[r, "Tf"] & TB \\ A \arrow[u, "\alpha"] \arrow[r, "f"] & B \arrow[u, "\beta"'] \end{tikzcd}$$

A homomorphism of coalgebras is a $\mathcal{C}$-morphism $f : A \longrightarrow B$ that makes the square commute, which we indicate by the triangle arrowhead. We mark the structure map α in the same way because it is a homomorphism to $(TA, T\alpha)$. We write $\mathbf{CoAlg}_T$ or just **CoAlg** for the category of coalgebras and homomorphisms.

This paper develops an entire theory that is remarkably similar to Set Theory, but just using a functor that need have hardly any of the properties of the powerset. This alone is a massive declaration of foundational autonomy. Nevertheless, to relate coalgebras to the background in Set Theory, we first need a full understanding of the powerset as a functor in a topos:

Notation 3.2 The ***covariant powerset functor*** $\mathcal{P} : \mathcal{S} \to \mathcal{S}$ is defined on an object X by $\mathcal{P}X \equiv \Omega^X$ and on a function $f : X \to Y$ by

$$\mathcal{P}fU \;\equiv\; \{fx \mid x \in U\} \;\equiv\; \{y : Y \mid \exists x{:}X.\ y = fx \wedge x \in U\} \subset Y$$

for $U \subset X$. We shall also need to define, for $V \subset Y$,

$$\begin{aligned} f^*V &\equiv \{x : X \mid fx \in V\} \\ f_*U &\equiv \{y : Y \mid \forall x{:}X.\ fx = y \Longrightarrow x \in U\}. \end{aligned}$$

These also provide the morphism parts of functors $\mathcal{S} \to \mathcal{S}$ that are respectively contravariant and covariant, since $(g\,;f)^*W = f^*(g^*W)$ and $(g\,;f)_*U = g_*(f_*U)$. More importantly for us, there are (order-)adjunctions

$$\begin{tikzcd} U \arrow[r, tail] & X \arrow[d, "f"] \\ V \arrow[r, tail] & Y \end{tikzcd} \qquad\qquad \begin{tikzcd} \mathcal{P}X \arrow[d, shift right=6, "\mathcal{P}f"'] \arrow[d, shift left=6, "f_*"] \\ \mathcal{P}Y. \arrow[u, "\dashv\ f^*\ \dashv" description] \end{tikzcd}$$

Diagrammatically, $\mathcal{P}f$ and f^* are given by composition and pullback respectively. The logical formulae that define $\mathcal{P}fU$ and f_*U are the same except that one involves an existential and the other a universal quantifier. We will use f_* in Section 9. $\square$

Gerhard Osius's principal insight was to characterise set-theoretic inclusions as homomorphisms of extensional recursive $\mathcal{P}$-coalgebras [Osi74, §6], although we will replace recursion with well-foundedness.

Lemma 3.3 A function $f : (B, \prec_B) \to (A, \prec_A)$ is a homomorphism of $\mathcal{P}$-coalgebras iff it is strictly monotone, *i.e.* it preserves the binary relation as in Proposition 1.4,

$$\forall b_1, b_2{:}B. \quad b_1 \prec_B b_2 \;\Longrightarrow\; fb_1 \prec_A fb_2,$$

and a simulation (Definition 2.6),

$$\forall a'{:}A.\ \forall b{:}B. \quad a' \prec_A fb \;\Longrightarrow\; \exists b'{:}B.\ a' = fb' \wedge b' \prec_B b.$$

$$\begin{tikzcd} B \arrow[r, "\beta"] \arrow[d, "f"'] \arrow[dr, phantom, "\supset"] & \mathcal{P}B \arrow[d, "\mathcal{P}f"] \\ A \arrow[r, "\alpha"] & \mathcal{P}A \end{tikzcd} \qquad \begin{tikzcd} \exists b' \arrow[r, dotted, "\prec_B"] \arrow[d, dotted, "f"'] & b \arrow[d, "f"] & B \arrow[d, "f"] \\ a \arrow[r, "\prec_A"] & fb & A \end{tikzcd}$$

In this case, the relation $(a \sim b) \equiv (a = fb)$ is actually a *bi*simulation.

Proof The inclusion $\beta\,;\mathcal{P}f \subset f\,;\alpha$ (as marked in the diagram on the left) holds iff f is strictly monotone and the reverse inclusion (illustrated on the right) iff f is a simulation. For a *bi*simulation we also require

$$\forall ab'.\ b' \prec_B b \;\wedge\; fb = a \Longrightarrow \exists a'.\ b' = fa' \;\wedge\; a' \prec_A a,$$

but this follows from strict monotonicity, with $a' \equiv fb' \prec_A fb = a$. $\square$

Corollary 3.4 If $f : B \subset A$ is a subcoalgebra inclusion then the lifting is unique, so being a homomorphism says that B carries the restriction of $\prec$ from A and is down-closed or an ***initial segment***,

$$\forall ab{:}A. \quad a \prec b \in B \implies a \in B,$$

just as we have used in the proof of the recursion theorem. $\square$

Observe that (infinitary) *directed* unions of initial segments can only build *ascending* $\prec$-sequences.

We are ready to formulate the two concepts that are connected by our main result.

Definition 3.5 A coalgebra $\alpha : A \longrightarrow TA$ is ***well founded*** if in any pullback diagram in the category $\mathcal{C}$ of the form

$$\begin{tikzcd} TU \arrow[r, tail, "Ti"] & & TA \\ H \arrow[u] \arrow[r, tail, "j"] & U \arrow[r, tail, "i"] & A \arrow[u, "\alpha"'] \end{tikzcd}$$

the maps i and therefore j are necessarily isomorphisms. To clarify, we mean that when we form the pullback H of Ti and α, the map $H \to A$ factors through $i : U \to A$.

We write $\mathbf{WfCoAlg}_T$ or just **WfCoAlg** for the category of well founded coalgebras and coalgebra homomorphisms. The "scheme" issues in Remark 1.2 will be considered in the next section.

Essentially this "broken pullback" appears (with $T \equiv \mathcal{P}$) on page 99 of [Mik76] and it is written symbolically as $\alpha^{-1}(\mathcal{P}U) \subset U \Longrightarrow U = A$ in [Osi74, §4] and [Osi75, Prop 6.1]. It was first given as the *definition* of well-foundedness in [Tay96b, Tay99].

The result that justifies this name is implicit in the work of Mikkelsen and Osius, but not very clearly expressed there:

Proposition 3.6 A binary relation $(A, \prec)$ is well founded in the earlier sense iff the corresponding (A, α) is a well founded $\mathcal{P}$-coalgebra.

Proof Write $U \equiv \{x : A \mid \phi x\}$ for some predicate ϕ defined on A.

An element $(a, V) \in H \subset A \times TU$ of the pullback consists of $a : A$ and $V \subset U \subset A$ such that

$$\alpha(a) \ \equiv \ \{x : A \mid x \prec a\} \ = \ V.$$

Thus V is determined uniquely by a (and the structure $\alpha : A \longrightarrow TA$), but for such a V to exist, a must satisfy

$$\{x : A \mid x \prec a\} \ \subset \ U, \qquad i.e. \ \ \forall x{:}A. \ \ x \prec a \Longrightarrow \phi x.$$

The pullback H therefore corresponds to the induction ***hypothesis*** (Definition 1.1).

The induction ***premise*** is that, for each such $a : A$ that satisfies the hypothesis, we have $a \in U$ or ϕa. In the diagram this means that $H \subset U$. The *tight* induction premise corresponds to having $H \cong U$ instead; this makes $U \subset A$ a subcoalgebra for which the square is a pullback.

Well-foundedness of the coalgebra says that whenever we have a diagram of this form then $U \cong A$, just as the induction *scheme* says that whenever the premise holds then we must have $\forall a{:}A. \ \phi a$. $\square$

We also have agreement with Definition 2.16:

Proposition 3.7 The relative successor can be defined on subobjects of the carrier of a coalgebra. Then such a subobject is a well founded element iff it is a well founded subcoalgebra.

$$\begin{tikzcd}
TU \arrow[rr, hook, "Tj"] & & TB \arrow[r, hook, "Ti"] & TA \\
sU \arrow[u] \arrow[rr, hook] \arrow[ur, phantom, "\urcorner", very near start] & & sB \arrow[u] \arrow[r, hook] \arrow[ur, phantom, "\urcorner", very near start] & A \arrow[u, "\alpha"'] \\
H \arrow[u, hook] \arrow[r, hook] \arrow[ur, phantom, "\urcorner", very near start] & U \arrow[r, hook] & B \arrow[u, hook] \arrow[r, "\mathsf{id}"] & B \arrow[u, hook, "i"'] \arrow[ull, "\beta"]
\end{tikzcd}$$

Proof Given a coalgebra $\alpha : A \longrightarrow TA$ and a subobject $i : B \hookrightarrow A$ of its carrier, the (upper right) pullback sB is the successor *subobject*. The condition $B \leq sB$ on subobjects is the lower right square; this is equivalent to having a subcoalgebra structure β, by composition and pullback.

Now let $U \hookrightarrow B$ be a subobject of the carrier B, so its successor sU is defined in the same way. Then the pullback H of TU and B factors through sU and states both

- $sU \cap B \equiv H \subset U$ for B to be a well founded *element* and
- the broken pullback for $(B\beta)$ to be a well founded *coalgebra*. $\square$

The other side of the main result is recursion:

Definition 3.8 A *coalgebra* $\alpha : A \longrightarrow TA$ obeys the ***recursion scheme*** if, for every *algebra* $\theta : T\Theta \to \Theta$, there is a unique map $f : A \to \Theta$ such that the square

$$\begin{tikzcd}
TA \arrow[r, "Tf"] & T\Theta \arrow[d, "\theta"] \\
A \arrow[u, "\alpha"] \arrow[r, "f"] & \Theta
\end{tikzcd}$$

commutes. The notion is a *scheme* because we only ever consider *particular* algebras (Θ, θ). A map of this kind has also been called a ***coalgebra-to-algebra homomorphism*** [Epp03].

To obtain ***parametric recursion***, in which the top line is replaced by

$$Tf \times \mathsf{id} : TA \times A \longrightarrow T\Theta \times A,$$

we just need to make Lemma 6.5 a bit more complicated. In fact Mikkelsen had an even more general recursion scheme than this, although still with $T \equiv \mathcal{P}$ [Mik76, pp 98–99] [Osi75, Def 6.2]. Osius's account of categorical Set Theory [Osi74] used recursion instead of well-foundedness (induction).

Example 3.9 The predecessor and test for zero function define a coalgebra on $\mathbb{N}$ for the functor $TX \equiv \mathbf{1} + X$ on $\mathcal{S}$. Then recursion defines $f : \mathbb{N} \to \Theta$ by the two cases

$$f0 \;=\; \theta(\star) \qquad \text{and} \qquad fn \;=\; \theta\big(f(n-1)\big).$$

In a topos, well-foundedness is *necessary* for recursion [Mik76, p 100] [Osi75, Prop 6.3] [Tay99, Exercise 6.14]:

Proposition 3.10 In a topos, if $\alpha : A \longrightarrow\!\!\!\!\triangleright\ TA$ obeys the recursion scheme then it is well founded.

Proof The subobject classifier (set of truth values) $\Theta \equiv \Omega \equiv \mathcal{P}(1)$ carries an algebra structure for any operation whatever, namely by interpreting it as (infinitary) *conjunction* or universal quantification. Then $f : A \to \Theta$ is a homomorphism iff

$$fa \iff \big(\forall x.\, x \prec a \Rightarrow fx\big).$$

This is the tight ($\Leftrightarrow$) version of the induction premise, whilst the constant function $f : a \mapsto \top$ is also a homomorphism. So *uniqueness* of f amounts to the induction scheme.

$$\begin{tikzcd}
H \arrow[r] \arrow[d, tail] \arrow[dr, phantom, very near start, "\lrcorner"] & TU \arrow[r] \arrow[d, tail, "Ti"'] \arrow[dr, phantom, very near start, "\lrcorner"] & T1 \arrow[r] \arrow[d, tail, "T\top"'] \arrow[dr, phantom, very near start, "\lrcorner"] & 1 \arrow[d, tail, "\top"] & U \arrow[l] \arrow[d, tail, "i"] \arrow[dl, phantom, very near start, "\llcorner"] \\
A \arrow[r, -{Triangle[open]}] & TA \arrow[r, "Tf"] & T\Omega \arrow[r, "\theta = \chi_{T\top}"] & \Omega & A \arrow[l, "f"']
\end{tikzcd}$$

This argument generalises. Let $\theta : T\Omega \to \Omega$ be the characteristic function of the subset $T\top : T1 \rightarrowtail T\Omega$, where $\top : 1 \rightarrowtail \Omega$ is the element "true". The induction premise is $\alpha\,;Tf\,;\theta \Rightarrow f$ and the tight premise has equality (bi-implication), but this is also satisfied by the constant function with value $\top$. $\square$

Remark 3.11 This result should be treated with circumspection, because taking the object of truth values as the target algebra means that we are relying on *higher order logic.* (This point is obscured classically by the identification of Ω with a discrete two-element set.)

For example, induction for the predicate $\phi x \equiv (x \not\prec x)$ shows that well founded relations must be *irreflexive.* However, this makes the idea too clumsy to analyse *fixed* points of iteration, as we might hope to do in future applications of the theory.

On the other hand, experience shows that we must count ourselves lucky to find a condition for termination of a heavily recursive program which is *sufficient* for the case at hand: asking for it to be *necessary* as well is too much.

Remark 4.18 replaces higher order Ω with similar objects for particular logical complexity levels.

In Remark 1.7 we made an *analogy* between Set Theory and term algebras. The tools that we now have already show us how to formalise this:

Lemma 3.12 Let $\kappa : T \to P$ be a natural transformation whose naturality squares are pullbacks, between endofunctors of a category with pullbacks. (Such κ is called a ***cartesian transformation***.) Then $\alpha : A \longrightarrow\!\!\!\!\triangleright\ TA$ is a well founded T-coalgebra iff $\alpha\,;\kappa_A : A \to PA$ is a well founded P-coalgebra. If κ_A is mono then the notions of extensionality coincide too.

$$\begin{tikzcd}
PU \arrow[rr, tail, "Pi"] & & PA \\
TU \arrow[u, "\kappa_U"] \arrow[rr, tail, "Ti"] \arrow[ur, phantom, very near start, "\urcorner"] & & TA \arrow[u, "\kappa_A"'] \\
H \arrow[u] \arrow[r, tail] \arrow[ur, phantom, very near start, "\urcorner"] & U \arrow[r, tail, "i"] & A \arrow[u, -{Triangle[open]}, "\alpha"']
\end{tikzcd}$$

Proof Since the upper rectangle is a pullback, the whole diagram is one iff the lower rectangle is. That is, the induction hypothesis H is the same for P as for T [Tay96b, Prop 7.4]. $\square$

Proposition 3.13 Let $T : \mathcal{S} \to \mathcal{S}$. Then there is a natural transformation $\kappa : T \to \mathcal{P}$ that is cartesian with respect to monos (as above) iff T preserves arbitrary intersections.

Such T was called an ***analytic functor*** in [Joy87] since it has a "power series" representation.

Proof For any set X and $t : TX$, define $\kappa_X : TX \to \mathcal{P}X$ by

$$\kappa_X(t) \quad \equiv \quad \bigcap\{U \subset X \mid t \in TU\} \quad \equiv \quad \{x : X \mid \forall U \subset X.\, t \in TU \Longrightarrow x \in U\}.$$

Now $(t, V) \in TA \times \mathcal{P}U$ lies in the pullback iff $V = \kappa_X(t) \subset U$. If T preserves intersections, this happens iff $t \in TU$.

$$\begin{tikzcd}
\mathcal{P}(\bigcap_i U_i) \arrow[r, equal] & \bigcap_i \mathcal{P}U_i \arrow[r, tail] & \mathcal{P}U_i \arrow[r, tail] & \mathcal{P}A \\
T(\bigcap_i U_i) \arrow[u, "\kappa_{\bigcap_i U_i}"] \arrow[r, tail] \arrow[ur, phantom, very near start, "\urcorner"] & \bigcap_i TU_i \arrow[u] \arrow[r, tail] & TU_i \arrow[u, "\kappa_{U_i}"'] \arrow[r, tail] \arrow[ur, phantom, very near start, "\urcorner"] & TA \arrow[u, "\kappa_A"']
\end{tikzcd}$$

The condition is necessary because $\mathcal{P}$ preserves arbitrary intersections and pullbacks commute with them [Tay96b, Prop 7.5]. ☐

4 Categorical requirements

Our theory applies to an endofunctor T that preserves monos, but we have not yet said anything about what we require of the category $\mathcal{C}$ on which it acts. Beyond that, as we generalise $\mathcal{C}$ further and further away from **Set**, we find that it have many different kinds of "inclusions" that (have but) are not necessarily characterised by the standard cancellation property that defines (what we shall call ***plain***) ***monos*** in a category.

Besides the functor T, the freedom to choose different categories and notions of mono in them gives considerable power to this theory.

We address those questions in this section, but really this is a technical analysis of the proof to follow. Therefore, even if you are proficient in categorical logic, it would still be better to understand the next two or three sections *grosso modo* before reading this one, so that you can see why the following subtleties are needed. So, this is like the configuration section of a piece of software, that logically has to come first, but which you must not touch until you know what you're doing.

On first reading, you should therefore simply take $\mathcal{C} \equiv \mathcal{S} \equiv \mathbf{Set}$, read both arrowtails as injective functions and assume that the functor $T : \mathbf{Set} \to \mathbf{Set}$ preserves them. Then you may omit this section.

The simplest statement of more general but sufficient conditions is this:

Provisional assumption 4.1 The category $\mathcal{C}$

(a) has inverse images (pullbacks) of monos along coalgebra homomorphisms;

(b) has an initial object $\emptyset$ and all maps $\emptyset \to X$ are mono;

(c) has directed unions of subobjects (Definition 2.1), and

(d) is well powered, whilst

(e) the functor $T : \mathcal{C} \to \mathcal{C}$ preserves monos.

Besides defining "unions" and "well powered", we also need to examine all of these assumptions more carefully.

We will use some other finite limits in $\mathcal{C}$, but only incidentally, not as part of the proof of our main theorem: Lemma 6.6 uses binary products to show how to handle parametric recursion.

Lemma 6.4 uses equalisers to prove uniqueness of recursion, but we can deduce that in another way, without using them. The terminal object $\mathbf{1}$ is never used.

Much of this section is about replacing the "monos" in (a,e) with some special class of maps that we use for "predicates" and those in (b,c,d) with another possibly smaller class of "initial segments".

Remark 4.2 Any category of finitary algebras satisfies (a,c,d), but part (b) is more delicate. Recall from universal algebra that, in an appropriately constructed category of algebras, the ***initial object*** typically arises as the collection of terms *generated* by a given set of symbols (*cf.* Remark 1.7).

We can mimic this for any object I of any category: Working instead with the (coslice or cocomma) category whose objects are monos $I \hookrightarrow X$ and whose morphisms are commutative triangles, the initial object is id_I and all maps out of it are monos. This construction leaves the other provisional assumptions intact, because the subobjects, inverse images and directed unions in the coslice are essentially the same as those in the original category.

For example, the category of fields does not meet our requirements as it stands, but cutting it down to *those of a particular characteristic* does: this selects one of the components of the category and then $\mathbb{Q}$ or $\mathbb{F}_p$ is the initial object. We also need to fix the characteristic if we want to work with rings (or commutative rings), because that ensures that all maps from the initial object ($\mathbb{Z}$ or $\mathbb{Z}_n$) are mono.

Our main Recursion Theorem 1.5 works by building up partial maps from the empty one. This means in particular that the initial *object* must serve as the least *subobject* of any object. This is why maps out of the initial object need to be mono, which is not the case for the initial ring $\mathbb{Z}$.

More generally, in order to *combine* partial maps we need to make the *colimits* of monos in the category behave like *unions* of subobjects of each object. So we first need to be clear what "unions" are in general; this is rather basic category theory (of the kind that makes it a far superior foundational tool to Set Theory) but I cannot find an attribution for it.

Definition 4.3 A ***union*** in a category is a diagram or its colimit such that
(a) the maps in the diagram are mono;
(b) the maps in the colimiting cocone are mono;
(c) for any other cocone consisting of monos, the colimit mediator is also mono.

Proposition 4.4 **Set** (or any topos $\mathcal{S}$) has directed unions.

Proof (Sketch) A colimit in **Set** is given by the quotient of a coproduct by an equivalence relation that is obtained from the diagram. The different components of a coproduct are disjoint.

Two elements are identified in the colimiting cocone iff they are linked by a finite zig-zag in the relation. Since the diagram is directed, it has some further stage (beyond the zig-zag but still within the diagram) that is a cocone over the zig-zag. Since this cocone consists of monos, the two elements were already equal.

Now consider the kernel (pullback against itself) of the mediator to any other cocone of monos. Since colimits are stable under pullback, this kernel is a doubly-indexed union. But since the diagram is directed, this is equivalent to a singly-indexed union, which is in fact the original diagram. Hence the projections from the kernel are isomorphisms and so the mediator is mono. □

In other categories, the second part of the argument shows that the mediators in Definition 4.3(c) are *plain* monos whenever colimits are stable under pullback. But this is not sufficient for other kinds of inclusions. We give the analogous results for pushouts in a (pre)topos more formally in Section 10.

To emphasise the importance of this property, we give an example of its failure:

Example 4.5 The union requirement fails for $\mathbf{Set}^{\mathsf{op}}$.

Proof It is clearer to discuss the dual categorical properties in **Set** itself.

Classically, all maps $X \to \mathbf{1}$ are epi, except when $X \equiv \emptyset$. All maps in a limiting cone over a cofiltered diagram of epis are epi, if we assume the axiom of choice.

However, Choice and excluded middle do not help in making the mediator epi too. Consider the following chain diagram, in which each column denotes a set and the successive maps between the finite sets squash the top two elements:

$$\begin{tikzcd}
 & \top \arrow[ddr, mapsto] & & & & \\
\vdots & \vdots & & & & \\
3 \arrow[r, mapsto, dotted] & 3 \arrow[r, mapsto] & \cdots \arrow[dr, mapsto] & & & \\
2 \arrow[r, mapsto, dotted] & 2 \arrow[r, mapsto] & \cdots \arrow[r, mapsto] & 2 \arrow[dr, mapsto] & & \\
1 \arrow[r, mapsto, dotted] & 1 \arrow[r, mapsto] & \cdots \arrow[r, mapsto] & 1 \arrow[r, mapsto] & 1 \arrow[dr, mapsto] & \\
0 \arrow[r, mapsto, dotted] & 0 \arrow[r, mapsto] & \cdots \arrow[r, mapsto] & 0 \arrow[r, mapsto] & 0 \arrow[r, mapsto] & 0 \\
\mathbb{N} \arrow[r, dotted] & \mathbb{N}^{\top} \arrow[r] & \cdots \arrow[r] & \mathbf{3} \arrow[r] & \mathbf{2} \arrow[r] & \mathbf{1}
\end{tikzcd}$$

Its limit is $\mathbb{N}^{\top}$, but there is also a cone of epis with vertex $\mathbb{N}$, but for which $\top$ is not in the image of the mediator, *i.e.* this is not surjective onto $\mathbb{N}^{\top}$.

Example 10.4 considers pullbacks. □

Remark 4.6 Venanzio Capretta, Tarmo Uustalu and Varmo Vene considered the categorical dual of our notion of well founded coalgebra, which they called an ***antifounded algebra*** [CUV09]. They presented a number of illuminating counterexamples that falsify our main recursion theorem unless we put other conditions on the category. Their simplest example is that $\mathsf{suc} : T\mathbb{N} \to \mathbb{N}$ is an antifounded algebra for $T \equiv \mathsf{id} : \mathbf{Set} \to \mathbf{Set}$, but there is no homomorphism from the trivial coalgebra $\mathsf{id} : \mathbf{1} \to T\mathbf{1}$, because its value should be the fixed point of suc, which we would like to be $\top$ in $\mathbb{N}^{\top}$. It would be instructive to compare their other counterexamples to our proofs, to see the necessity of the conditions in this section. □

Definition 4.7 A category is ***well powered*** if, for each object X, there is a "set" of isomorphism classes of monos $U \hookrightarrow X$.

On the face of it, the word "set" is an embarrassment, given that we aim to eliminate Set Theory from mathematical foundations. But, as mathematicians, we pay our words extra to mean what *we* want them to mean [Car72, Chapter 6]. In general, we do this by specifying the ways in which we intend to use them, *i.e.* the axioms.

.

A "set" of objects is not a chaotic jumble but a *single* object that is *dependent* on some *parameter*. In the geometric tradition, this arose as the object (such as a tangent space) *varied* from one place to another in a space. In type theory (and indeed longstanding symbolic usage in real analysis), it simply means a formula containing an unknown.

What we require of dependency is just to be able to *substitute* other formulae for the unknown parameter. This parameter has a certain *type*. Such types and their formulae form a category $\mathcal{S}$, called the ***base***, which may be **Set**, an elementary topos or even something simpler. Then, for each type Γ in $\mathcal{S}$, the objects whose parameter is of type Γ together form the ***fibre*** over Γ.

Substitution of a formula for a parameter (or along a morphism f) is an operation f^* on dependent objects. There are two techniques for capturing how f^* takes one fibre to another:

(a) if we consider the fibres as *separate* structures, they are the object part and f^* is the morphism part of a *functor* that is contravariant in f, giving an ***indexed*** structure; but

(b) the fibres may be combined into a single structure, called a ***fibration***, in which f^* acts by *pullback*.

The account that develops well-poweredness in most detail, in the indexed style, is [PS78], although its goal is the adjoint functor theorem rather than our needs. The indexed approach has to contend with choices of isomorphic objects, which the fibred one avoids, but at greater learning cost. Brief accounts in the fibred style are in [Joh02, Example B1.3.14] and [Str05, §11]. Unfortunately, both techniques have rather obscure notation and huge diagrams, so, since we already have some very complicated ones, we will content ourselves a verbal description of how they work.

Definition 4.8 A ***generic*** object G is a parametric one that has the universal property that any *particular* object P is obtained as $P \cong f^*G$, by substitution of a value for the parameter in the generic one, The morphism f that achieves this is called the ***name*** of P.

The *type* of names is an object of the base category $\mathcal{S}$ and the generic object belongs to the fibre over this type. In particular, when the "objects" in question are monos $i : U \rightarrowtail X$ targeted at a particular object X of $\mathcal{C}$, the type of names is called $\mathsf{Sub}\,(X)$ and the generic subobject of X belongs to the fibre over this type.

Using the definition of genericity, any external structure that respects substitution induces an ***internal structure*** on the type of names in $\mathcal{S}$. For example, triangles of monos in $\mathcal{C}$ give rise to an internal order on $\mathsf{Sub}\,(X)$. In this sense we say that the external structure is ***equivalent*** to an internal one.

It is instructive to draw a few of these diagrams to show how, for example, pullbacks in $\mathcal{C}$ yield meets in $\mathsf{Sub}\,(X)$, making it an internal semilattice in $\mathcal{S}$. Then you will see that $\mathsf{Sub}\,(X)$ is like the handle of a marionette, with manoeuvres linked to the actions of the doll. With practice, we can just describe what the doll does, so long as we remember *how* it does it. We don't write out the diagram of strings because it conveys comparatively little information per cm^2 and is not really needed. In fact, the doll is *well powered* exactly when it is *impotent*, being able to do no more nor less than what the puppeteer makes it do.

Nevertheless, there is perhaps a PhD in collecting the applications of well powered categories from the literature and formalising results such as Proposition 4.10. This account would be analogous to those by Osius and others on the logic of a topos; indeed the subobject classifier provides the generic mono in a topos. We do not use universes (types of types) here, but they can be presented categorically in a similar way, although without uniqueness of names.

Corollary 4.9 Any construction on a generic object that respects substitution corresponds uniquely to a morphism of the base category. In particular, the construction of one subobject of X from another corresponds to an endomorphism of $\mathsf{Sub}\,(X)$.

Proof An operation on a parametric object yields another object with the same parameter, *i.e.* in the same fibre, whilst binary operations such as categorical products combine the parameters using pullbacks in $\mathcal{S}$. We then use the universal property of the *generic* object of the resulting kind to define the morphism of the base category. □

So far we have only discussed *finitary* structure such as composition and pullback. The original reason for requiring a "small" set of subobjects was so that we could legitimately form their union.

Proposition 4.10 External $\mathcal{S}$-indexed unions in a well powered category $\mathcal{C}$ correspond to joins in $\mathsf{Sub}\,(X)$.

Proof Any of the accounts of indexed and fibred categories explains how they handle colimits. Of course the "set" of objects of which we form the colimit is a single parametric one as before. In fact, the union operation is left adjoint to substitution and has an even simpler characterisation in that the opposite of the fibration functor is also a fibration.

The universal property of the generic subobject translates this into a join in the internal poset $\mathsf{Sub}\,(X)$. □

Remark 4.11 Pataraia's Theorem 2.3 is for *internal* ipos in $\mathcal{S}$. The role of the union and well powered conditions that we have described is to provide an *equivalence* amongst external colimits and unions and internal joins. The same link also relates constructions in $\mathcal{C}$ to morphisms between objects of $\mathcal{S}$. In particular, the "relative successor" that we construct in the category in

Constructions 5.6ff and 6.5 corresponds to a monotone inflationary endofunction of the internal ipo.

This has a fixed point by Pataraia's Theorem, which is valid precisely because the well powered condition turns colimits into joins in the object $\mathsf{Sub}(X)$ that is an internal poset in a topos. We translate this back into the category, as an object on which the construction yields an isomorphic object. $\square$

Remark 4.12 There is yet another reason why we need a "set" of subobjects, namely to justify universal quantification over them as predicates. (In Set Theory this distinction is known as ***unbounded*** *versus* ***bounded quantification***.)

When we introduced well-foundedness in Definitions 1.1 and 3.5, we called it a *scheme*, which means a property that we assert for *each individual* predicate ϕ. We will develop the *general* theory of well-foundedness in this way.

On the other hand, when we come to *apply* well-foundedness in the proof of our main theorem, we need it to be a *single* legitimate property in the logic of an elementary topos. For this it cannot be a scheme but must be *quantified* over all predicates ϕ.

Once again, by a "set" of predicates we mean a single generic predicate with a parameter. Well-foundedness with respect to a particular predicate ϕ is expressed in $\mathsf{Sub}(X)$ as above, with a parameter ϕ. Universal quantification over ϕ is now the *right* adjoint to substitution for ϕ, as is amply explained in the topos literature, *cf.* Notation 3.2. $\square$

Remark 4.13 We now turn to investigating the classes of "inclusions" that we might use in place of plain categorical monos when applying our ideas to objects with richer structure than sets have. We will use inclusions for three purposes in this paper:

(a) as the extents of *predicates* that test well-foundedness;

(b) as the inclusions of subcoalgebras that are the *supports* of attempts; and

(c) as the structure maps of *extensional* coalgebras.

All supports must be predicates to prove totality of recursion (Lemma 5.8), whilst supports and extensionality are thoroughly mixed up in Construction 7.5, so we must treat these as the same thing. Therefore we potentially have *two* classes of inclusions, one contained in the other, and we write

$$\rightarrowtail \text{ for predicates} \quad \text{and} \quad \hookrightarrow \text{ for supports and extensionality.}$$

It is tempting (thinking in terms of so-called Descriptive Set Theory) to call $U \rightarrowtail X$ a *subspace* and $U \hookrightarrow X$ an *open* subspace of X. Unfortunately, this need not be the same as an open subspace in whatever topology the object X might carry.

Beware that these two classes of monos are *additional structure* for the situation, along with the category $\mathcal{C}$ and functor T. Since our primary interest is likely to be in $\mathcal{C}$ and T, we are at liberty to choose the two classes of monos in whatever way yields the optimum results, although we may then want to show that these are independent of the choices.

Remark 4.14 As you will see in the next section, we have some conflict in the objectives for this paper between proving the central recursion theorem and developing the whole theory of well founded coalgebras. For the general theory, we might typically want

(a) a large class of predicates so that we can make liberal use of induction, but

(b) a small class of supports.

For the proof of the recursion theorem, it turns out that
we only need to do induction over the supports,
so the two classes are the same.

We simply need a usable class of extensional well founded coalgebras that contains the iterates of the functor T applied to the initial object, as tightly as possible.

Therefore, in the particular application category, we would like to find some notion of inclusion that is both tractable and restrictive.

It is straightforward to substitute these chosen inclusions for the "monos" in the definitions above of unions and being well powered. However, Proposition 4.4 only works for plain monos and so needs to be replaced with some other argument, which is why we formulated Definition 4.3 instead of just asking that colimits be stable under pullback.

It may be possible to control the unions even further, such as by making the diagrams computable in some sense, using techniques from various forms of synthetic domain theory, but we leave that for another day.

For the general theory we do distinguish between the classes and so need to axiomatise them separately. In doing this, it is convenient to make an auxiliary definition for the closure conditions that are common to both classes:

Definition 4.15 A ***class of*** T***-monos*** $\mathcal{M}$ must
(a) contain all isomorphisms;
(b) contain all maps from the initial object (*cf.* Remark 4.2);
(c) be closed under composition;
(d) be preserved by the functor T;
(e) be preserved by pullback along T-coalgebra homomorphisms; and
(f) satisfy the cancellation property for plain monos, $\forall f g.\ f\,;i = g\,;i \Longrightarrow f = g$.

The reason why we need the cancellation property is this: For the initial object $A \equiv \emptyset$, there is a map $p : A \to U$ with $p\,;m = \mathsf{id}_A$. The ubiquitous idiom in using well-foundedness gives the same thing. We use the cancellation property to deduce that $m\,;p = \mathsf{id}_U$.

Another easy but useful property that is also known as ***cancellation*** may be deduced as a "warm-up" exercise in the kind of diagram-chasing that we shall use throughout this paper:

Lemma 4.16 For any class of T-monos $\mathcal{M}$,
(a) if $i\,;m \in \mathcal{M}$ and m is a plain mono then $i \in \mathcal{M}$ too; and
(b) in the broken pullback for the induction premise (Definition 3.5), if the predicate $U \overset{i}{\rightarrowtail} X$ belongs to $\mathcal{M}$ then so does $H \overset{j}{\rightarrowtail} U$.

Proof Hint: The maps id, $(i\,;m)$, i and m form a pullback square. □

We are now ready to state the conditions for the two classes:

Assumption 4.17 The maps $\rightarrowtail$ used for predicates form a class of T-monos $\mathcal{M}$ for which also
(a) $\mathcal{M}$ includes all inclusions of initial segments $\hookrightarrow$; and
(b) each map $i \in \mathcal{M}$ belongs to some well-powered subclass $\mathcal{M}' \subset \mathcal{M}$ of T-monos.

For additional results beyond the main recursion theorem,
(c) the class could include all regular monos (equalisers, *cf.* Lemma 6.4);
(d) the functor T could preserve inverse image diagrams; or
(e) the inverse image operators f^* applied to predicates could have right adjoints f_* (Section 9).

Recall that, in categorical logic, inverse images correspond to substitution, equalisers to equations, composition of monos to existential quantification and the right adjoint f_* to universal quantification, *cf.* Notation 3.2. The conditions above are therefore natural and very flexible for considering precise restrictions on the logical strength of the predicates over which we may perform induction. This is possible (contrary to what was said in [Tay96b, Prop 6.7]) because we are making a distinction between the roles of predicates and initial segments.

Remark 4.18 Suppose that the class $\mathcal{M}$ has a ***dominance*** $\top : \mathbf{1} \rightarrowtail \Sigma$ [Ros86]. This means a map of which every $\mathcal{M}$-map is the inverse image along a unique map, like Ω for all monos in a topos and the Sierpiński space Σ for open inclusions of topological spaces. Then Proposition 3.10 specialises to $\mathcal{M}$ with $\Theta \equiv \Sigma$. $\square$

Example 4.19 The category **Pos** of posets and monotone functions has two suitable classes of monos:

(a) inclusions of arbitrary subsets that carry the restriction of the order relation, which we call $\mathcal{R}$; and

(b) inclusions of lower subsets, again with the restricted order, which we call $\mathcal{L}$.

The class $\mathcal{R}$ includes (is) that of regular monos and this is not contained in $\mathcal{L}$. However, $\mathcal{L}$ has but $\mathcal{R}$ fails the other extra properties [Tay23].

Now we turn to the other use of "monos" in the theory.

Assumption 4.20 The maps $\hookrightarrow$ used for inclusions of subcoalgebras and for structure maps of extensional coalgebras must form a class of T-monos (Definition 4.15) that

(a) is contained in the class of used for predicates;

(b) admits directed unions (Definitions 2.1 and 4.3); and

(c) is well powered (Definition 4.7).

Again, for additional results we may also assume that

(d) this class is part of a factorisation system (Section 8); or

(e) these monos admit pushouts that are unions (Section 10).

The classes $\mathcal{R}$ and $\mathcal{L}$ in Example 4.19 enjoy all of these properties, except that pushouts for $\mathcal{R}$ are not as well behaved as inclusions of sets [Tay23].

Remark 4.21 The monos in this class will often also be coalgebra homomorphisms and so will be written $\hookrightarrow\!\!\!\triangleright$. We will just call them ***initial segments***, to exploit the intuition from ordinals. However, the two ends of the arrow signify different things:

(a) the triangle arrow*head* ($\longrightarrow\!\!\!\triangleright$) says that the map is a coalgebra homomorphism, which captures the *traditional* order-theoretic ideas (*cf.* Lemma 3.3); whilst

(b) the hook *tail* ($\hookrightarrow$) says that the underlying $\mathcal{C}$-map belongs to a special class of monos: this aspect is a *novelty* in this paper.

5 Generating well founded coalgebras

In this section we study how the category of well founded coalgebras is built up, which is roughly analogous to the von Neumann hierarchy V_α in Set Theory. However, we stress that we assume no more about the underlying category $\mathcal{C}$ than that it is an elementary topos and in fact the results are more widely applicable, as explained in the previous section.

The first result is the categorical proof of Lemma 2.10, because coalgebra homomorphisms generalise bisimulations (Lemma 3.3) and pullback captures substitution.

Lemma 5.1 The induction premise (broken pullback) is stable under pullback against coalgebra homomorphisms.

$$
\begin{tikzcd}
TV \arrow[rrrr, tail, "Tj"] \arrow[dr] & & & & TB \arrow[dr, "Tf"] & \\
K \arrow[u] \arrow[drrrr, tail] \arrow[drr, dotted] \arrow[ddr, dashed] \arrow[ur, phantom, very near start, "\lrcorner"] & TU \arrow[rrrr, tail, "Ti"] & & & & TA \\
& & V \arrow[rr, tail, "j", "\text{pbk}"'] \arrow[dr] & & B \arrow[uu, "\beta"] \arrow[dr, "f"] & \\
& H \arrow[uu] \arrow[rr, tail] \arrow[ur, phantom, very near start, "\lrcorner"] & & U \arrow[rr, tail, "i"] & & A \arrow[uu, "\alpha"']
\end{tikzcd}
$$

Beware that we are saying nothing about i being an isomorphism.

Proof The thick lines show the homomorphism $f : B \longrightarrow A$ and the given induction premise $H \rightarrowtail U$ for the predicate $i : U \rightarrowtail A$.

Let $j : V \rightarrowtail B$ be the inverse image of i along f. Apply T to this pullback, to give the *parallelogram* at the top, although we are not assuming that this is a pullback.

Form the inverse image $K \rightarrowtail B$ of Tj along β, so that K is the induction hypothesis for $V \rightarrowtail B$.

The top, back and right quadrilaterals commute (from K to TA), so there is a pullback mediator $K \to H$ that makes the left and bottom quadrilaterals commute, *i.e.* from K to TU and to A. The map $K \to H$ deduces the induction hypothesis for U from that for V.

Because of this, there is a pullback mediator $K \to V$ that makes everything commute, in particular from K to B. Then $K \to V$ is the required induction premise. $\square$

The proof of the recursion theorem forms the union of attempts, so we consider colimits next, but see Definition 4.3 for the relationship between colimits and unions in general categories.

Note, however, that we are not asking for *new* colimits: we are merely *enhancing* the properties of those that *already* exist in the category $\mathcal{C}$, by showing that the categories of coalgebras and of well founded coalgebras inherit them.

Although we state the Proposition for general colimits, we only use directed unions (Definition 2.1) in our main proof of the recursion theorem. We will consider pushouts in Section 10.

Lemma 5.2 The initial object $\emptyset$ of $\mathcal{C}$ carries a unique T-coalgebra structure, which is well founded and is the least subcoalgebra of any coalgebra.

Proof Easy, but *cf.* Theorem 1.5(a), Remark 4.2 and Definition 4.15(b,f). $\square$

Proposition 5.3 The forgetful functors **WfCoAlg** $\longrightarrow$ **CoAlg** $\longrightarrow \mathcal{C}$ create colimits. That is, the colimit of any diagram of coalgebras and homomorphisms is given by the colimit of their carriers, if this exists, and then the structure map is uniquely determined. If the individual coalgebras are well founded then so is their colimit (*cf.* Theorem 1.5(b)).

Proof The structure map α on the colimit is the colimit mediator, as shown in the diagram on the left, where the colimiting cocone consists of coalgebra homomorphisms, *i.e.* the parallelograms from A_i to TA commute.

$$\begin{tikzcd}
&&& TA \\
TA_i \arrow[urrr, -Triangle] \arrow[rr, -Triangle] && TA_j \arrow[ur, -Triangle] & \\
&&& A \arrow[uu, dotted, -Triangle, "\alpha"'] \\
A_i \arrow[uu, -Triangle, "\alpha_i"] \arrow[rr, -Triangle] \arrow[urrr, -Triangle] && A_j \arrow[uu, -Triangle, "\alpha_j"] \arrow[ur, -Triangle] &
\end{tikzcd}
\qquad
\begin{tikzcd}
& TU \arrow[rrrr, tail, "Ti"] &&&& TA \\
TU_i \arrow[ur, dotted] \arrow[rrrr, tail, dotted, "Tk_i"] &&&& TA_i \arrow[ur, -Triangle] & \\
& H \arrow[uu] \arrow[rr, tail] \arrow[dr, phantom, "\lrcorner"] && U \arrow[rr, tail, "k"] && A \arrow[uu, -Triangle, "\alpha"'] \\
K_i \arrow[uu, dotted] \arrow[ur, dotted] \arrow[rr, tail, dotted] && U_i \arrow[ur, dotted] \arrow[rr, tail, dotted, "k_i"', "\text{pbk}"] && A_i \arrow[uu, -Triangle, "\alpha_i"] \arrow[ur, -Triangle] &
\end{tikzcd}$$

Now suppose that the α_i are well founded and let $k : U \rightarrowtail A$ be a predicate satisfying the induction premise for the colimit α (the back rectangle, from H to TA).

Form the inverse images K_i of this induction premise against the homomorphisms $A_i \longrightarrow\!\!\triangleright\, A$ of the colimiting cocone, using Lemma 5.1.

Since each A_i is well founded, $k_i : U_i \cong A_i$.

Now U is the vertex of a cocone over the diagram A_i, so it has a mediator from the colimit A, and $i : U \cong A$ as required [Tay96b, Prop 6.6]. □

This establishes the order-theoretic setting for the fixed point theorems from Section 2.

Corollary 5.4 The category of subcoalgebras of any coalgebra (A, α) and inclusions between them is equivalent to an $\mathcal{S}$-internal ipo $\mathsf{Seg}\,(A, \alpha)$. The well founded subcoalgebras form a subipo

$$\mathsf{WfSeg}\,(A, \alpha) \;\subset\; \mathsf{Seg}\,(A, \alpha)$$

of this, *i.e.* a subset ($\mathcal{S}$-subobject) that contains the least element and is closed under directed joins.

Proof Assumption 4.20, Lemma 5.2 and Proposition 5.3 provide the colimits in $\mathcal{C}$, **CoAlg** and **WfCoAlg**. However, we need Definition 4.3 to make these colimits agree with unions of subcoalgebras and then the well powered condition (Proposition 4.10) to link the external unions with the internal joins.

Finally, the well powered condition is used again to justify quantification over the class of predicates in the definition of well-foundedness (Remark 4.12); note here that, for the main recursion theorem, we will only use initial segments for these predicates. □

The next four results are about the "successor" operation that we introduced in Proposition 3.7

Lemma 5.5 The functor T preserves well founded coalgebras.

Proof A special case of Lemma 5.9 with $c \equiv \mathsf{id}$. □

Construction 5.6 Let $i : (B, \beta) \hookrightarrow\!\!\triangleright\, (A, \alpha)$ be a subcoalgebra. Then its ***relative successor***

$k : (C, \gamma) \hookrightarrow (A, \alpha)$ is given by pullback of α and Ti.

$$
\begin{tikzcd}
& A \arrow[r, "\alpha"] & TA & \\
& C \arrow[u, hook, "k"'] \arrow[r, "c"] \arrow[rrd, "\gamma"] \arrow[ur, phantom, very near start, "\lrcorner"] & TB \arrow[u, hook, "Ti"] \arrow[rd, hook, "Tj"] & \\
B \arrow[ruu, hook, "i"] \arrow[ru, "j"] \arrow[rru, "\beta"'] & & & TC \arrow[luu, "Tk"']
\end{tikzcd}
$$

The pullback mediator $j : B \to C$ makes $(B, \beta) \hookrightarrow (C, \gamma) \hookrightarrow (A, \alpha)$ as subcoalgebras (initial segments) when we define $\gamma \equiv c\,;Tj$.

We will write sB for the relative successor (C); the operation s is inflationary because $j : B \hookrightarrow sB$. It has been called the ***next time operator*** elsewhere.

Proof Since the functor T and inverse images preserve initial segments (Definition 4.15) and the latter obey the cancellation property (Lemma 4.16), if i is an initial segment then so successively are Ti, k, j, Tk and Tj. Finally,

$$k\,;\alpha = c\,;Ti = c\,;Tj\,;Tk = \gamma\,;Tk \qquad \text{and} \qquad j\,;\gamma = j\,;c\,;Tj = \beta\,;Tj,$$

so j and k are coalgebra homomorphisms. □

Lemma 5.7 The relative successor construction s is monotone (functorial) in B.

$$
\begin{tikzcd}
& A \arrow[r, "\alpha"] & TA & \\
B \arrow[ru, hook, "i"] \arrow[r, hook, "j"] & sB \arrow[u, hook, "k"'] \arrow[r, "c"] \arrow[ur, phantom, very near start, "\lrcorner"] & TB \arrow[u, hook, "Ti"] \arrow[r, hook, "Tj"] & TsB \arrow[lu, hook', "Tk"'] \\
B' \arrow[u, hook, "\ell"] \arrow[r, hook, "j'"] & sB' \arrow[u, hook] \arrow[r, "c'"] \arrow[ur, phantom, very near start, "\lrcorner"] & TB' \arrow[u, hook] \arrow[r, hook, "Tj'"] & TsB' \arrow[u, hook, "T\ell"']
\end{tikzcd}
$$

Proof Given initial segments $B' \hookrightarrow B \hookrightarrow A$, apply T and then pullback; the one giving sB' factors uniquely through the one for sB. □

The next result provides the special condition for Pataraia's Theorem 2.3 and is actually the sole place in our proof of the recursion theorem where we use well-foundedness. (Indeed, we only use the definition and none of the theory above.) The induction predicate is the initial segment $i : B \hookrightarrow A$, *cf.* Theorem 1.5(g).

Lemma 5.8 In the previous diagram, if (B, β) is well founded and both it and (B', β') are fixed by the relative successor ($j : B \cong sB$ and $j' : B' \cong sB'$) then $\ell : B' \cong B$.

Proof B, TB, TB' and B' form a pullback. It is the one in Definition 3.5 of well-foundedness, except that $K = U = B'$. Therefore $B' \cong B$. □

In the case of the covariant powerset, any subcoalgebra of a well founded coalgebra is again well founded, by Proposition 1.4. Using this, we could deduce well-foundedness of $sB \equiv C$ from that of TB and hence from that of B by Lemma 5.5. However, since we have chosen to use weaker conditions in our account, we need a slightly more complicated result at this point, which we call the ***sandwich lemma***.

Lemma 5.9 Let (B, β) be a well founded coalgebra and $j : B \to C$ and $c : C \to TB$ maps such that $\beta = j\,;c$. Put $\gamma \equiv c\,;Tj$. Then (C, γ) is also a well founded coalgebra and j and c are homomorphisms.

Proof They are homomorphisms because

$$j\,;\gamma \;\equiv\; j\,;c\,;Tj \;=\; \beta\,;Tj \qquad \text{and} \qquad \gamma\,;Tc \;\equiv\; c\,;Tj\,;Tc \;=\; c\,;T\beta.$$

Now let $k : W \rightarrowtail C$ satisfy the induction premise given by the pullback H and form the inverse image of this along j, using Lemma 5.1. This gives the induction premise K for the predicate $\ell : V \rightarrowtail B$:

$$\begin{tikzcd}
 & TW \arrow[rrrr, tail, "Tk"] & & & & TC \\
TV \arrow[ur, dotted] \arrow[rrrr, dotted, tail, "T\ell"] & & & & TB \arrow[ulll, dotted, "Th"'] \arrow[ur, "Tj"] & \\
 & H \arrow[uu] \arrow[rr, tail] & & W \arrow[rr, tail, "k"] & & C \arrow[ul, "c"'] \arrow[uu, "\gamma"'] \\
K \arrow[uu, dotted] \arrow[ur, dotted] \arrow[rr, dotted, tail] & & V \arrow[ur, dotted] \arrow[rr, dotted, tail, "\ell"] & & B \arrow[uu, "\beta"] \arrow[ul, dotted, "h"] \arrow[ur, "j"'] &
\end{tikzcd}$$

Since B is well founded, $\ell : V \cong B$ and so there is a map $h : B \to W$ making the triangle with C commute. The one with TB, TW and TC also commutes.

The triangle on the right commutes too ($\gamma = c\,;Tj$), so the maps $C \to TB \to TW$ and $\mathsf{id} : C \to C$ form a commutative square at TC. This factors through the pullback H, splitting the inclusion $H \rightarrowtail W \rightarrowtail C$ as required [Tay96b, Lemma 8.2]. $\square$

Recall that we gave slightly different arguments for (the first part of) Theorem 2.4 and for Lemma 2.5, as the revised forms of von Neumann's Recursion Theorem 1.5(g). The simpler one provides exactly what we will require in the next section and only uses the *definition* of well-foundedness with respect to initial segments, not any of the theory that we have developed or Pataraia's Theorem.

Lemma 5.10 For any well founded coalgebra (A, α), the relative successor defines an endofunction of the ipo $\mathsf{Seg}\,(A, \alpha)$ whose unique fixed point is the top element, A itself.

Proof The well powered requirement that we used to define the ipo in Corollary 5.4 also says that categorical constructions correspond to endofunctions of it (Corollary 4.9). By Lemmas 5.6 and 5.7, the relative successor therefore defines a monotone inflationary function $s : \mathsf{Seg}\,(A, \alpha) \to \mathsf{Seg}\,(A, \alpha)$.

By construction, the ipo has a top element (A) and this is a fixed point of successor. Since A is well founded, Lemma 5.8 says directly that it is the only fixed point. Note that this statement makes a quantification over subcoalgebras, which also requires the well powered condition (Remark 4.12). $\square$

The second version applies to *general* coalgebras. This does exploit Pataraia's Theorem and the theory of well-foundedness that we have developed,

Proposition 5.11 Any coalgebra $A \xrightarrow{\alpha} TA$ has a greatest well founded subcoalgebra,

$$\begin{tikzcd} WA \arrow[r, "\omega"] \arrow[d, hook] & T(WA) \arrow[d, hook] \\ A \arrow[r, "\alpha"] & TA \end{tikzcd}$$

and this is independent of the choice of classes of predicates and initial segments.

Proof Corollary 5.4 also defined the subipo $\mathsf{WfSeg}\,(A,\alpha) \subset \mathsf{Seg}\,(A,\alpha)$ of well founded subcoalgebras. By Lemma 5.9, the relative successor restricts to an endofunction of the smaller ipo, where it is inflationary and monotone.

Next we verify the the special condition in Pataraia's Theorem 2.3: By Lemma 5.8, if the subcoalgebra B is well founded with respect to initial segments and $B' \subset B$ are subcoalgebras that are each fixed by the successor then $B' = B$. Hence $\mathsf{WfSeg}\,(A,\alpha)$ has a top element, say (WA, ω).

As an element of the larger ipo $\mathsf{Seg}\,(A,\alpha)$, WA is characterised as the *least* fixed point of the successor. The statement of this property is independent of the notion of well-foundedness: If we re-define the latter for a larger class of predicates that satisfies Definition 4.15, even though there may be fewer well founded subcoalgebras, WA is still one of them and the proof that it is the largest one also remains valid. □

In Section 8 we improve this greatest subcoalgebra to an adjoint, on an additional assumption.

The results of this section also bear some resemblance to the way sets are built up in Zermelo's Set Theory [Zer08b], where the Sandwich Lemma corresponds to subsets of powersets, so we call the next result ***Zermelo induction***:

Theorem 5.12 Any property that holds of the initial coalgebra and is preserved by directed unions and sandwiching *à la* Lemma 5.9 holds of all well founded coalgebras.

Proof Although this appears to be about the *class* of well founded coalgebras, it is a *scheme* of results about $(A, \prec)$ as usual, because we just require $\Phi(B)$ be a predicate on $B \in \mathsf{Seg}\,(TA)$.

The sandwich property, that $\Phi(B)$ implies $\Phi(C)$ whenever C splits $\beta : B \longrightarrow C \longrightarrow TB$, means that the relative successor for initial segments of A preserves Φ. Moreover, A itself is the only fixed point. We deduce $\Phi(A)$ by Pataraia induction. □

6 The recursion theorem

Now we are ready to prove the recursion theorem for well founded coalgebras. Since we only assume that the functor T preserves monos and not their inverse images, the proof is more subtle than the one in [Tay99, §6.3]. It is based on our *revised* argument for well founded relations in Theorem 2.4, which uses *directed* unions of attempts, instead of forming the union of *all* of them. In particular, we do not assume that we have *binary* unions, although we will return to them in Section 10.

The proof has similar components to the constructions in the previous section, dealing with the empty attempt, successors, directed unions and the special condition for Pataraia's Theorem. The particular novelty of our proof is the more careful analysis of the successor.

Remark 6.1 An *attempt* from a coalgebra $\alpha : A \longrightarrow TA$ to an algebra $\theta : T\Theta \to \Theta$ is intended to be a partial map $f : A \rightharpoonup \Theta$ that is a subhomomorphism in the sense that

$$\begin{tikzcd}
TA \arrow[r, "Tf", harpoon] & T\Theta \arrow[d, "\theta"] \\
A \arrow[u, "\alpha"] \arrow[r, "f"', harpoon] \arrow[ur, phantom, "\sqsubseteq"] & \Theta
\end{tikzcd}$$

i.e. if the composite *via* TA is defined then so is that *via* Θ and then they are equal, *cf.* the definition in Theorem 1.5.

Composition of partial functions in a category uses inverse images. In order to define a category of coalgebras and *partial* homomorphisms, the functor T should therefore *preserve* inverse image diagrams, as the powerset and term algebra functors do.

However, the structure maps α and θ are total and we never need to compose partial maps. The notion of attempt therefore has a simple equivalent form that is sufficient to carry out the proof of the theorem:

Definition 6.2 An ***attempt*** from A to Θ is a diagram of the form

$$\begin{tikzcd}
TA & TB \arrow[l, "Ti"', hook'] \arrow[r, "Tf"] & T\Theta \arrow[d, "\theta"] \\
A \arrow[u, "\alpha"] & B \arrow[l, "i", hook'] \arrow[u, "\beta"'] \arrow[r, "f"] & \Theta
\end{tikzcd}$$

That is, a subcoalgebra inclusion (initial segment) $i : B \hookrightarrow A$ together with coalgebra-to-algebra homomorphism $f : B \longrightarrow \Theta$. A map f satisfies the recursion scheme (Definition 3.8) exactly when it is a ***total attempt***, with $i : B \cong A$. We call the attempt ***well founded*** if the ***support*** (B, β) is.

We also need a well powered assumption for attempts, which is easily adapted from that for initial segments (Definition 4.7ff). Alternatively we may consider them as subobjects of $A \times \Theta$ instead of those of A. Then, for any given coalgebra and algebra, there is a set or $\mathcal{S}$-object $\mathsf{Att}\,(A, \alpha, \Theta, \theta)$ of attempts from A to Θ, *cf.* $\mathsf{Seg}\,(A, \alpha)$ in Corollary 5.4.

Lemma 6.3 There is a "support" function (morphism of $\mathcal{S}$)

$$\mathsf{supp}\ : \mathsf{Att}\,(A, \alpha, \Theta, \theta) \longrightarrow \mathsf{Seg}\,(A, \alpha) \quad \text{by} \quad (A \xleftarrow{i} B \xrightarrow{f} \Theta) \longmapsto (B \xhookrightarrow{i} A).$$

Proof Corollary 4.9. □

One way to show that attempts are *unique* is by an easy application of well-foundedness:

Lemma 6.4 Let A be a well founded coalgebra, Θ an algebra and $f, g : A \rightrightarrows \Theta$ be total attempts. Then $f = g$ (*cf.* Theorem 1.5(d)).

Proof The two parallel squares on the right commute since f and g are total attempts. Let $i : E \rightarrowtail A \rightrightarrows \Theta$ be the equaliser in $\mathcal{C}$.

$$\begin{tikzcd}
TE \arrow[rr, "Ti", tail] & & TA \arrow[r, shift left, "Tg"] \arrow[r, shift right, "Tf"'] & T\Theta \arrow[d, "\theta"] \\
& E \arrow[dr, "i", tail] & & \\
H \arrow[uu] \arrow[ur, dotted] \arrow[rr, tail] \arrow[uurr, phantom, very near start, "\lrcorner"] & & A \arrow[uu, "\alpha"] \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & \Theta
\end{tikzcd}$$

Form the pullback H of $A \to TA \leftarrow TE$; the composites $H \rightrightarrows T\Theta$ are equal by construction, so those $H \rightarrowtail A \rightrightarrows \Theta$ are also equal. Then $H \rightarrowtail A$ factors through the equaliser, so $H \rightarrowtail E \rightarrowtail A$. Hence $i : E \cong A$ by well-foundedness of A and so $f = g$. [Mik76, page 99] [Osi74, Prop 6.5] [Osi75, Prop 6.3] [Tay96a, 2.5] [Tay96b, Prop 6.5] [Tay99, Prop 6.3.9]. $\square$

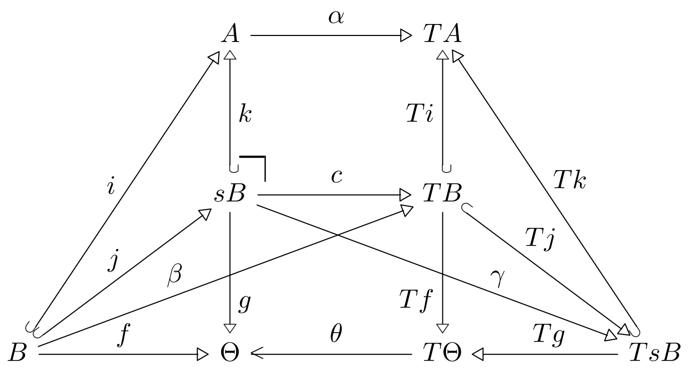
You will object that we did not ask for equalisers in Section 4, either in the category itself or in the class of predicates over which we may perform induction. This lemma is valid in **Set**, and also in **Pos** if we use $\mathcal{R}$ for predicates, but not using $\mathcal{L}$ (Example 4.19).

However, it transpires that there is a more subtle proof by induction on the structure of subcoalgebras that does not need equalisers after all:

Lemma 6.5 There is a bijection between attempts

$$A \xhookleftarrow{i} B \xrightarrow{f} \Theta \qquad \text{and} \qquad A \xhookleftarrow{j} sB \xrightarrow{g} \Theta,$$

where sB is the relative successor of B (Lemma 5.6). Hence the successor lifts not only the existence but also the uniqueness of an attempt.



Proof Let $(A, \alpha) \xhookleftarrow{i} (B, \beta) \xrightarrow{f} (\Theta, \theta)$ be an attempt, so

$$i\,;\alpha \;=\; \beta\,; T\alpha \qquad \text{and} \qquad f \;=\; \beta\,; Tf\;\theta$$

Then the relative successor attempt is defined by

$$\gamma \;\equiv\; c\,; Tj \qquad \text{and} \qquad g \;\equiv\; c\,; Tf\,; \theta$$

and satisfies

$$\begin{aligned} f &\;=\; \beta : Tf\,; \theta \;=\; j\,; c\,; Tf\,; \theta \;=\; j\,; g \\ g &\;\equiv\; c\,; Tf\,; \theta \;=\; c\,; Tj\,; Tc\,; TTf\,; T\theta\,; \theta \;=\; c\,; Tj\,; Tg\,; \theta \;\equiv\; \gamma\,; Tg\,; \theta. \end{aligned}$$

So $(A, \alpha) \xhookleftarrow{j} (sB, \gamma) \xrightarrow{g} (\Theta, \theta)$ is also an attempt, extending f.

Conversely, $f \equiv i\,; g$ satisfies

$$\begin{aligned} f &\;\equiv\; j\,; g \;=\; j'\gamma\,; Tg\,; \theta \;=\; j\,; c\,; Tj\,; Tg\,; \theta \;=\; \beta\,; Tf\,; \theta \\ g' &\;\equiv\; c\,; Tf\,; \theta \;=\; c\,; Tj\,; Tg\,; \theta \;=\; \gamma\,; Tg\,; \theta \;=\; g, \end{aligned}$$

establishing the bijection. $\square$

The parametric version is similar and is the only place where we use binary products:

Lemma 6.6 There is a bijection between parametric attempts given by

$$g \;\equiv\; \langle k, (c\,; Tf)\rangle\,; \theta \qquad \text{and} \qquad f \;=\; j\,; g. \qquad \square$$

$$
\begin{tikzcd}
B \arrow[r, "\langle i,\beta\rangle"] \arrow[d, hook, "j"'] \arrow[dd, bend right=60, "f"'] & A\times TB \arrow[d, "A\times Tj"] \arrow[dd, bend left=60, "A\times Tf"] \\
sB \arrow[r, "\langle k,\gamma\rangle"'] \arrow[ur, "\langle k,c\rangle"] \arrow[d, dotted, "g"'] & A\times TsB \arrow[d, dotted, "A\times Tg"] \\
\Theta & A\times T\Theta \arrow[l, "\theta"']
\end{tikzcd}
$$

Lemma 6.7 The initial object is the support of a unique attempt. For any directed diagram of subcoalgebras, if each member is the support of a unique attempt then so is the union of the diagram.

Proof The statements are the universal properties of the initial object and filtered colimits, but we need to say that they are unions (Definition 4.3). Also *cf.* Remark 4.2, Lemma 5.2 and Proposition 5.3. $\square$

We can now achieve our principal goal, the ***Recursion Theorem***, based on Theorem 2.4 for well founded relations.

Theorem 6.8 From any well founded coalgebra (A, α) to any algebra there is a unique total attempt.

Proof By Lemma 5.10, $\mathsf{Seg}\,(A,\alpha)$ is an ipo on which the relative successor defines an endofunction, whose unique fixed point is the top element, A itself. The uses of the well powered condition that we make here were explained there.

Lemma 6.5 defined an endomorphism of $\mathsf{Att}\,(A,\alpha,\Theta,\theta)$ called relative successor and $\mathsf{supp} : \mathsf{Att} \to \mathsf{Seg}$ commutes with the two successors (Corollary 4.9). Hence this situation is wholly about objects and morphisms of the topos $\mathcal{S}$.

Consider the subset $U \subset \mathsf{Seg}\,(A,\alpha)$ consisting of those initial segments $i : B \hookrightarrow A$ such that there is a unique attempt with support B. That is,

$$U \quad \equiv \quad \{B \in \mathsf{Seg}\,(A,\alpha) \mid \exists! a \in \mathsf{Att}\,(A,\alpha,\Theta,\theta).\ \mathsf{supp}\,(a) = i\}.$$

Then $\emptyset \in U$ and it is closed under directed unions by Lemma 6.7, whilst $s : U \to U$ by Lemma 6.5.

Therefore, by Pataraia induction (Theorem 2.3(c)) U contains the least fixed point of the successor, which is A itself. This means that there is a unique attempt with support A, *i.e.* a total one or solution to the recursion equation.

The *statement* of the Theorem is independent of the notion of initial segment that we choose. Also, if we enlarge the class of predicates then there are just fewer well founded coalgebras and the result remains the same [Mik76, pp 101–4] [Osi75, Prop 6.5] [Tay99, Thm 6.3.13] $\square$

We have developed the theory of well founded *co*algebras to approximate the initial *algebra* when the functor T does not have one, such as in the case of the powerset. When the initial algebra does exist, we therefore need to link the two accounts together.

Two of the steps in the circular equivalence below are based on observations by Joachim Lambek [Lam68] and by Daniel Lehmann and Michael Smyth [LS81, §5.2]. Lambek discusses systems of coherently commuting functors and gives a criterion for the existence of a fixed point.

Proposition 6.9 The structure maps of the initial algebra, final coalgebra and final well founded coalgebra, if they exist, are isomorphisms.

$$
\begin{tikzcd}
T\Theta \arrow[d, "\theta"'] \arrow[r, dotted, shift right, "T\alpha"'] & TT\Theta \arrow[l, shift right, "T\theta"'] \arrow[d, "T\theta"] \\
\Theta \arrow[r, dotted, shift right, "\alpha"'] & T\Theta \arrow[l, shift right, "\theta"']
\end{tikzcd}
\qquad
\begin{tikzcd}
TA \arrow[r, shift left, "T\alpha"] & TTA \arrow[l, dotted, shift left, "T\theta"] \\
A \arrow[u, "\alpha"] \arrow[r, shift left, "\alpha"] & TA \arrow[u, "T\alpha"'] \arrow[l, dotted, shift left, "\theta"]
\end{tikzcd}
$$

These objects are therefore both algebras and coalgebras and we call them ***fixed points*** of the functor. Coalgebra-to-algebra homomorphisms from or to them are respectively the same as plain algebra or coalgebra homomorphisms.

The successor relative (Lemma 5.6) to the initial algebra is just the functor T.

Proof This is illustrated by the diagrams. It also applies to the final *well founded* coalgebra because the functor T preserves well-foundedness by Lemma 5.5. In Lemma 5.6, since $A \cong TA$ also $C \cong TB$. $\square$

Proposition 6.10 The initial algebra A is well founded *quâ* coalgebra.

$$
\begin{tikzcd}
TU \arrow[rr, tail, shift left, "Ti"] & & TA \arrow[ll, dotted, shift left, "Tp"] \\
H \arrow[u, "\cong"] \arrow[r, tail, "j"] \arrow[ur, phantom, very near start, "\lrcorner"] & U \arrow[r, tail, shift left, "i"] & A \arrow[u, "\cong"', "\alpha"] \arrow[l, dotted, shift left, "p"]
\end{tikzcd}
$$

Proof Since the structure map is invertible, so is its pullback, so $TU \cong H \stackrel{j}{\rightarrowtail} U$ makes U an algebra and $i : U \rightarrowtail A$ an algebra monomorphism. But this is split since A is initial *quâ* algebra. Hence A is well founded *quâ* coalgebra. $\square$

Corollary 6.11 If any of the following exists then it satisfies the other properties too:

(a) a final well founded coalgebra;

(b) a well founded coalgebra whose structure map is an isomorphism;

(c) an initial fixed point;

(d) an initial algebra.

Moreover, it is unique up to unique isomorphism.

Proof The Recursion Theorem 6.8 says that the final well founded coalgebra has the universal property of the initial algebra, so b⇒c. Proposition 6.10 is almost the converse, d⇒c, where Proposition 6.9 fills in the gaps, a⇒b and c⇔d. $\square$

Corollary 6.12 If T has a final coalgebra F then its greatest well founded subcoalgebra $A \equiv WF$ is the initial algebra.

Proof The structure map of F is an isomorphism, so by cancellation of monos, that of A is mono too. But TA is another well founded subcoalgebra of F, so $A \cong TA$, whence A is the initial algebra. $\square$

This seems to have been known in some form for a long time, in a sense since [Mir17a]. That there is a homomorphism $A \to F$ follows from Lambek's lemma and his paper says more about the category of fixed points, in which A and F are initial and final. However, that $A \to T$ is mono must depend on the assumptions in Section 4. Peter Freyd considered the situation where $A \cong F$, which he called ***algebraic compactness*** [Fre91].

7 Extensionality

Now that we have some understanding of the Axiom of Foundation generalised to coalgebras, we turn to the Axiom of Extensionality.

Definition 7.1 A coalgebra $\alpha : A \longrightarrow TA$ ***extensional*** if α is an initial segment.

That is, it belongs to the same class of monos that we used for subcoalgebras in our proof of the recursion theorem (Assumption 4.20). If you skipped Section 4, you should simply understand α to be mono, *cf.* Definition 1.8 when $T \equiv \mathcal{P}$.

An ***ensemble*** is an extensional well founded coalgebra. If need be, the name could be qualified by stating the category, functor and two classes of monos that are used in the definition.

Remark 7.2 Zermelo's generalisation from well *ordered* to well *founded* systems [Zer35] introduced "noise" in the form of repetition. Extensionality removes this *so thoroughly* that there are no automorphisms aside from the identity. Extensional well founded coalgebras are fragments of the initial algebra (if there is one) and behave very much like Set Theory, even in a much more general setting. This justifies the name *ensemble*.

As we said in Remark 1.10, the "sets" that we are mimicking here are those that are called *transitive* in Set Theory. So Gerhard Osius used the name ***transitive set object*** for our extensional well founded coalgebras [Osi74, §§6,7]. He defined a general "set" to be an $\mathcal{S}$-subobject of some transitive set object and developed Set Theory following [Zer08b], in fact giving a logical subtopos of $\mathcal{S}$. See [Tay96a, §3] for another account of this. We shall not go any further in this paper with modelling the rest of Zermelo's axioms and have no generalisation of the Kuratowski–Wiener pair formula $\{x, \{x, y\}\}$.

Instead we pick up on the idea that a "set" is a *fragment* of the unattainable universe that would be the free algebra for the powerset functor. We will show that these features of Set Theory are shared by ensembles for more general functors.

We need first to adapt the Recursion Theorem from the previous section to *partial* coalgebra-to-*co*algebra homomorphisms. It becomes rather more complicated, but we handle this by relying more heavily on Pataraia's Theorem 2.3.

Definition 7.3 An ***attempt*** from one coalgebra (A, α) to another (D, δ) is a pair (i, f) of coalgebra homomorphisms, also known as a ***span***,

$$\begin{tikzcd} TA & TB \arrow[l, hook', "Ti"'] \arrow[r, "Tf"] & TD \\ A \arrow[u, "\alpha"] & B \arrow[l, hook', "i"'] \arrow[r, "f"] \arrow[u, "\beta"] & D \arrow[u, hook, "\delta"'] \end{tikzcd}$$

This is the same as Definition 6.2, apart from reversing the arrow δ. The relationship is that we think of (D, δ) as a *partial* algebra whose evaluation part is id_D.

We call the attempt ***well founded*** if the ***support*** B is.

By the time we get to the Theorem, all of the arrows will be initial segments (monos), but we need slightly more generality at first (*cf.* Lemma 7.11): We will assume that δ and i are initial segments, so D is extensional, but *â priori* α and j need not be monos.

Lemma 7.4 Consider the category whose objects are attempts from A to D and whose morphisms are coalgebra homomorphisms $B' \longrightarrow B$ that make commutative triangles. This category is equivalent to an $\mathcal{S}$-ipo and it has a subipo of well founded attempts,

$$\mathsf{WfAtt}\,(A, \alpha, D, \delta) \quad \subset \quad \mathsf{Att}\,(A, \alpha, D, \delta).$$

Proof For the same reasons as in Corollary 5.4, relying on the assumptions about unions and being well powered. □

The next diagram may be daunting, but it is just the adaptation of Construction 6.5 for successor attempts to the situation where the target is a partial algebra or extensional coalgebra. It is more complicated because we have to trim the support according to the partial target.

If you would like to try to compare this with the relational version in Proposition 2.9, recall from Lemma 3.3 that coalgebra homomorphisms define *bi*simulations. Since such relations are composable and reversible, the span of homomorphisms becomes a single relation.

Construction 7.5 The ***relative successor*** of an attempt from a coalgebra (A, α) to an extensional coalgebra (D, δ).

Proof Let $(A, \alpha) \xleftarrow{i} (B, \beta) \xrightarrow{f} (D, \delta)$ be an attempt, where i and δ are initial segments, as shown in the bold lines in the diagram (which is rotated relative to the one in Definition 7.3):

$$\begin{tikzcd}
 & & A \arrow[r, "\alpha"] & TA & TC_1 \arrow[l, hook', "Tj"'] \\
 & & C_1 \arrow[u, hook, "j"] \arrow[dr, "a"] & & \\
B \arrow[uurr, hook, "i"] \arrow[r, hook, dotted, "\ell"] \arrow[rrr, bend left, "\beta"] \arrow[ddrr, "f"'] & C \arrow[ur, hook, "k"] \arrow[dr, "h"] \arrow[r, phantom, "\mathrm{pbk}"] \arrow[rrr, bend right, "\gamma"'] & & TB \arrow[uu, hook', "Ti"'] \arrow[r, hook, dotted, "T\ell"] \arrow[dd, "Tf"] & TC \arrow[uu, hook', "Tk"'] \arrow[dd, "Th"] \\
 & & C_2 \arrow[ur, hook, "d"] \arrow[d, "g"] & & \\
 & & D \arrow[r, hook, "\delta"] & TD & TC_2 \arrow[l, "Tg"']
\end{tikzcd}$$

Let C_1, C_2 and C be the pullbacks shown. Then Ti, j, d, k and $k\,;j$ are initial segments because T, pullback and composition preserve them. Notice that we have used both the subcoalgebra inclusion $B \overset{i}{\hookrightarrow} A$ and the extensional structure map $D \overset{\delta}{\hookrightarrow} TD$ to do this, so it is not possible to separate these two uses of "monos", *cf.* Remark 4.13.

This construction makes C the limit of the W-diagram

$$\begin{tikzcd}
 & A \arrow[dr, "\alpha"] & \\
 & & TA \\
sB \equiv C \arrow[uur, hook] \arrow[r] \arrow[ddr] & TB \arrow[ur, hook, "Ti"] \arrow[dr, "Tf"] & \\
 & & TD \\
 & D \arrow[ur, hook, "\delta"'] &
\end{tikzcd}$$

Now B is the vertex of another cone over the W, with arrows i, β and f. Hence there is a unique mediator $\ell : B \to C$ to the limit, with

$$i \;=\; \ell\,;k\,;j, \qquad \beta \;=\; \ell\,;k\,;a \;=\; \ell\,;h\,;d \qquad \text{and} \qquad f \;=\; \ell\,;h\,;g.$$

Then ℓ is an initial segment by the Cancellation Lemma 4.16 since i and $k\,;j$ are.

Now we make C a coalgebra by defining $\gamma \equiv k\,;a\,;T\ell$. Then ℓ is a homomorphism because

$$\ell\,;\gamma \;\equiv\; \ell\,;k\,;a\,;T\ell \;=\; \beta\,;T\ell.$$

The new attempt with support C is given by the composites $k\,;j : C \to A$ and $h\,;g : C \to D$, whose composites with ℓ are i and f. Then $k\,;j$ and $h\,;g$ are homomorphisms because

$$(k\,;j)\,;\alpha \;=\; k\,;a\,;Ti \;=\; k\,;a\,;T\ell\,;T(k\,;j) \;=\; \gamma\,;T(k\,;j)$$

and $$(h\,;g)\,;\delta \;=\; h\,;d\,;Tf \;=\; h\,;d\,;T\ell\,;T(h\,;g) \;=\; \gamma\,;T(h\,;g).$$

The map $\ell : B \hookrightarrow C$ makes the successor inflationary. $\square$

Lemma 7.6 The relative successor s is monotone (functorial) in $B' \longrightarrow B$.

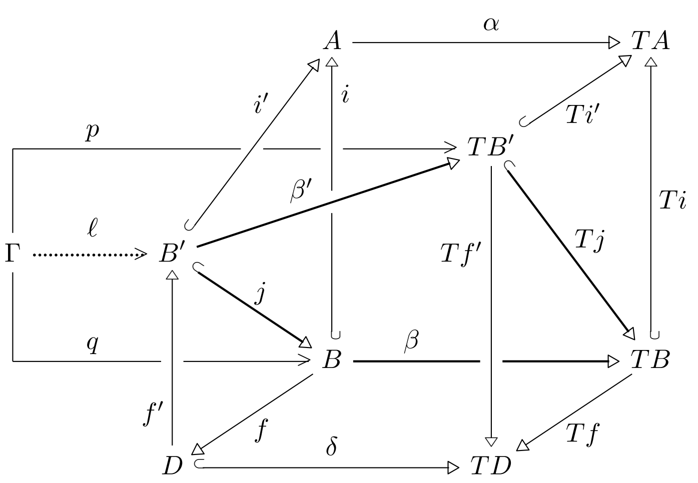
Proof The proof amounts to the mediator between two W-limits that share the nodes A, TA, D and TD but differ on $TB' \longrightarrow TB$, *cf.* the next diagram. $\square$

Lemma 7.7 If B is well founded then so is $C \equiv sB$.

Proof By Lemma 5.9, since C is sandwiched between B and TB. $\square$

The next result is the special condition for Pataraia's Theorem 2.3:

Lemma 7.8 If B is well founded and $B' \cong sB' \xhookrightarrow{j} B \cong sB$ then $j : B' \cong B$.



Proof That B' is a fixed point means that it is already the limit of the W that defines its successor, namely

$$A \xrightarrow{\alpha} TA \xhookleftarrow{Ti'} TB' \xrightarrow{Tf'} TD \xhookleftarrow{\delta} D.$$

We claim that the homomorphism quadrilateral for $B' \xhookrightarrow{j} B$ (shown in bold) is a pullback, so let Γ be the vertex of a cone, with $p\,;Tj = q\,;\beta$. Then

$$q\,;i : \Gamma \to A, \qquad p : \Gamma \to TB' \qquad \text{and} \qquad q\,;f : \Gamma \to D$$

define a cone over the W for B' because

$$q\,;i\,;\alpha \;=\; q\,;\beta\,;Ti \;=\; p\,;Tj\,;Ti \;=\; p\,;Ti'$$

and $$q\,;f\,;\delta \;=\; q\,;\beta\,;Tf \;=\; p\,;Tj\,;Tf \;=\; p\,;Tf'.$$

Since B' is the limit, there is a unique mediator $\ell : \Gamma \to B'$ with

$$\ell\,;i' \;=\; \ell\,;j\,;i \;=\; q\,;i, \qquad \ell\,;\beta' \;=\; p \qquad \text{and} \qquad \ell\,;f' \;=\; q\,;f$$

whence $\ell\,;j = q$ since i is mono. Thus ℓ provides the mediator that is required for B' to be the pullback. However, B is well founded by hypothesis, so any such pullback degenerates, making $j : B' \cong B$. $\square$

Corollary 7.9 There is a greatest well founded attempt from (A,α) to (D,δ).

Proof $\mathsf{WfAtt}\,(A,\alpha,D,\delta)$ has a top element by Pataraia's Theorem 2.3. $\square$

Lemma 7.10 If (A,α) and (D,δ) are both extensional then the greatest attempt with well founded support consists of a pair of initial segments.

Proof We use Pataraia induction (Theorem 2.3(c)) for the property that the evaluation part of the attempt is an initial segment (mono).

The least attempt, whose support is the initial object, satisfies this property.

The successor Construction 7.5 preserves it: if α and f are initial segments then so too are Tf, g, a, h and $h\,;g$.

Directed unions also preserve it (Definition 2.1).

Therefore the greatest attempt has it too. $\square$

Lemma 7.11 Any coalgebra homomorphism $f : A \longrightarrow D$ between ensembles is an initial segment. There is at most one such homomorphism. If there are homomorphisms in both directions then they are inverse.

Proof Without using the hypothesis that A is extensional, the homomorphism f provides an attempt $(\mathsf{id}\,, f)$, in the sense of Definition 7.3, which also says that (i,f) is well founded iff A is.

This attempt is fixed by the relative successor (Construction 7.5): i, Ti and j are isomorphisms, so the initial segments ℓ and k are too since initial segments are plain monos.

Hence the attempt $(\mathsf{id}\,, f)$ coincides with the greatest one (Lemma 7.8), which is a pair of initial segments by Lemma 7.10. Moreover this greatest attempt is unique.

In particular, the only *endo*morphism of an ensemble is the identity, so if there are homomorphisms both ways between ensembles then they must be inverse. $\square$

Corollary 8.11 strengthens this to say that *any* outgoing homomorphism from an extensional well founded coalgebra is an initial segment, *whatever* the codomain. (It is proved there on much stronger assumptions, but it should be possible to prove it under the present ones, *cf.* Lemma 2.12.) An ensemble is therefore a very ***rigid*** structure.

Corollary 7.12 Ens is a preorder with binary meets. Moreover, whenever a meet-span of ensembles is part of a commutative square of them then this is a pullback.

$$\begin{tikzcd}
C \arrow[dr, dotted] \arrow[drr, hook] \arrow[ddr, hook] & & \\
& B \arrow[r, hook] \arrow[d, hook] & D \arrow[d, hook] \\
& A \arrow[r, hook] & E
\end{tikzcd}$$

Proof Let $A \hookleftarrow B \hookrightarrow D$ be the greatest attempt from A to D, these being ensembles that are both contained in some ensemble E. Let $A \hookleftarrow C \hookrightarrow D$ be another pair of homomorphisms from an ensemble C. That these commute at E is automatic by the Lemma. The maps from C define an attempt from A to D, which lies below (factors through) the greatest one, B. The universal property of a pullback holds irrespective of its root. $\square$

Under stronger assumptions, this is actually a pullback in **WfCoAlg** by Theorem 8.9 and even in **CoAlg** and $\mathcal{C}$ by Proposition 9.6.

Remark 7.13 The meet therefore has the property that we would normally call a *product* in a category. We avoid that word because the construction looks like set-theoretic *intersection* and nothing like the *Cartesian* or Kuratowski–Wiener product. Set-theoretically, the maps $A \leftarrow C \to D$ do have to be subset inclusions and not arbitrary functions. Categorically, $A \leftarrow C \to D$ must be coalgebra homomorphisms and not just $\mathcal{C}$-maps.

This corollary was a bonus that we should not have expected unless T preserves inverse images (Proposition 9.6). However, we can't take it any further unless that is the case; in particular, we do not yet have binary *joins* of ensembles, but we will study them under additional assumptions in Section 10. $\square$

Theorem 7.14 The category **Ens** of ensembles and coalgebra homomorphisms
(a) is a preorder with
(b) a least (isomorphism class of) object(s),
(c) directed unions,
(d) binary meets and
(e) an inflationary monotone successor, namely the functor T.
Moreover,
(f) the greatest ensemble is the initial algebra (Corollary 6.11), if either of these exists, and is the unique fixed point of T $\square$

In the last part, the successor coalgebra relative to the initial algebra is just given by the functor T.

Proposition 7.15 Suppose that the category $\mathcal{C}$ has set-indexed filtered colimits. Then the functor T has an initial algebra iff there is a set rather than a proper class of isomorphism classes of ensembles.

Proof Since an ensemble is an initial segment of the initial algebra, the forward direction follows from the well powered Assumption 4.20. Conversely, another way of stating the "size" condition is that the preorder **Ens** is equivalent to an *internal* poset in $\mathcal{S}$. The initial object, endofunctor and filtered colimits in **Ens** become the least element, directed joins and endofunction of the poset. By Pataraia's Theorem 2.3, there is a fixed point, which is the top element, and this corresponds to the initial algebra. $\square$

Corollary 7.16 If there is an initial algebra, it satisfies any property of coalgebras that holds of the initial object and is preserved by isomorphism, the functor and filtered colimits.

Proof By Pataraia induction, Theorem 2.3(c). $\square$

8 Imposing the properties

In this section we show how to turn a general coalgebra into a well founded one and then make it extensional too. That is, we will find adjoints to the inclusions $\mathbf{Ens} \rightarrowtail \mathbf{WfCoAlg} \rightarrowtail \mathbf{CoAlg}$.

The key idea in doing this is (the categorical abstraction of) the fact that any function can be expressed as the composite of a surjection and the inclusion of its image. One of the earliest achievements of category theory, or rather of Modern or Universal Algebra, was to bring together the various "isomorphism theorems" relating these for groups, rings, vector spaces, *etc.* The abstract formulation was given by Peter Freyd and Max Kelly [FK72]:

Definition 8.1 Two maps $e : X \twoheadrightarrow Q$ and $m : V \hookrightarrow Y$ in any category are ***orthogonal***, written $e \perp m$, if, for any two maps f and g such that the square commutes, there is a unique morphism $p : Q \to V$ making the two triangles commute:

$$\begin{tikzcd} X \arrow[r, two heads, "e"] \arrow[d, "f"'] & Q \arrow[d, "g"] \arrow[dl, dotted, "p"'] \\ V \arrow[r, hook, "m"] & Y \end{tikzcd}$$

Then a ***factorisation system*** is a pair of classes of morphisms $(\mathcal{E}, \mathcal{M})$ such that
(a) the classes $\mathcal{E}$ and $\mathcal{M}$ each contain all isomorphisms;
(b) they are each closed under composition;
(c) $e \perp m$ for every $e \in \mathcal{E}$ and $m \in \mathcal{M}$ and
(d) every morphism $f : X \to Y$ can be expressed as $f = e\,;m$ with $e \in \mathcal{E}$ and $m \in \mathcal{M}$.

Lemma 8.2 In a factorisation system,
(a) if $f \in \mathcal{E} \cap \mathcal{M}$ then f is an isomorphism;
(b) if the pullback of an $\mathcal{M}$-map exists in the category then it is also in $\mathcal{M}$ (so Assumptions 4.1(a) and 4.15(e) become redundant);
(c) $\mathcal{E}$ has the cancellation property that if $f,\ (f\,;e) \in \mathcal{E}$ then $e \in \mathcal{E}$, *cf.* Lemma 4.16;
(d) if $f \in \mathcal{C}$ has $f \perp m$ for all $m \in \mathcal{M}$ then $f \in \mathcal{E}$ (in fact quantification over $\mathcal{M}$ is not needed: we only use this for the $\mathcal{M}$-part of the factorisation of f);
(e) if the maps in a directed or pushout diagram are all in $\mathcal{E}$ then so are those in the colimiting cocone; and
(f) the mediator from such a colimit to a cocone consisting of $\mathcal{E}$-maps is also in $\mathcal{E}$.

$$\begin{tikzcd} X_0 \arrow[r, two heads] \arrow[d] \arrow[rrr, bend left] & X_i \arrow[r] \arrow[dl, dotted] & \mathsf{colim} \arrow[r] \arrow[d] & \Theta \\ U \arrow[rr, tail] & & Y & \end{tikzcd}$$

Proof (a–d) Easy, but see *e.g.* Lemma 5.7.6 and Proposition 5.7.7 in [Tay99]. (e) Any pushout has a *root* X_0 (with maps to all of the other vertices of the diagram), and any directed diagram is equivalent to one with a root. Using $(X_0 \twoheadrightarrow X_i) \perp (U \rightarrowtail Y)$, there is a unique mediator $X_i \to U$. These maps form a cocone, with mediator $\mathsf{colim} \to U$. Finally, (f) follows from (d,e). $\square$

Examples 8.3
(a) Inclusions (1–1 maps, monomorphisms) and surjections (onto maps, epimorphisms) in **Set** or a topos, where surjections are quotients by equivalence relations and this class is stable under pullbacks.
(b) More generally in type theories, if the factorisation is stable under pullback then the $\mathcal{E}$ class is associated with an existential quantifier [HP89] [Tay99, §9.3].

(c) In a general category with inverse images, a map e that is orthogonal to all monos is called an ***extremal epi*** and is characterised by $\forall m \in \mathcal{M}.\, e = m\,;f \implies m = \mathsf{id}$.

(d) The classes of "monos" $\mathcal{R}$ and $\mathcal{L}$ in Example 4.19 belong to factorisation systems in **Pos**, whose "epis" are functions that are respectively surjective on points and cofinal. The first of these classes is well co-powered but the second is not.

Assumption 8.4 In this section we require that the category $\mathcal{C}$

(a) have a factorisation system $(\mathcal{E}, \mathcal{M})$ in which $\mathcal{M}$ is the class of initial segment maps that we have been using,

(b) be well *co*powered with respect to $\mathcal{E}$-maps, and

(c) have filtered colimits of $\mathcal{E}$-homomorphisms.

Theorem 8.7 only requires the first of these, but we use the others for Theorem 8.9.

The notion of being well *co*powered is the obvious analogue of being well powered (Definition 4.7): that the (pre-ordered) external category of outgoing $\mathcal{E}$-maps is equivalent to an internal poset in the base topos $\mathcal{S}$.

Ivan Di Liberti and Jiři Rosický have given the name *convenient* factorisation system to the combination of our union property for $\mathcal{M}$ (Definition 4.3) and $\mathcal{E}$ being well co-powered [DLR09].

Remark 8.5 We will call $\mathcal{E}$-maps ***cofinal***. As in Remark 4.21 for initial segments, this term is intended to hint at certain intuitions, which are linked to the fact that all of the maps that we call cofinal are coalgebra homomorphisms. However, the notion is not necessarily the same as the traditional order-theoretic one. It is a novelty of this work that $\mathcal{E}$ can be a special class of morphisms in a category that need not be a topos.

The following is the categorical version of Corollary 2.11:

Lemma 8.6 Let E be a well founded coalgebra and $e : E \twoheadrightarrow C$ be a cofinal homomorphism. Then C is also well founded.

$$\begin{tikzcd}
TW \arrow[rrr, hook, dotted, "Tj"] \arrow[dr, dotted] & & & TE \arrow[dr, two heads, "Te"] & \\
& TV \arrow[rrr, hook, "Ti"] & & & TC \\
K \arrow[u, dotted] \arrow[rr, hook, dotted] \arrow[dr, dotted] & & W \arrow[r, hook, dotted, "j"] \arrow[d, dotted, "\text{pbk}"'] & E \arrow[uu, "\epsilon"] \arrow[dr, two heads, "e"] & \\
& H \arrow[uu] \arrow[r, hook] & V \arrow[rr, hook, "i"] & & C \arrow[uu, "\gamma"']
\end{tikzcd}$$

Proof Let $i : V \hookrightarrow C$ be an initial segment that satisfies the induction premise given by the broken pullback from H to TC (at the front).

Pull this back along the homomorphism $e : E \twoheadrightarrow C$, using Lemma 5.1.

By well-foundedness of E, we have $j : W \cong E$.

Since $e : E \twoheadrightarrow C$ is cofinal and it factors through the initial segment $i : V \hookrightarrow C$, the latter is also an isomorphism [Tay96b, Prop 7.8]. $\square$

Theorem 8.7 The inclusion $\mathbf{WfCoAlg} \rightarrowtail \mathbf{CoAlg}$ has a right adjoint (coreflection), whose counit is an initial segment. This is independent of the choices of classes of predicates, initial segments and cofinal maps.

$$
\begin{tikzcd}
E \arrow[r, "f"] \arrow[d, two heads, "e"'] & A \\
C \arrow[r, hook, "k"] \arrow[ur, hook, "j"] & WA \arrow[u, hook, "i"']
\end{tikzcd}
$$

Proof We claim that the largest well founded subcoalgebra $i : WA \hookrightarrow A$ (Proposition 5.11) provides the adjoint. That is, any coalgebra homomorphism $f : E \longrightarrow A$ with E well founded factors uniquely through i.

Let $E \overset{e}{\twoheadrightarrow} C \overset{j}{\hookrightarrow} A$ be the factorisation in $\mathcal{C}$ of f as a cofinal map followed by an initial segment. Since Tj is also an initial segment, the orthogonality mediator provides a coalgebra structure on C such that e and j are homomorphisms. Then C is well founded by Lemma 8.6 and it is a subcoalgebra of A by construction.

It is therefore a subcoalgebra of WA, since WA was the largest such. The map $E \longrightarrow WA$ is unique since $i : WA \hookrightarrow A$ is mono.

Proposition 5.11 said that the largest well founded subcoalgebra WA is independent of the classes. This is also true of the factorisation, because a stronger notion of well-foundedness just replaces **WfCoAlg** with a full subcategory. □

Now we turn to imposing extensionality, which is our categorical version of Theorem 2.13 and of Mostowski's theorem, but formulating this using a factorisation system gives a much more general result.

In the more familiar setting where (Proposition 9.6 holds and) $\mathcal{E}$ consists of regular epis in a topos (or effective regular category), such maps correspond to their kernels. In the earlier version, kernels were expressed as equivalence bisimulation relations, but in a categorical style they are spans of homomorphisms similar to Definition 7.3.

Construction 8.8 The ***successor quotient*** (C, γ) of any coalgebra (B, β) is given by factorising β as a cofinal homomorphism followed by an initial segment, as shown below. Then $e : B \cong C$ iff (B, β) is extensional. If B is well founded then so is C. Any homomorphism $f : (B, \beta) \longrightarrow (E, \epsilon)$ to an extensional coalgebra factors uniquely through C.

$$
\begin{tikzcd}
TTB & & \\
& TC \arrow[ul, hook, "Ti"'] \arrow[dr, dotted, "Tg"] & \\
TB \arrow[uu, "T\beta"] \arrow[ur, "Te"] \arrow[rr, "Tf"'] & & TE \\
& C \arrow[uu, "\gamma"] \arrow[ul, hook, "i"'] \arrow[dr, dotted, "g"] & \\
B \arrow[uu, "\beta"] \arrow[ur, two heads, "e"] \arrow[rr, "f"] & & E \arrow[uu, hook, "\epsilon"']
\end{tikzcd}
$$

Proof Let $\beta = e\,; i$ be the factorisation, *via* C, and put $\gamma \equiv i\,; Te$. Then the three triangles on the left commute, so $e : B \longrightarrow C$ and $i : C \longrightarrow TB$ are coalgebra homomorphisms.

If $e : B \cong C$ then $\beta \cong i \in \mathcal{M}$, so B is extensional, and conversely.

If B is well founded then so is C by either Lemma 5.9 or 8.6.

Since $e : B \twoheadrightarrow C$ is orthogonal to $\epsilon : E \hookrightarrow TE$, there is a unique map $g : C \to E$ such that $e\,;g = f$ and $i\,;Tf = g\,;\epsilon$. Hence g is a homomorphism:

$$\gamma\,;Tg \;\equiv\; i\,;Te\,;Tg \;=\; i\,;Tf \;=\; g\,;\epsilon. \qquad \square$$

Theorem 8.9 The inclusion **Ens** $\longrightarrow$ **WfCoAlg** has a left adjoint, called the ***extensional reflection***, whose unit is cofinal.

Proof Let (A, α) be a well founded coalgebra. Since the category is well copowered, we may consider the poset of isomorphism classes of cofinal maps $A \twoheadrightarrow B$ (and commutative triangles), ordered such that the identity $\mathsf{id} : A \to A$ is the least element.

Since $\mathcal{C}$ has filtered colimits and Lemma 8.2 says that these are joins, this poset is directed complete, so it is an ipo.

The successor quotient (Construction 8.8) defines an inflationary monotone endofunction, so we consider the special condition in Pataraia's Theorem 2.3:

(a) An object B is fixed by the successor iff it is extensional, by the Construction.

(b) Any coalgebra homomorphism $B' \longrightarrow B$ between extensional well founded coalgebras is an initial segment by Lemma 7.11 (which requires well-foundedness), but the present construction only uses cofinal maps, so $B' \cong B$ (Lemma 8.2).

Hence the ipo has a greatest element and this is the unique fixed point of the successor, so it is an extensional well founded coalgebra.

Now let $f : A \longrightarrow E$ be a homomorphism to an extensional coalgebra. We repeat the construction, but now using factorisations $A \twoheadrightarrow B \hookrightarrow E$ of f. This contains $\bot \equiv (\mathsf{id}_A, f)$ and is closed under successor quotient and filtered colimits. It embeds in the simpler version and so contains $A \twoheadrightarrow D$ by Pataraia induction (Theorem 2.3(c)). The corresponding $A \twoheadrightarrow D \hookrightarrow B$ is the required factorisation that shows that D is the extensional reflection of A. $\square$

Corollary 8.10 If $\mathcal{C}$ has pushouts then so does **Ens**.

Proof **WfCoAlg** $\to \mathcal{C}$ creates them (Proposition 5.3) and any left adjoint preserves them. That is, the pushout in **Ens** is the extensional reflection of that in **WfCoAlg**, which is actually calculated in $\mathcal{C}$. $\square$

Beware, however, that the result could be vastly larger than the given objects or $\mathcal{C}$-pushout; Theorem 10.6 looks at when the pushout in **WfCoAlg** is already extensional.

The adjoints in Theorems 8.7 and 8.9 do not commute because the extensional reflection requires well-foundedness. However, we can put them together to deduce an even stronger rigidity property:

Corollary 8.11 Any coalgebra homomorphism $E \longrightarrow C$ from an ensemble to any coalgebra whatever is an initial segment, *cf.* Lemma 2.12. However, there could be multiple maps $E \rightrightarrows C$.

$$\begin{tikzcd}
E \arrow[r] \arrow[d, hook, dotted] \arrow[dr, dotted] & C \\
R & WC \arrow[l, two heads, dotted] \arrow[u, hook, dotted]
\end{tikzcd}$$

Proof Let WC be the largest well founded subcoalgebra of C, so $WC \hookrightarrow C$ is an initial segment and $E \longrightarrow C$ factors through it by Theorem 8.7. Then let R be the extensional reflection of WC, so by Lemma 7.11 the composite $E \longrightarrow WC \longrightarrow R$ is also an initial segment. By cancellation (Lemma 4.16), so is $E \hookrightarrow WC$ and by composition $E \hookrightarrow C$ is too. Finally, consider $C \equiv \mathbf{2} \times E$. $\square$

Warning 8.12 Assumption 8.4 that the category be well copowered with respect to a class $\mathcal{E}$ of cofinal maps that are "surjective" in only the most tenuous of senses must not be taken lightly. Indeed, we will propose the existence of the extensional reflection when $\mathcal{E}$ fails to be well co-powered as a candidate for the categorical form of the axiom-scheme of replacement [Tay23].

Remark 8.13 There may be some other adjoints, but they are not very interesting:
(a) The forgetful functor $\mathbf{CoAlg} \to \mathcal{C}$ has a right adjoint iff there is a final T-coalgebra in each slice $\mathcal{C}/X$.
(b) If $\mathbf{CoAlg} \to \mathcal{C}$ has a left adjoint and there is a final coalgebra then it must be $T\mathbf{1} \cong \mathbf{1}$. Then the initial algebra and all ensembles are subobjects of $\mathbf{1}$.
(c) If the forgetful functor $U : \mathbf{WfCoAlg} \to \mathbf{CoAlg}$ has left adjoint L, the unit $\eta : A \to ULA$ would provide a map from *any* coalgebra to a well founded one. If Proposition 9.3 holds, every coalgebra would be well founded. If there is a final coalgebra then it would also be the initial algebra, *cf.* Peter Freyd's principle of ***algebraic compactness*** [Fre91]. □

9 When the functor preserves pullbacks

Previous work on this subject required the functor to preserve pullbacks, or at least inverse images of monos, but this new account has only used preservation of the monos themselves. We now re-impose the stronger assumption and prove the relatively few earlier results that depend on it. Principal amongst these is Proposition 1.4, which is a very important result for the way that well founded relations are used across mathematics.

There is a second essential requirement for this proof, namely the *universal quantifier*. In the categorical formulation this appears in the form of the adjunction $f^* \dashv f_*$. Gerhard Osius noted this in his version of the result [Osi74, Prop 6.3(a)]. Any topos has it (Notation 3.2), but since we are considering more general categories, we state it as a further assumption on the subobjects:

Assumption 9.1 In addition to the assumptions in Section 4,
(a) the functor $T : \mathcal{C} \to \mathcal{C}$ must preserve inverse image diagrams of predicates along coalgebra homomorphisms; and
(b) each inverse image operation f^* must have a right adjoint f_* on predicates, at least when f is a coalgebra homomorphism.

As an aid to understanding our categorical proof, we first give it for well founded *relations*, in as similar a form as possible. See [Tay99, Prop 2.6.2] for a box-style proof in natural deduction for well founded relations.

Proposition 9.2 Let $(A, \prec)$ be a well founded relation and $f : (B, <) \to (A, \prec)$ a ***strictly monotone*** function in the sense that

$$\forall bb' : B. \quad b' < b \implies fb' \prec fb$$

then $(B, <)$ is also well founded.

Proof Let ψ be a predicate on B satisfying the induction premise

$$\forall b. \qquad (\forall b'.\, b' < b \Rightarrow \psi b') \Longrightarrow \psi b.$$

For comparison with the categorical proof below, *cf.* Proposition 3.6, this is $K \subset V$, where

$$K \;\equiv\; \{b : B \mid \forall b'.\, b' < b \Rightarrow \psi b'\} \subset B \qquad \text{and} \qquad V \;\equiv\; \{b : B \mid \psi b\} \subset B.$$

The key step is to define $f_* V \equiv \{a : A \mid \phi a\} \subset A$, where $\phi a \;\equiv\; (\forall b'.\, fb' = a \Rightarrow \psi b)$, and

$$H \;\equiv\; \{a : A \mid \forall a'.\, a' \prec a \Rightarrow \phi a'\} \;\equiv\; \{a : A \mid \forall b'.\, b' \prec a \Longrightarrow \psi b'\} \;\subset\; A.$$

Strict monotonicity and the induction premise give $f^*H \subset K \subset V$, which is

$$\forall b. \qquad (\forall b'.\, fb' \prec fb \Rightarrow \psi b') \Longrightarrow (\forall b'.\, b' < b \Rightarrow \psi b') \Longrightarrow \psi b.$$

Quantifying over $\{b' \mid fb' = a\}$, we obtain $H \subset f_*V$, which is

$$\forall a. \qquad (\forall a'.\, a' \prec a \Rightarrow \phi a') \iff (\forall b'.\, fb' \prec a \Rightarrow \psi b') \Longrightarrow (\forall b'.\, fb' = a \Rightarrow \psi b') \;\equiv\; \phi a.$$

Then $\forall a.\, \phi a$ since $(A, \prec)$ is well founded, whence $\forall b.\, \psi b$ as required. $\square$

We now prove the result for general functors that preserve inverse images and coalgebra homomorphisms that are equipped with f_*. Notice, however, that the hypothesis that f be a coalgebra homomorphism is actually stronger (in the case of $T \equiv \mathcal{P}$) than being strictly monotone (*cf.* Lemma 3.3).

Theorem 9.3 Let $f : (B, \beta) \longrightarrow\!\!\triangleright\, (A, \alpha)$ be a coalgebra homomorphism with f_*, where (A, α) is well founded. Then (B, β) is also well founded.

Proof Given the diagram marked in thick lines, apply the right adjoint f_* to $j : V \rightarrowtail B$, to get $i : f_*V \rightarrowtail A$. The counit of this adjunction is $\epsilon : f^*f_*V \to V$ and makes the little triangle $(*)$ commute, where f^* is given by pullback (inverse image) of i along f. The upper part of the diagram is the T-image of the lower part, including this pullback but not K. Let $H \equiv \alpha^*T(f_*V)$ be the pullback of Ti and α and f^*H its pullback along f.

$$\begin{tikzcd}
 & (Tf)^*T(f_*V) = T(f^*f_*V) \arrow[rr, "\mathrm{pbk}"] \arrow[dl, "T\epsilon"'] \arrow[dr, tail] & & T(f_*V) \arrow[dr, tail, "Ti"] & \\
TV \arrow[rr, tail, "Tj"] & & TB \arrow[rr, "Tf"] & & TA \\
 & f^*H \arrow[uu, dotted] \arrow[rr] \arrow[dr, tail, "\mathrm{pbk}"'] \arrow[dl, dotted] & & H \arrow[uuu] \arrow[dr, tail] \arrow[dd, dotted] \arrow[r, phantom, "\mathrm{pbk}"] & \\
K \arrow[uu] \arrow[r, tail] & V \arrow[r, tail, "j"] \arrow[r, phantom, "(*)"', shift right=3] & B \arrow[uu, "\beta"'] \arrow[rr, "f"] & & A \arrow[uu, "\alpha"'] \\
 & f^*f_*V \arrow[u, "\epsilon"] \arrow[ur, tail] \arrow[rr, "\mathrm{pbk}"] & & f_*V \arrow[ur, tail, "i"] &
\end{tikzcd}$$

By construction, the whole diagram of solid lines commutes from f^*H to TA. In particular, $f^*H \rightarrowtail B \to TB$ and $f^*H \to H \to T(f_*V)$ agree at TA, so there is a pullback mediator $f^*H \to T(f^*f_*V)$. Then $f^*H \to T(f^*f_*V) \to TV$ agrees with $f^*H \rightarrowtail B$ at TB, so there is also a pullback mediator $f^*H \to K$.

This shows that $f^*H \subset V$ as $\mathcal{C}$-subobjects of B. Therefore, by the adjunction $f^* \dashv f_*$, we have $H \subset f_*V$ as subobjects of A.

That is, there is a map $H \to f_*V$ that makes the right-hand part of the diagram into a broken pullback. Now, since A is well founded, $i : f_*V \cong A$, so $f^*f_*V \cong B$ and $j : V \cong B$ [Tay96b, Prop 7.3]. $\square$

Examples 9.4 To show that the additional hypotheses are necessary, we substitute preorders for categories in the whole theory, so a well founded *coalgebra* becomes a well founded *element* in the sense of Proposition 2.16.

$$
\begin{array}{ccc}
y & \leq & sy = ss\bot \\
\text{\rotatebox{90}{$\geq$}} & & \text{\rotatebox{90}{$\geq$}} \\
\bot & \leq & s\bot
\end{array}
\qquad\qquad
\begin{array}{ccccccccccc}
y & \leq & sy & \leq & ssy & \leq & sssy & \leq & \cdots & \leq & s^\omega y \\
\text{\rotatebox{90}{$\geq$}} & & \text{\rotatebox{90}{$\geq$}} & & \text{\rotatebox{90}{$\geq$}} & & \text{\rotatebox{90}{$\geq$}} & & & & \| \\
\bot & \leq & s\bot & \leq & ss\bot & \leq & sss\bot & \leq & \cdots & \leq & s^\omega\bot
\end{array}
$$

In both diagrams, the elements $s^n\bot$ and $s^\omega\bot$ are well founded, but y is not, because $s\bot \wedge y \leq \bot$ but $y \not\leq \bot$.

The first example is a Heyting semilattice, but s does not preserve the meet $y \wedge s\bot = \bot$.

The second is also distributive but it is not a Heyting semilattice, since $y \wedge (-)$ does not preserve the directed join $\bigvee^{\uparrow} s^n\bot$. However, s preserves meets because, for $n < \omega$ and $m \leq \omega$,

$$s^n\bot \;\wedge\; s^m y \quad = \quad s^{\mathsf{min}(n,m)}\bot. \qquad \square$$

Question 9.5 You may think that this is not a "real" counterexample, because it uses posets instead of categories, but by the extensional reflection (Theorem 8.9) the issue is really what happens in **Ens**, which is a preorder. But could there be a *cofinal* homomorphism $f : A \twoheadrightarrow B$ where B is well founded and extensional, but A is not well founded (*cf.* Lemma 8.6)?

The Theorem seems to be needed to construct binary pullbacks of well founded coalgebras.

Proposition 9.6 The functors **Ens** $\longrightarrow$ **WfCoAlg** $\longrightarrow$ **CoAlg** $\longrightarrow \mathcal{C}$ create pullbacks.

$$
\begin{tikzcd}
 & TD \arrow[rr] \arrow[dd] \arrow[dr, phantom, "\lrcorner", very near start] & & TC \arrow[dd, "Tg"] \\
D \arrow[rr] \arrow[dd] \arrow[ur, dotted, "\delta"] \arrow[dr, phantom, "\lrcorner", very near start] & & C \arrow[ur, "\gamma"'] \arrow[dd, "g", near start] & \\
 & TB \arrow[rr, "Tf", near start] & & TA \\
B \arrow[rr, "f"] \arrow[ur, "\beta"] & & A \arrow[ur, "\alpha"'] &
\end{tikzcd}
$$

Proof The diagram shows how to compute the pullback (D, δ) of coalgebras $B \longrightarrow A \longleftarrow C$ for a functor T that preserves them.

If the given coalgebras are well founded then so is the pullback, by Theorem 9.3.

If they are extensional then all of structure maps and homomorphisms in the cube are initial segments (by Lemma 7.11, composition and cancellation), so the pullback is extensional too. $\square$

As an application of this, we may construct the well founded part of any T-coalgebra (Theorem 8.7) in a uniform way:

Corollary 9.7 If T has a final coalgebra F (and hence an initial algebra I by Corollary 6.12) then the well founded part WA of any coalgebra A is given by the inverse image on the left:

$$
\begin{tikzcd}
WA \arrow[r] \arrow[d, hook, "f^*i"'] \arrow[dr, phantom, "\lrcorner", very near start] & I = WF \arrow[d, hook, "i"] \\
A \arrow[r, "f"] & F
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\mathbf{WfCoAlg} \arrow[d, tail, shift right] \arrow[r, phantom, "\simeq"] & \mathbf{CoAlg}/I \arrow[d, tail, shift right] \\
\mathbf{CoAlg} \arrow[u, shift right, "W"', "\dashv"] \arrow[r, phantom, "\simeq"] & \mathbf{CoAlg}/F \arrow[u, shift right, "i^*"', "\dashv"]
\end{tikzcd}
$$

Proof Any category with a terminal object is equivalent to the slice by it. By Proposition 6.10, the initial algebra I is the terminal well founded coalgebra, whilst by Theorem 9.3, any coalgebra having a homomorphism to I is well founded. Hence we have equivalences as shown on the right, commuting with the forgetful functors. The latter both have right adjoints, which must also be equivalent. $\square$

Question 9.8 Can W have a further right adjoint? What would it mean?

10 Pushouts

Besides the preservation of pullbacks, we have also avoided using pushouts (binary joins) in this work, by developing a much more delicate proof for *directed* joins using Pataraia's fixed point theorem. We now restore the pushouts, inspired by Gerhard Osius's treatment of them in his reconstruction of Set Theory within a topos.

We will still pay attention to our analysis of special classes of monos in the base category $\mathcal{C}$, but now the additional assumptions make it much more like **Set** or a topos than we have so far needed. Of the classes of "monos" in **Pos** (Example 4.19), $\mathcal{L}$ satisfies them but $\mathcal{R}$ does not [Tay23].

In addition to the Assumptions that we have accumulated, we need a "union" property for pushouts analogous to Proposition 4.4 for directed unions. See [Bar87] for another account of this property in toposes and Abelian categories.

$$\begin{tikzcd}
& B \arrow[dr, "i"] \arrow[drr, tail, "f"] & & \\
A \arrow[ur, tail, "m"] \arrow[dr, tail, "n"'] \arrow[r, phantom, "\mathrm{pbk}"] & {} \arrow[r, phantom, "\mathrm{psh}"] & D \arrow[r, "d"] & \Theta \\
& C \arrow[ur, "j"'] \arrow[urr, tail, "g"'] & &
\end{tikzcd}$$

We will discuss these properties of pushouts in **Set** and $\mathbf{Set}^{\mathsf{op}}$. The first result is known as the ***Amalgamation Lemma***:

Lemma 10.1 In **Set** or any pretopos, the pushout of a pair of monos $B \xleftarrow{m} A \xrightarrow{n} C$ is another pair of monos and is also a pullback.

Proof The following is a congruence:

$$(A+B)+(A+C) \underset{[m;\nu_1,\ \nu_0,\ n;\nu_0,\ \nu_1]}{\overset{[m;\nu_0,\ \nu_0,\ n;\nu_1,\ \nu_1]}{\rightrightarrows}} B+C.$$

If $f : B \to \Theta$ and $g : C \to \Theta$ make a commutative square then $[f, g] : B + C \to \Theta$ coequalises the congruence. Since the quotient is effective, to verify monos and equalisers, it suffices to inspect the congruence [FS90, 1.651] [Tay99, 5.8.10]. $\square$

Lemma 10.2 The dual property also holds in **Set** or any effective regular category, such as a category of finitary algebras.

Proof The pullback of $B \overset{m}{\twoheadrightarrow} A \overset{n}{\twoheadleftarrow} C$ is $D \equiv \{(b, c) \mid mb = nc : A\}$. Suppose $B \overset{u}{\leftarrow} E \overset{v}{\rightarrow} C$ make a commutative square from D. For each $a : A$, since m and n are surjective there are $b : B$ and $c : C$ with $a = mb = nc$, so $(b, c) \in D$ and $ub = vc$. Then if $a = mb' = nc'$ too, also $ub = vc' = ub' = vc$. Hence we may unambiguously define the mediator $e : A \to E$ by $ea \equiv ub$. $\square$

Lemma 10.3 In **Set** or any pretopos, if A, B, Θ and C form a pullback and A, B, D and C form a pushout, with all these maps mono, then the mediator $d : D \to \Theta$ is also mono.

Proof Regarding pullbacks, first note that if the square rooted at D is one then so is that to Θ, but the converse requires $D \to \Theta$ to be (plain) mono.

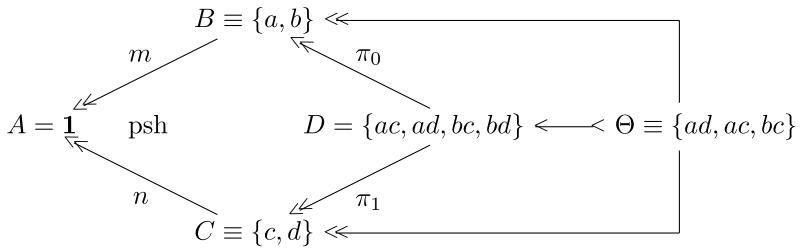
We consider the ***kernel*** of d (the pullback of d against itself), $K \subset D \times D$.

Since D is the union of its subobjects B and C and the pullback d^* preserves unions, $D \times D$ is the union of four parts, $B \times B$, $B \times C$, $C \times B$ and $C \times C$, and K is the union of their intersections with it. Putting these parts together, we have a surjection

$$\left.\begin{array}{lclcl} \mathsf{ker}\,(i\,;d) & = & K \cap B \times B & = & \Delta_B \\ \mathsf{pbk}\,(i\,;d,\ j\,;d) & = & K \cap B \times C & = & \Delta_A \\ \mathsf{pbk}\,(j\,;d,\ i\,;d) & = & K \cap C \times B & = & \Delta_A \\ \mathsf{ker}\,(j\,;d) & = & K \cap C \times C & = & \Delta_C \end{array}\right\} \longrightarrow\!\!\!\!\!\!\twoheadrightarrow K \ \subset\ D \times D$$

so the kernel $K \subset D \times D$ is $\Delta_B \cup \Delta_C$, which is the diagonal Δ_D. Hence d is mono, as required. $\square$

Example 10.4 The dual of this Lemma fails in **Set**.



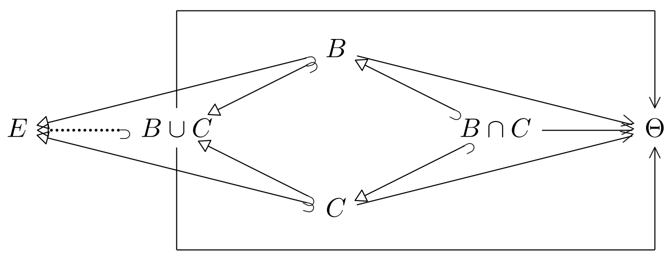
Proof Any pullback D rooted at $\mathbf{1}$ is a product, so $D = B \times C$. For the whole diagram to commute $A \leftleftarrows \Theta$, using the same ideas as in Lemma 10.2, the three elements of Θ give the three equations

$$nd \;=\; ma \;=\; nc \;=\; mb \;:A,$$

whence the pushout rooted at Θ is $\mathbf{1}$. However, the pullback mediator $D \leftarrow \Theta$ is not epi. $\square$

So long as the base category $\mathcal{C}$ also has these properties, we can paste two attempts together, as in Theorem 1.5(e):

Lemma 10.5 Well founded subcoalgebras and attempts admit binary joins.



Proof Suppose that the outer diamond defines two attempts with well founded supports B and C. Let $B \cap C$ be the intersection (pullback) of these subobjects of E, so $B \cap C$ is a well founded coalgebra by Proposition 9.6. By Lemma 6.4, the restrictions $B \cap C \to B \to \Theta$ and $B \cap C \to C \to \Theta$ agree. By the union property we have $B \cup C \to \Theta$. $\square$

Recall that Lemma 6.4 used equalisers and so assumed that regular monos are predicates admitting induction, whereas in that section we went on to prove (uniqueness in) the recursion theorem by another argument, without using this. In other words, we could avoid using equalisers here by relying instead on the main Theorem, for which this was intended to be a lemma.

Finally we give the categorical explanation of the strange "overlapping union" in Set Theory: putting B and C together does not yield a *coproduct* $B+C$ but their *pushout* rooted at their meet $A \equiv B \cap C$ from Corollary 7.12.

We already know from Proposition 5.3 that the functors **WfCoAlg** $\longrightarrow$ **CoAlg** $\longrightarrow \mathcal{C}$ create colimits, whilst by Theorem 7.14, **Ens** $\to$ **WfCoAlg** creates *filtered* colimits and the initial object. Corollary 8.10 showed that **Ens** has binary joins, but relied on the extensional reflection for this. What we show now is that they are inherited from $\mathcal{C}$, under the additional assumptions of this section.

Theorem 10.6 The preorder **Ens** has binary joins, given by pushout in $\mathcal{C}$ over the binary meet.

$$\begin{tikzcd}
 & B \arrow[rr, hook, "\beta"] \arrow[dr, hook, "i"] & & TB \arrow[dr, hook, "Ti"] & \\
A \equiv B \cap C \arrow[ur, hook] \arrow[dr, hook] & & D \arrow[rr, "\delta"] & & TD \\
 & C \arrow[rr, hook, "\gamma"] \arrow[ur, hook, "j"] & & TC \arrow[ur, hook, "Tj"] &
\end{tikzcd}$$

Proof Let (B, β) and (C, γ) be ensembles, so β and γ are initial segments.

By Theorem 7.14, they have a meet $A \equiv B \cap C$, and the maps $B \leftarrow A \to C$ are initial segments.

Let D be the pushout in $\mathcal{C}$; it is well founded by Proposition 5.3. By the union assumption (*cf.* Lemma 10.1), i and j are initial segments, as are Ti, Tj, $\beta\,;Ti$ and $\gamma\,;Tj$.

The key point is that $B \cap C$ is the pullback rooted at *either* D or TD, by Corollary 7.12.

Therefore $\delta : D \to TD$ is an initial segment by the union assumption (*cf.* Lemma 10.3), making D extensional. $\square$

The argument that Osius gave for this [Osi74, Thm 6.6] is rather more complicated (with a big diagram). Throughout his paper he used recursion instead of well-foundedness (*cf.* Proposition 3.10) and of course $T \equiv \mathcal{P}$, but for this particular result he used the partial map classifier $\tilde{C}$ (nowadays written $C_\perp$) in a topos.

There are no more adjoints amongst these categories.

Example 10.7 Ens $\to$ **WfCoAlg** does not create or even preserve colimits and so does not have a right adjoint: Binary coproducts are idempotent ($A + A \cong A$) in **Ens** but disjoint ($A \cap A = \emptyset$) (in $\mathcal{C}$ by hypothesis and so also) in **WfCoAlg**. For a concrete example, the extensional well founded relation $0 \prec 1$ is embedded twice in V. $\square$

Example 10.8 The rank functor $R :$ **WfCoAlg** $\to$ **Ens** does not have a left adjoint because it takes the pullback

$$\begin{tikzcd}
\mathbf{0} \arrow[r, hook] \arrow[d, hook] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{1} \arrow[d, hook] & & \mathbf{0} \arrow[r, hook] \arrow[d, hook] & \mathbf{1} \arrow[d, equal] \\
\mathbf{1} \arrow[r, hook] & \mathbf{2} & \text{to} & \mathbf{1} \arrow[r, equal] & \mathbf{1}
\end{tikzcd}$$

where $\mathbf{2}$ carries the empty relation, which is well founded but not extensional.

Further work

The original purpose of this work was to provide an intuitionistic categorical account of transfinite recursion for my book [Tay99]. However, there was no way to use Hartogs' Lemma constructively, and then out of the blue came Pataraia's far simpler proof of the fixed point theorem, but domain theorists ought to have found it much earlier.

What we did learn from the intuitionistic ordinals [JM95, Tay96a] is that their irreflexive membership and reflexive containment relations must be considered separately. In symbolic logic this at least doubles the work, but category theory was invented to *organise* such difficulties, by isolating the essential *argument*, whilst wrapping the *complications* in an appropriate choice of categories and functors.

Therefore the next task is to apply the present work to the category of posets instead of sets [Tay23]; we haven't included that here because there are too many order-theoretic facts to check. Then the ideas can be extended to other categories, which might have fixpoint objects [CP92] or accommodate corecursion alongside recursion, *cf.* Remark 4.6.

This is why we went to some trouble in Section 4 to pin down just what we were using in the original setting. In other categories there are many alternatives to the naïve ideas of 1–1 and onto mappings that we could use for the predicates over which we do induction and for defining extensionality, although changing the former doesn't seem to be very fruitful.

Extensionality is not as innocent as it looks: equality is like marriage in that it transfers any property of one partner to the other. Dana Scott showed that it is essential for giving the axiom of replacement its power: *without* it, that is provably consistent in *Zermelo* Set Theory [Sco66].

We expect to see even more dramatic results from applying the extensional reflection (Theorem 8.9) to other categorical settings. Using various notions of "initial segment" and "cofinal map" turns sets ($\in$-structures) into ordinals and thin ordinals into plump ones. Transfinite iteration of functors and Jean-Yves Girard's *dilators* [Gir81] are also examples of this process.

To perform these over Set Theory requires Replacement, but, being adjoints, they are expressed in the mother tongue of category theory, so we can regard them as *candidates for new axioms* to replace Replacement.

We have also explained how extensional well founded coalgebras are "fragments" of the initial algebra, whether that exists or not. Even if it does, it may be very complicated, whilst it may be easier to characterise its fragments instead.

Since there is plenty to do in "concrete" categories, it is not really an issue that we haven't fully explained how they are well powered. Any fibration defines a factorisation system, in which we would require prone maps to be initial segments and cofinal ones to be vertical. If we are going that deep into foundations, we should also deconstruct what is needed of the base category $\mathcal{S}$ to prove Pataraia's Theorem, in particular the directed completeness and impredicativity.

All of these considerations come together when the algebra is some type theory. There is a categorical construction called *gluing* [Tay99, §7.7] or *logical relations* that apparently magically proves consistency and termination results. It invokes the universal property of the free algebra, *i.e. recursion* over the *entirety* of its world of types, terms and proofs. How it manages to do this ought to raise eyebrows in the light of Kurt Gödel's incompleteness theorems.

The symbolic approach to such things is to turn the syntax of proofs into an ordinal, which to a categorist is vandalism because it thows the algebraic structure away. In fact proof theorists also exploit their *arithmetic* of ordinals to keep track of iterated transformations of proofs. One might hope to develop methods that retain both the algebra of the type theory and that of proof-manipulation.

Above all we must escape from the idea that ordinals are linear orders for counting beyond infinity.

The first version of this work was presented at *Category Theory 1995* in Cambridge and at *Logical Foundations of Mathematics, Computer Science and Physics — Kurt Gödel's Legacy (Gödel '96)* in Brno. Although it did not appear in the proceedings of either meeting, [Tay96b] was circulated there and available on my web page from 1996 to 2003 and from 2006. Summaries of the results were published in Sections 2.5, 6.3, 6.7 and 9.5 of [Tay99]. Work was resumed in 2019 in answer to a demand from those studying coalgebras to weaken the conditions on the functor.

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References

[Acz88] Peter Aczel. *Non-well-founded sets.* Number 14 in CSLI Lecture Notes. Center for the Study of Language and Information, 1988.

[Awo13] Steven Awodey. Bibliography for algebraic set theory. `www.phil.cmu.edu/projects/ast/`, 2013.

[Bar87] Michael Barr. On categories with effective unions. In Francis Borceux, editor, *Categorical Algebra and its Applications*, number 1348 in Lecture Notes in Mathematics, pages 19–35, 1987.

[Bau09] Andrej Bauer. On the failure of fixed-point theorems for chain-complete lattices in the effective topos. *Electronic Notes in Theoretical Computer Science*, 249:157–167, August 2009.

[Ber58] Paul Bernays. *Axiomatic Set Theory.* Studies in Logic and the Foundations of Mathematics. North Holland, 1958. With historical introduction by Abraham Fraekel. Second edition 1968, Reprinted by Dover 1991.

[BF97] Cesare Burali-Forti. Una questione sui numeri transfiniti. *Rendiconti del Circolo matematico di Palermo*, 11:154–164, 1897. English translation in [vH67], pages 104–111.

[BL12] Andrej Bauer and Peter LeFanu Lumsdaine. On the Bourbaki–Witt principle in toposes. arXiv:1201.0340, 2012.

[Bla14] Frédéric Blanqui. A point on fixpoints in posets. arXiv:1502.06021, 2014.

[Bou49] Nicolas Bourbaki. Sur le théorème de Zorn. *Archiv der Mathematik*, 2:434–7, 1949.

[Cam78] Paul Campbell. The origin of Zorn's lemma. *Historia Mathematica*, 5:77–89, 1978.

[Can95] Georg Cantor. Beiträge zur Begründung der transfiniten Mengenlehre (Erste Artikel). *Mathematische Annalen*, 46:481–512, 1895. Reprinted in [Can32]; English translation in [Can15].

[Can97] Georg Cantor. Beiträge zur Begründung der transfiniten Mengenlehre (Zweiter Artikel). *Mathematische Annalen*, 49:207–246, 1897. Reprinted in [Can32]; English translation in [Can15].

[Can15] Georg Cantor. *Contributions to the Founding of the Theory of Transfinite Numbers.* Open Court, 1915. Translated and with a historical introduction by Philip Jourdain; republished by Dover, 1955.

[Can32] Georg Cantor. *Gesammelte Abhandlung mathematischen und philosophischen Inhalts.* Springer-Verlag, 1932. Edited by Ernst Zermelo.

[Car72] Lewis Carroll. *Through the Looking Glass.* Macmillan, 1872.

[CP92] Roy L. Crole and Andrew M. Pitts. New foundations for fixpoint computations: FIX-hyperdoctrines and the FIX-logic. *Information and Computation*, 98(2):171–210, June 1992.

[CUV09] Venanzio Capretta, Tarmo Uustalu, and Varmo Vene. Corecursive algebras: A study of general structured corecursion. In Marcel Oliveira and Jim Woodcock, editors, *Brazilian Symposium on Formal Methods*, number 5902 in Lecture Notes in Computer Science, pages 84–100, 2009.

[DLR09] Ivan Di Liberti and Jiři Rosický. Enriched locally generated categories. arxiv.org/abs/2009.10980, 2009.

[Epp03] Adam Eppendahl. *Categories and Types for Axiomatic Domain Theory.* PhD thesis, Queen Mary, University of London, 2003.

[Esc03] Martín Escardó. Joins in the complete Heyting algebra of nuclei. *Applied Categorical Structures*, 11:117–124, 2003.

[Fel62] Walter Felscher. Doppelte Hülleninduktion und ein Satz von Hessenberg und Bourbaki. *Archiv der Mathematik*, 13:160–5, 1962.

[FK72] Peter Freyd and Max Kelly. Categories of continuous functors, I. *Journal of Pure and Applied Algebra*, 2:169–191, 1972.

[Fow94] David Fowler. Could the Greeks have used mathematical induction? Did they use it? *Physis*, 31:252–265, 1994.

[Fre91] Peter Freyd. Algebraically complete categories. In Aurelio Carboni, Maria Pedicchio, and Giuseppe Rosolini, editors, *Como Category Theory Conference*, number 1488 in Lecture Notes in Mathematics, pages 95–104, Berlin, 1991. Springer-Verlag.

[FS90] Peter Freyd and Andre Scedrov. *Categories, Allegories.* Number 39 in Mathematical Library. North-Holland, 1990.

[Gir81] Jean-Yves Girard. Π^1_2 logic: Part I, dilators. *Ann. Math. Logic*, 21:75–219, 1981.

[Har15] Friedrich Hartogs. Über das Problem der Wohlordnung. *Mathematische Annalen*, 76:590–5, 1915.

[Har40] G. H. Hardy. *A Mathematician's Apology.* Cambridge University Press, 1940. Reprinted 1992.

[Hes08] Gerhard Hessenberg. Kettentheorie und Wohlordnung. *Journal für die reine und angewandte Mathematik (Crelle's Journal)*, 135:81–133, 1908.

[HK66] William Howard and Georg Kreisel. Transfinite induction and bar induction of types zero and one, and the role of continuity in intuitionistic analysis. *Journal of Symbolic Logic*, 31(3):325–358, September 1966.

[HP89] Martin Hyland and Andrew Pitts. The theory of constructions: Categorical semantics and topos-theoretic models. In John Gray and Andre Scedrov, editors, *Categories in Computer Science and Logic*, volume 92 of *Contemporary Mathematics*, pages 137–199, 1989.

[Jib11] Mamuka Jibladze. Obituary of Dito Pataraia, 2011. categories email forum, 23 December.

[JM95] André Joyal and Ieke Moerdijk. *Algebraic Set Theory.* Number 220 in London Mathematical Society Lecture Notes in Mathematics. Cambridge University Press, 1995.

[Joh02] Peter Johnstone. *Sketches of an Elephant: a Topos Theory Compendium.* Oxford logic guides. Oxford University Press, 2002.

[Joy87] André Joyal. Foncteurs analytiques et espèces de structures. In *Combinatoire énumerative*, volume 1234 of *Lecture Notes in Mathematics*, pages 126–159. Springer Verlag, 1987.

[JS97] Mamuka Jibladze and Alex Simpson. A proof of the fixpoint theorem for dcpos, 1997. Exchange of emails, 20–21 January.

[KM74] Anders Kock and Christian Mikkelsen. Non-standard extensions in the theory of toposes. In Albert Hurd and Peter Loeb, editors, *Victoria Symposium on Nonstandard Analysis*, number 369 in Lecture Notes in Mathematics, pages 122–143. Springer-Verlag, 1974.

[Kna28] Bronisław Knaster. Un théorème sur les functions d'ensembles. *Ann Soc Polon Math*, 6:133–4, 1928.

[Kur22] Kazimierz Kuratowski. Une méthode d'élimination des nombres transfinis des raisonnements mathématiques. *Fundamenta Mathematicae*, 3:76–108, 1922.

[Kur61] Kazimierz Kuratowski. *Introduction To Set Theory and Topology.* Pergamon Press, 1961.

[Lam68] Joachim Lambek. A fixpoint theorem for complete categories. *Mathematische Zeitschrift*, 103:151–161, April 1968.

[Law70] F. William Lawvere. Quantifiers and sheaves. In *Actes du Congrès International des Mathématiciens*, volume 1, pages 329–334, 1970.

[LS81] Daniel Lehmann and Michael Smyth. Algebraic specifications of data types: a synthetic approach. *Mathematical Systems Theory*, 14:97–139, 1981.

[Mik76] Christian Mikkelsen. *Lattice-theoretic and Logical Aspects of Elementary Topoi.* PhD thesis, Aarhus Universitet, 1976. `www.chrisjuul.dk/math/LL.pdf`.

[Mir17a] Dimitry Mirimanoff. Les antinomies de Russell et de Burali-Forti, et le problème fondemental de la théorie des ensembles. *Enseignement Mathématique*, 19:37–52, 1917.

[Mir17b] Dimitry Mirimanoff. Remarques sur la théorie des ensembles et les antinomies Cantoriennes, I. *Enseignement Mathématique*, 19:209–217, 1917.

[Mir19] Dimitry Mirimanoff. Remarques sur la théorie des ensembles et les antinomies Cantoriennes, II. *Enseignement Mathématique*, 21:29–52, 1919.

[Mos49] Andrzej Mostowski. An undecidable arithmetical statement. *Fundamenta Mathematica*, 36:143–164, 1949.

[Osi74] Gerhard Osius. Categorical set theory: a characterisation of the category of sets. *Journal of Pure and Applied Algebra*, 4:79–119, 1974. (his Habilitationsschrift).

[Osi75] Gerhard Osius. Logical and set theoretical tools in elementary topoi. In *Model Theory and Topoi*, volume 445 of *Lecture Notes in Mathematics*, pages 297–346. Springer-Verlag, 1975.

[Par74] Robert Paré. Colimits in topoi. *Bulletin of the American Mathematical Society*, 80(3):556–561, May 1974.

[Pea89] Giuseppe Peano. *Arithmetices Principia, Nova Methodo Exposita.* Fratres Bocca, Turin, 1889. English translation, *The Principles of Arithmetic, presented by a new method*, in [vH67], pages 20–55.

[PS78] Robert Paré and Dietmar Schumacher. Abstract families and the adjoint functor theorems. In *Indexed Categories and Their Applications*, volume 661 of *Lecture Notes in Mathematics.* Springer-Verlag, 1978.

[Ros86] Giuseppe Rosolini. *Continuity and Effectiveness in Topoi.* PhD thesis, University of Oxford, 1986.

[San11] Davide Sangiorgi. Origins of bisimulation and coinduction. In Davide Sangiorgi and Jan Rutten, editors, *Advanced Topics in Bisimulation and Coinduction*, pages 18–19. Cambridge University Press, 2011.

[Sco66] Dana Scott. More on the axiom of extensionality. In Yehoshua Bar-Hillel, E. I. J. Poznanski, M. O. Rabin, and Abraham Robinson, editors, *Essays of the Foundations of Mathematics*, pages 115–139. Magnes Press, Hebrew University, 1966. Distributed by Oxford University Press.

[Str05] Thomas Streicher. Fibred categories à la Bénabou. https://arxiv.org/pdf/1801.02927.pdf, 2005.

[Tar55] Alfred Tarski. A lattice-theoretic fixed point theorem and its applications. *Pacific J Math*, 5:285–309, 1955.

[Tay96a] Paul Taylor. Intuitionistic sets and ordinals. *Journal of Symbolic Logic*, 61:705–744, September 1996.

[Tay96b] Paul Taylor. Towards a unified treatment of induction I: the general recursion theorem. `www.paultaylor.eu/ordinals`, entirely superseded by this paper, 1996.

[Tay99] Paul Taylor. *Practical Foundations of Mathematics.* Number 59 in Cambridge Studies in Advanced Mathematics. Cambridge University Press, 1999.

[Tay23] Paul Taylor. Ordinals in categories. Work in progress, 2023.

[vH67] Jan van Heijenoort. *From Frege to Gödel: a Source Book in Mathematical Logic, 1879–1931.* Harvard University Press, 1967.

[vN25] John von Neumann. Eine Axiomatisierung der Mengenlehre. *Journal für Mathematik*, 154:219–240, 1925.

[vN28] John von Neumann. Über die Definition durch transfinite Induktion und verwandte Fragen der allgemeinen Mengenlehre. *Mathematisches Annalen*, 99:373–393, 1928.

[Wil01] Todd Wilson. An intuitionistic version of Zermelo's proof that every choice set can be well-ordered. *Journal of Symbolic Logic*, 66:1121–6, 2001.

[Wit51] Ernst Witt. Beweisstudien zum Satz von M. Zorn. *Mathematische Nachrichten*, 4:434–8, 1951.

[Zer08a] Ernst Zermelo. Neuer Beweis für die Möglichkeit einer Wohlordnung. *Mathematisches Annalen*, 65:107–128, 1908. English translation in [vH67], pages 183–198.

[Zer08b] Ernst Zermelo. Untersuchungen über die Grundlagen der Mengenlehre I. *Mathematische Annalen*, 59:261–281, 1908. English translation in [vH67], pages 199–215.

[Zer30] Ernst Zermelo. Über Grenzzahlen und Mengenbereiche. *Fundamenta Mathematicae*, 16:29–47, 1930.

[Zer35] Ernest Zermelo. Grundlagen einer allgemeinen Theorie der mathematischen Satzsysteme. *Fundamenta Mathematicae*, 25:135–146, 1935.

[Zor35] Max Zorn. A remark on transfinite algebra. *Bull. Amer. Math. Soc.*, 41(10), 1935.