Well Founded Coalgebras and Recursion

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Abstract

We define well founded coalgebras and prove the recursion theorem for them: that there is a unique coalgebra-to-algebra homomorphism to any algebra for the same functor. Earlier, this was proved intuitionistically by adapting the traditional argument for well founded relations or ordinals in set theory, which considers the union of partial solutions. However, that required subobjects to form a complete Heyting algebra and the functor to preserve inverse images. Our new argument for functors that just preserve monos exploits Pataraia's fixed point theorem and only uses directed unions of subobjects.

1 Background

Categorical set theory seeks to incorporate the ideas of set theory into modern mainstream mathematics, after stripping them of their foundational pretensions and obscure coding. The value of these ideas is that they provide ways of expressing very strong principles of *induction* and *recursion*, which may be used to prove the *consistency* of other logical systems. More recently, the methods that we discuss in this paper have also been applied to reasoning about computational processes.

Set theory has two traditions: one that employs *completed infinities* (classes, universes, inaccessible cardinals) and another that eschews them, developing *potential infinities* instead.

Completed infinities feature in ordinary mathematics in the form of *free algebras*. André Joyal and Ieke Moerdijk [JM95], approaching the subject from this point of view, treated the universes of sets and of (three kinds of intuitionistic) ordinals as the free algebras for the powerset functor together with "successor" functions having various properties. They modelled the small/large distinction using ideas that had been developed in topology and sheaf theory to handle *open maps*. Their *algebraic set theory* has been developed further by a number of authors [Awo13] and now gives a categorical account of several highly powerful notions in set theory.

Type theories also commonly include (multiple) *universes*: when the motivations are symbolic formulae, it is quite natural to internalise the whole system within itself. This is also used to provide results that would otherwise be obtained *impredicatively*.

Our view, on the other hand, is in the tradition of *potential* infinities. In particular, since we cannot solve $X \cong \mathcal{P}(X)$, *i.e.* the covariant powerset functor has no free algebra, we instead characterise and work with *fragments* of it. These are the $\in$-structures of traditional set theory. We study these *for themselves*, in the same way that we might study groups or topological spaces, and not as ballast for other mathematics.

In set-theoretic terms, we are only interested in the axioms of *foundation* and *extensionality*. The axiom of foundation and the notion of a well *founded* relation are the (to us, natural) generalisation of the well-*orderings* or *ordinals* that Georg Cantor discussed from the early days [Can95, Can97]. He stated the defining property in two ways:

(a) every non-empty subset $U \subset X$ has a $\prec$-least element; or

(b) there is no infinite descending sequence $\cdots \prec d \prec c \prec b \prec a$,

the second of which had been invoked for the natural numbers in Euclid's *Elements* VII 31.

It took some time to recognise the generalisation to well-foundedness and its role in showing that Zermelo's set theory and infinitary proofs are not vulnerable to circular arguments like Russell's Paradox. Dimitry Mirimanoff seems to have been the first to do this [Mir17]. John von Neumann proposed a meta-axiom, that the system of set theory be the minimal one [vN25]. Ernst Zermelo asserted the two properties above for $\in$ as his axiom of foundation [Zer30]. He then introduced the general notion of a well founded relation and applied it to proof theory [Zer35].

If we state either of the properties (a,b) as the definition, we have to make frequent use of *excluded middle* or *dependent choice*, respectively. For the intuitionistic definition, we identify what we actually want to *do* with the notion. (I do not know who first stated this as the intuitionistic definition, despite asking on MathOverflow.)

Definition 1.1 A binary relation $\prec$ on a carrier A is ***well founded*** if it obeys the ***induction scheme***

$$\frac{\forall a\!:\!A.\ \ (\forall b\!:\!A.\ \ b \prec a \Rightarrow \phi b) \Longrightarrow \phi a}{\forall a\!:\!A.\ \ \phi a}$$

for any predicate ϕ on A. It will be convenient to dissect this by saying that the sub-formula

$$\forall b\!:\!A.\ b \prec a \Longrightarrow \phi b$$

is the ***induction hypothesis*** (for ϕ at a) and the whole of the top line is the ***induction premise*** (for ϕ). If it happens that the outer implication on the top is two-way then we refer to the ***strict induction premise***.

Although for simplicity we will speak of well-foundedness as if it were quantified over all ϕ, the word *scheme* indicates that, in proof theory in particular, one may restrict attention to predicates of a certain logical complexity, such as with at most a particular number of alternations of quantifiers. Our categorical structure will be able to accommodate this generalisation (Assumption) and we indicate for what predicates we are using induction, but we shall not get involved in proof theory in this paper.

Example 1.2 With $n \prec n+1$ on the natural numbers, the induction scheme is known as Peano induction:

$$\frac{\phi 0 \qquad \forall n\!:\!\mathbb{N}.\ \ \phi n \Longrightarrow \phi(n+1)}{\forall n\!:\!\mathbb{N}.\ \ \phi n}$$

although this idiom predates Giuseppe Peano by at least three centuries.

Whilst the general notion of well-foundedness is natural and long-established, many mathematicians seem to be reluctant to use it. Instead they say that they are doing induction or recursion on the *length* of a string, the *height* of a tree, its *depth* in computer science, or some other such numerical measure. This is also the way in which iterative or recursive programs are shown to terminate.

The general result that is being invoked is this:

Proposition 1.3 If $(A, \prec)$ is well founded and $f : (B, <) \to (A, \prec)$ is ***strictly monotone*** in the sense that

$$\forall b_1 b_2\!:\!B.\quad b_1 < b_2 \implies f b_1 \prec f b_2$$

then $(B, <)$ is also well founded.

Proof If B has an infinite descending sequence then so does A, which is forbidden. Alternatively, if $\emptyset \neq V \subset B$ then $\emptyset \neq fV \subset A$, so there is a minimal $a \in fV$, where $a = fb$ for some $b \in V$ and this is minimal there. The more difficult intuitionistic proof is given in Proposition 9.2. $\square$

From the ability to *prove a predicate by induction* we may derive that of *defining a function by recursion.* Putting this ***recursion theorem*** into a modern form is the main goal of this paper. John von Neumann proved it for the ordinals in his reformulation of their theory that became the classic one [vN28, § III]. The following is the (mild) adaptation of his argument to intuitionistic well founded relations:

Theorem 1.4 Let $(A, \prec)$ be a carrier with a well founded binary relation and Θ another carrier with a function $\theta : \mathcal{P}\Theta \to \Theta$ that takes an arbitrary subset of Θ as its argument and returns a single element. Then there is a unique function $f : A \to \Theta$ such that

$$\forall a\!:\!A. \quad fa \;=\; \theta(\{fb \mid b \prec a\}).$$

We call this equation the ***recursion scheme***, because we only ever consider *particular* (Θ, θ).

Proof An ***initial segment*** of A is a subset $B \subset A$ such that

$$\forall bc\!:\!A.\;\; c \prec b \in B \Longrightarrow c \in B$$

and an ***attempt*** is a partial function $f : A \rightharpoonup \Theta$ whose support (domain of definition) $B \subset A$ is an initial segment and

$$\forall b\!:\!A.\;\; b \in B \;\Longrightarrow\; fb \;=\; \theta(\{fc \mid c \prec b\}).$$

(a) There is a unique attempt with empty support.
(b) The union of any directed family of initial segments or attempts is another such.
(c) The restriction of $\prec$ to any initial segment is well founded.
(d) Any two attempts f, g with the same support B are equal, by induction over B for the predicate $\phi b \equiv (fb = gb)$.
(e) Hence any two attempts with supports B_1 and B_2 agree on $B_1 \cap B_2$ and so may be amalgamated into an attempt with support $B_1 \cup B_2$.
(f) Given any attempt f with support B, there is a successor attempt g with support

$$C \;\equiv\; \{c : A \mid \forall b\!:\!A.\;\; b \prec c \Longrightarrow b \in B\} \qquad \text{given by} \qquad gc \;\equiv\; \theta\{b : A \mid b \prec c\}.$$

(g) In this construction, $C = B$ iff $B = A$, by induction over A for the predicate $\phi a \equiv (a \in B)$.
(h) The required solution to the recursion equation is the union of all of the attempts (a,b,e); this is total because it is fixed by the successor operation (g) and unique by (d). $\square$

It is essential to understand the steps of this traditional proof before proceeding with the rest of this paper. We label them because they will each be the subject of lemmas in our categorical proof.

However, we shall give this in a generality in which Proposition 1.3 *fails* — even though that is plainly an extremely important property of well founded relations. We therefore lose steps (c) and (e) of the proof and so cannot simply form the union of all attempts in the final part. This will oblige us to find more sophisticated ways of dealing with the issues. In fact, we also prove part (d) in a different way.

The handicap actually turns out to be an advantage, because we can only mimic step (e) in categories that are rather like **Set**, so when we are freed of this we may apply the result much more generally.

Remark 1.5 Steps (a) and (f) in the traditional proof provide the initial and next attempts, so by Peano recursion we can define the nth one for all $n : \mathbb{N}$. Can we not then just use step (b) at limit stages to continue this through the ordinals?

No.

First of all, ordinals are not "transfinite numbers" but require a proof to justify recursion over them, namely the result due to von Neumann that we have just stated. *Using* ordinals to do this would therefore be begging the question.

Secondly, the ordinals go on "forever" — Cesare Burali-Forti [BF97] showed early on that they do not form a "set". So when do we stop?

This is answered by a crucial but frequently overlooked lemma, due to Friedrich Hartogs [Har15], which is this: For any set X, let λ be the set of isomorphism classes of well-orderings of subsets of X. Then λ is well ordered and there is no injection $\lambda \rightarrowtail X$. In the application, we deduce that λ is a fixed point of the construction.

Hartogs' proof was one of the earliest formal applications of Zermelo's set theory [Zer08b] and he set out the prerequisites from that and Cantor's original work [Can97] very clearly. Principal amongst them is that, for any two well ordered sets, one is uniquely isomorphic to an initial segment of the other; we would now say that this is a consequence of von Neumann's (later) recursion theorem (Remark 7.13), but Cantor had actually given a valid direct proof of it.

It was one of the other innovations of von Neumann's paper to use the global set-theoretic membership relation $\in$ for the order on an *ordinal*, but for Cantor, Hartogs and us, the relation $\prec$ on a *well ordered* or *well founded* set is superstructure.

Proposition 3.5 provides a lightweight alternative to Hartogs' Lemma for obtaining the ordinal λ.

Thirdly, the traditional theory of the ordinals depends *very heavily* on excluded middle. There are two existing intuitionistic accounts [JM95, Tay96a], which show that there are several different notions. Even so, Hartogs' lemma remains irretrievably classical.

Finally, the ordinals themselves are significant applications of the generalisations that our categorical approach will offer, but we will only give them a passing mention here (Examples **??**) because they deserve a treatment of their own: their *successor* and *limit* operations may be seen categorically as the results of the *unit* and *multiplication* of a monad structure, where we only consider functors in this paper. We will show that there are even more types of ordinals, but we will put them in a more organised form.

Remark 1.6 In order to start generalising these ideas, consider first the recursion scheme: θ is the evaluation operation for some sort of *algebra* Θ. In taking a *set* of arguments instead of a *list*, we are saying that θ is *idempotent* and *commutative* with respect to them, but these conditions are inessential.

Indeed, we can consider any ***free theory***, *i.e.* one with no equations at all, but a (possibly infinite) collection Σ of operation symbols, each r of which has a (possibly infinite) ***arity*** $\mathsf{ar}\,(r)$. Then for any set X (of constants, generators, indeterminates or variables as you please), there is a set

$$TX \quad \equiv \quad \coprod_{r:\Sigma} X^{\mathsf{ar}\,(r)}$$

of ***terms*** of depth 1 built from these generators and operation symbols. With no generators, $T\emptyset$ is the set of constants or nullary operation-symbols. Of course TTX is the set of terms of depth 2 and so on.

An ***algebra*** for these operation *symbols* is a carrier Θ that is equipped with an *operation* $\Theta^{\mathsf{ar}\,(r)} \to \Theta$ for each symbol $r : \Sigma$. These may be combined into a single function on the disjoint union:

$$\theta : T\Theta \longrightarrow \Theta.$$

In particular, at least in the case where all of the arities are finite, there is a ***term-*** or ***free algebra*** that is obtained by forming the union A of all of the iterates of T, applied to the empty set. Since we have already done so exhaustively, applying T again to A yields the same thing, so

$$TA \underset{\mathsf{parse}}{\overset{\mathsf{ev}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\cong}}}} A,$$

where ev and parse are the operations of wrapping and unwrapping the outermost symbol of a term.

Therefore,

$$b \prec a \quad \equiv \quad (r, b) \in \mathsf{parse}\,(a)$$

defines the ***immediate sub-term relation*** on A. Since A only consists of expressions that are formed by repeated application of the operation symbols, this relation clearly satisfies the "descending sequence" definition of well-foundedness. Indeed, the intuitionistic definition is called ***structural induction*** and the Theorem yields ***structural recursion***.

The $\in$ relation in set theory satisfies another, apparently innocent, property:

Definition 1.7 A (well founded) binary relation $\prec$ such that

$$\forall ab{:}A. \quad (\forall c{:}A.\ \ c \prec a \iff c \prec b) \implies a = b$$

is called ***extensional***. The analogous property of sub-terms in a free algebra is that the parse map be one-to-one.

Well-foundedness and extensionality together explain many characteristic features of set theory. They are also important properties of term algebras, underlying the algorithm for *unification*, *i.e.* for assigning (sub-)terms to indeterminates in two or more terms so that they match.

In set theory, when we form the "union" of two supposedly independent objects, we may find that they already overlap. (Besides being bizarre from the point of view of any other kind of mathematics, this is irritating if we try to use set theory as a foundation.) Unification "matches up" sub-terms in a similar way.

We shall find in Section 7 that the *category* of extensional well founded structures and the appropriate homomorphisms is actually a *pre-order*, *i.e.* there is at most one map between any two objects. When we put two objects together, they (typically) have a non-empty intersection (meet in this order) and therefore an "overlapping" union.

Any collection of terms that includes all of their sub-terms is a part of the free algebra. Similarly, if we imagine for a moment that there could be a universal set, any set would be a subset of it.

Remark 1.8 Continuing with this fiction of a universal set as the free algebra, for any well founded relation $(A, \prec)$ we may use the recursion theorem to define

$$fa \qquad \text{as the set ($\in$-structure)} \qquad \{fb \mid b \prec a\}.$$

Even if $(A, \prec)$ was not extensional, the result is. Therefore, following Andrzej Mostowski [Mos49, Thm 3],

(a) any extensional well founded relation is isomorphic to a unique set ($\in$-structure); and

(b) any well founded relation has an extensional quotient, with a suitable universal property.

The first of these obliges us to subscribe to the belief that a set *is* some particular thing, instead of having a mathematical property that is shared by any isomorphic structure (*cf.* the distinction between von Neumann's *ordinals* and Cantor's *well ordered* sets). Moreover, if we admit that then we commit ourselves even more deeply, because this $\in$-structure is not defined within Zermelo's original set theory [Zer08b], but requires the axiom-scheme of replacement — the F (for Fraenkel) in ZF.

On the other hand, the second statement is an ordinary theorem of higher order logic. It's a quotient, so we may construct it using an equivalence relation, albeit one that has a *co-recursive* definition. This is done in [Tay96a, Thm 2.11] and we prove a more general result in Theorem 8.9.

Remark 1.9 This discussion of whether or not Mostowski's theorem requires Replacement is a distraction. There undoubtedly are constructions that ordinary mathematicians do, but which are not available in Zermelo set theory or its modern substitutes:

(a) It is common to *iterate* constructions, either over $\mathbb{N}$ or an ordinal, the simplest case being $\coprod \mathcal{P}^n(\mathbb{N})$.

(b) By methods variously known as realisability, gluing or logical relations, one can compare the term model of a logic system with a semantic one to prove consistency or completeness. Since this seems to conflict with Gödel's Incompleteness Theorems, the recursion over the term model must be one that goes beyond what that logic can prove for itself.

Giving a categorical account of the axiom-scheme of replacement is a topic for a PhD thesis: it goes well beyond what we can consider in this paper, although we make some suggestions.

We merely demonstrate how our categorical methods can *define* transfinite iteration of functors. Of course we cannot *construct* this: we are simply adding a new tool to the categorical lexicon. This lies alongside, for example, the *definition* of the subobject classifier in a category with finite limits, which defines but does not construct an elementary topos.

Mostowski's theorem is nevertheless the conceptual key to this, because our definition of transfinite iteration is another example of the extensional quotient. However, this is in a context where we have used categorical tools to generalise the notions of "injective" and "surjective" functions.

In the next section we outline how coalgebras for a functor account for these properties of set theory and term algebras. Section 3 gives a brief history of order-theoretic fixed point results, since a simple union of attempts will be inadequate for the generality that we consider for the main categorical results. Section 4 states the categorical conditions under which we can prove (generalisations of) the recursion theorem. Section 5 shows how well founded coalgebras are generated. Section 6 proves our central result, the recursion theorem. Section 7 introduces extensional well founded coalgebras and shows how they behave like set theory.

Section 9 gives the intuitionistic and categorical proofs of Proposition 1.3, along with examples of its failure without the extra assumptions that we chose not to make. Section 9 considers binary joins or pushouts under the stronger conditions.

2 Well founded coalgebras

We now show how the ideas from set theory and universal algebra in the previous section can be expressed in category theory. This builds on the work of Christian Mikkelsen and Gerhard Osius.

They were working in the years following the introduction of the notion of an elementary topos by Bill Lawvere and Myles Tierney, when the key issues were to optimise the categorical axioms and show that toposes could do anything that sets could do.

The main part of Mikkelsen's thesis [Mik76] was an important step towards the former, showing that colimits could be derived from limits (although this was eclipsed by Robert Paré's monadicity result [Par74]), after his supervisor Anders Kock had derived exponentials from powersets. As an appendix, he gave the first proof of the recursion theorem in a topos, in a very "diagrammatic" style.

Gerhard Osius was one of several people who demonstrated how to interpret "ordinary" mathematical notation (higher order logic) in a topos. For the present work, Osius's key contribution was to take $\in$-structures seriously in a categorical setting [Osi74, §§4&6]. He also summarised Mikkelsen's proof of the recursion theorem in more familiar notation [Osi75, §6].

It is a pity that neither of them continued studying categorical logic: Osius became a professor of statistics and Mikkelsen a schoolteacher.

The extension of their theory to any endofunctor T of a topos that preserves inverse images was made in [Tay99, §6.3] and for other categories in [Tay96b]. In this paper we weaken the requirement on T to preservation of mono(morphism)s, although Section 4 considers more carefully what we mean by these.

We give precise references to the corresponding results in these earlier works, for historical comparison, but ours are often much more general.

We work throughout in the logic of an elementary topos $\mathcal{S}$, remembering to thank Osius and others for allowing us to write this in the vernacular of mathematics. You may therefore treat $\mathcal{S}$ as **Set**, except that we do not use Excluded Middle or the Axiom of Choice, although we do use Impredicative constructions.

In the leading example, we have a binary relation on a carrier A. There are many ways of representing this in category or type theory, but the one that we choose is as a function (morphism)

$$A \xrightarrow{\alpha} \mathcal{P}A \qquad \text{by} \qquad a \longmapsto \{b \mid b \prec a\} \subset A.$$

That is, just as we re-interpreted the set-theoretic notion of *membership* in a free algebra as the *immediate sub-term relation* in Remark 1.6, so now we take the idea of the parse operation back to well founded relations.

For a general functor T, the $\prec$ relation becomes this:

Definition 2.1 A ***coalgebra*** for an endofunctor $T : \mathcal{C} \to \mathcal{C}$ of any category is an object A of $\mathcal{C}$ together with a morphism $\alpha : A \longrightarrow TA$. We say (provisionally) that (A, α) is ***extensional*** if α is mono in $\mathcal{C}$, *cf.* Definition 1.7.

$$\begin{tikzcd} TA \arrow[r, "Tf"] & TB \\ A \arrow[u, "\alpha"] \arrow[r, "f"] & B \arrow[u, "\beta"'] \end{tikzcd}$$

A homomorphism of coalgebras is a $\mathcal{C}$-morphism $f : A \longrightarrow B$ that makes the square commute, which we indicate by the triangle arrowhead. We mark the structure map α in the same way because it is a homomorphism $\alpha : (A, \alpha) \to (TA, T\alpha)$. We write $\mathbf{CoAlg}_T$ or just $\mathbf{CoAlg}$ for the category of coalgebras and homomorphisms.

To relate this idea back to set theory, we first need a full understanding of the powerset as a functor in a topos:

Notation 2.2 The ***covariant powerset functor*** $\mathcal{P} : \mathcal{S} \to \mathcal{S}$ is defined on an object X by $\mathcal{P}X \equiv \Omega^X$ and on a function $f : X \to Y$ by

$$\mathcal{P}fU \;\equiv\; \{fx \mid x \in U\} \;\equiv\; \{y : Y \mid \exists x{:}X.\, y = fx \wedge x \in U\} \subset Y$$

for $U \subset X$. We shall also need to define, for $V \subset Y$,

$$\begin{aligned} f^*V &\;\equiv\; \{x : X \mid fx \in V\} \\ f_*U &\;\equiv\; \{y : Y \mid \forall x{:}X.\, fx = y \Longrightarrow x \in U\}. \end{aligned}$$

These also provide the morphism parts of functors $\mathcal{S} \to \mathcal{S}$ that are respectively contravariant and covariant, since $(g\,;f)^*W = f^*(g^*W)$ and $(g\,;f)_*U = g_*(f_*U)$. More importantly for us, there are (order-)adjunctions

$$\begin{tikzcd} U \arrow[r, tail] & X \arrow[d, "f"] \\ V \arrow[r, tail] & Y \end{tikzcd} \qquad\qquad \begin{tikzcd}[column sep=small] & \mathcal{P}X \arrow[dl, phantom] & \\ \mathcal{P}X \arrow[d, "\mathcal{P}f"'] \arrow[r, phantom, "\dashv"] & \mathcal{P}Y \arrow[l, phantom] & \mathcal{P}X \arrow[d, "f_*"] \\ \mathcal{P}Y & & \mathcal{P}Y \end{tikzcd}$$

Diagrammatically, $\mathcal{P}f$ and f^* are given by composition and pullback respectively. The logical formulae that define $\mathcal{P}fU$ and f_*U are the same except that one involves an existential and the other a universal quantifier. We shall use f_* in Section 9. $\square$

Gerhard Osius's insight was to characterise set-theoretic inclusions as homomorphisms of extensional well founded $\mathcal{P}$-coalgebras [Osi74, §6], although strictly speaking this depends on the recursion theorem.

Remark 2.3 The inclusion $\beta\,;\mathcal{P}f \subset f\,;\alpha$ (as marked in the diagram on the left) holds iff

$$\forall b_1, b_2{:}B.\quad b_1 \prec_B b_2 \;\Longrightarrow\; fb_1 \prec_A fb_2,$$

i.e. f is ***strictly monotone*** or preserves the binary relation, *cf.* Proposition 1.3.

$$\begin{tikzcd} B \arrow[r, "\beta"] \arrow[d, "f"'] \arrow[dr, phantom, "\supset"] & \mathcal{P}B \arrow[d, "\mathcal{P}f"] \\ A \arrow[r, "\alpha"'] & \mathcal{P}A \end{tikzcd} \qquad \begin{tikzcd} \exists b' \arrow[r, dotted, "\prec_B"] \arrow[d, dotted, "f"'] & b \arrow[d, "f"] \\ a \arrow[r, "\prec_A"'] & fb \end{tikzcd} \quad \begin{tikzcd} B \arrow[d, "f"] \\ A \end{tikzcd}$$

The reverse inclusion is

$$\forall a{:}A.\, \forall b{:}B.\quad a \prec_A fb \;\Longrightarrow\; \exists b'{:}B.\, a = fb' \wedge b' \prec_B b,$$

which is a "lifting" property similar to that defining a fibration, as illustrated on the right. In process algebra a function f with this property is known as a ***simulation*** [Acz88].

If $f : B \subset A$ is a subcoalgebra inclusion then the lifting is unique and being a simulation says that B is down-closed or an ***initial segment***,

$$\forall ab{:}A.\quad a \prec b \in B \;\Longrightarrow\; a \in B. \qquad \square$$

Now we come to the concept that is one side of our main result.

Definition 2.4 A coalgebra $\alpha : A \longrightarrow\!\!\!\triangleright\ TA$ is ***well founded*** if in any pullback diagram in the category $\mathcal{C}$ of the form

$$\begin{tikzcd} TU \arrow[rr, tail, "Ti"] & & TA \\ H \arrow[u] \arrow[r, tail, "j"] \arrow[ur, phantom, very near start, "\urcorner"] & U \arrow[r, tail, "i"] & A \arrow[u, "\alpha"'] \end{tikzcd}$$

the maps i and therefore j are necessarily isomorphisms. We write $\mathbf{WfCoAlg}_T$ or just $\mathbf{WfCoAlg}$ for the category of well founded coalgebras and coalgebra homomorphisms.

Essentially this "broken pullback" appears (with $T \equiv \mathcal{P}$) on page 99 of [Mik76] and it is written symbolically as $\alpha^{-1}(\mathcal{P}U) \subset U \Longrightarrow U = A$ in [Osi74, §4] and [Osi75, Prop 6.1]. It was first given as the *definition* of well-foundedness in [Tay96b, Tay99].

As in Definition 1.1, this definition is a *scheme* with respect to monos $i : U \rightarrowtail A$.

Lemma 2.5 A binary relation $(A, \prec)$ is well founded in the earlier sense iff the corresponding (A, α) is a well founded $\mathcal{P}$-coalgebra.

Proof Write $U \equiv \{x : A \mid \phi x\}$ for some predicate ϕ defined on A.

An element $(a, V) \in H \subset A \times TU$ of the pullback consists of $a : A$ and $V \subset U \subset A$ such that

$$\alpha(a) \;\equiv\; \{x : A \mid x \prec a\} \;=\; V.$$

Thus V is determined uniquely by a (and the structure $\alpha : A \longrightarrow\!\!\!\triangleright\ TA$), but for such a V to exist, a must satisfy

$$\{x : A \mid x \prec a\} \;\subset\; U, \qquad i.e.\;\; \forall x{:}A.\;\; x \prec a \Longrightarrow \phi x.$$

The pullback H therefore corresponds to the induction *hypothesis*.

The induction *premise* is that, for each such $a : A$ that satisfies the hypothesis, we have $a \in U$ or ϕa. In the diagram this means that $H \subset U$. The *strict* induction premise corresponds to having $H \cong U$ instead; this makes $U \subset A$ a subcoalgebra for which the square is a pullback.

Well-foundedness of the coalgebra says that whenever we have a diagram of this form then $U \cong A$, just as the induction *scheme* says that whenever the premise holds then we must have $\forall a{:}A.\ \phi a$. □

Proposition 1.3 said that well-foundedness is reflected by order-preserving functions and in particular is inherited by initial segments, *cf.* Theorem 1.4(c). Whereas that only generalises to coalgebras on assumptions that we will not make until Section 9, here is a simpler result about the induction *premise* that will be an important tool (Lemma 5.1) in our categorical construction.

Lemma 2.6 Substitution along simulations preserves the induction premise.

Proof Let $f : (B, \prec) \to (A, <)$ be a simulation and ϕ a predicate on A that satisfies the induction premise

$$\forall a. \quad (\forall a'.\ a' < a \Rightarrow \phi a') \Longrightarrow \phi a.$$

Put $\psi \equiv f^*\phi$, so $\psi b \equiv \phi(fb)$, and suppose that it satisfies the induction hypothesis

$$\forall b'.\ b' \prec b \Longrightarrow \psi b'$$

for $b : B$. Let $a' : A$ be such that $a' < a \equiv fb$. Then, since f is a simulation, there is some $b' : B$ with $a' = fb'$ and $b' \prec b$. By the induction hypothesis for B at b, this satisfies $\psi b'$, which is $\phi(fb')$ or $\phi a'$.

Hence we have proved the induction hypothesis for ϕ on A at $a \equiv fb$. It follows from the induction premise for A that $\phi a \equiv \phi(fb) \equiv \psi b$. Therefore we have proved that

$$\forall b. \quad (\forall b'.\ b' \prec b \Rightarrow \psi b') \Longrightarrow \psi b,$$

which is the induction premise for B. $\square$

Corollary 2.7 Surjective simulations preserve well-foundedness.

Proof If $(B, <)$ is well founded, f is surjective and ϕ obeys the induction premise for A in the Lemma then $\forall b.\ \psi b$ and $\forall a.\ \exists b.\ a = fb$, whence $\forall a.\ \phi a$. $\square$

The other side of the main result is recursion:

Definition 2.8 A *coalgebra* $\alpha : A \longrightarrow\!\!\!\triangleright\ TA$ obeys the ***recursion scheme*** if, for every *algebra* $\theta : T\Theta \to \Theta$, there is a unique map $f : A \to \Theta$ such that the square

$$\begin{tikzcd} TA \arrow[r, "Tf"] & T\Theta \arrow[d, "\theta"] \\ A \arrow[u, "\alpha"] \arrow[r, "f"] & \Theta \end{tikzcd}$$

commutes.

To obtain ***parametric recursion***, in which the top line is replaced by

$$Tf \times \mathsf{id}\ : TA \times A \longrightarrow T\Theta \times A,$$

we just need to make Lemma 6.6 a bit more complicated. In fact Mikkelsen had an even more general recursion scheme than this, although still with $T \equiv \mathcal{P}$ [Mik76, pp 98–99] [Osi75, Def 6.2]. Osius's account of categorical set theory [Osi74] largely used recursion instead of well-foundedness (induction). The notion is a *scheme* because we only ever consider *particular* algebras (Θ, θ). A map of this kind has also been called a ***coalgebra-to-algebra homomorphism*** [Epp03].

Example 2.9 The predecessor and test for zero function define a coalgebra on $\mathbb{N}$ for the functor $TX \equiv \mathbf{1} + X$ on $\mathcal{S}$. Then recursion defines $f : \mathbb{N} \to \Theta$ by the two cases

$$f0 \;=\; \theta(\star) \qquad \text{and} \qquad fn \;=\; \theta\big(f(n-1)\big).$$

This idiom dates from the 1650s or maybe even earlier. $\square$

In a topos, well-foundedness is *necessary* for recursion [Mik76, p 100] [Osi75, Prop 6.3] [Tay99, Exercise 6.14]:

Proposition 2.10 In a topos, if $\alpha : A \longrightarrow\!\!\!\triangleright\ TA$ obeys the recursion scheme then it is well founded.

Proof The subobject classifier (set of truth values) $\Theta \equiv \Omega \equiv \mathcal{P}(1)$ carries an algebra structure for any operation whatever, namely by interpreting it as (infinitary) *conjunction* or universal quantification. Then

$$fa \iff \big(\forall x.\, x \prec a \Rightarrow fx\big),$$

which is the strict ($\Leftrightarrow$) version of the induction premise. On the other hand, it is easy to see that the constant function $f : a \mapsto \top$ satisfies the recursion property in this case, so *uniqueness* of f amounts to the induction scheme.

$$\begin{tikzcd}
H \arrow[r] \arrow[d, tail] \arrow[dr, phantom, "\lrcorner", very near start] & TU \arrow[r] \arrow[d, tail, "Ti"] \arrow[dr, phantom, "\lrcorner", very near start] & T1 \arrow[r] \arrow[d, tail, "T\top"'] \arrow[dr, phantom, "\lrcorner", very near start] & 1 \arrow[d, tail, "\top"] & U \arrow[l] \arrow[d, tail, "i"] \arrow[dl, phantom, "\llcorner", very near start] \\
A \arrow[r, "\alpha"] & TA \arrow[r, "Tf"] & T\Omega \arrow[r, "\theta = \chi_{T\top}"] & \Omega & A \arrow[l, "f"']
\end{tikzcd}$$

This argument generalises. Let $\theta : T\Omega \to \Omega$ be the characteristic function of the subset $T\top : T1 \rightarrowtail T\Omega$, where $\top : 1 \rightarrowtail \Omega$ is the element "true". The induction premise is $\alpha\,; Tf\,; \theta \Rightarrow f$ and the strict premise has equality (bi-implication), but this is also satisfied by the constant function with value $\top$. ☐

Remark 2.11 This result should be treated with circumspection, because taking the object of truth values as the target algebra means that we are relying on *higher order logic*. (This point is obscured classically by the identification of Ω with a discrete two-element set.)

For example, induction for the predicate $\phi x \equiv (x \not\prec x)$ shows that well founded relations must be *irreflexive*, and therefore too clumsy to analyse *fixed* points of iteration.

On the other hand, experience shows that we must count ourselves lucky to find a condition for termination of a heavily recursive program which is *sufficient* for the case at hand: asking for it to be *necessary* as well is too much.

In Remark 1.6 we made an *analogy* between set theory and term algebras. The tools that we have already show us how to formalise this:

Lemma 2.12 Let $\kappa : T \to P$ be a natural transformation whose naturality squares are pullbacks, between endofunctors of a category with pullbacks. (Such κ is called a ***cartesian transformation***.) Then $\alpha : A \longrightarrow TA$ is a well founded T-coalgebra iff $\alpha\,; \kappa_A : A \to PA$ is a well founded P-coalgebra. If κ_A is mono then the notions of extensionality coincide too.

$$\begin{tikzcd}
PU \arrow[rr, tail, "Pi"] & & PA \\
TU \arrow[u, "\kappa_U"] \arrow[rr, tail, "Ti"] \arrow[urr, phantom, "\urcorner", very near start] & & TA \arrow[u, "\kappa_A"'] \\
H \arrow[u] \arrow[r, tail] \arrow[ur, phantom, "\urcorner", very near start] & U \arrow[r, tail, "i"] & A \arrow[u, "\alpha"']
\end{tikzcd}$$

Proof Since the upper rectangle is a pullback, the whole diagram is one iff the lower rectangle is. That is, the induction hypothesis H is the same for P as for T [Tay96b, Prop 7.4]. ☐

Proposition 2.13 Let $T : \mathcal{S} \to \mathcal{S}$. Then there is a natural transformation $\kappa : T \to \mathcal{P}$ which is cartesian with respect to monos (as above) iff T preserves arbitrary intersections.

Such T was called an ***analytic functor*** in [Joy87] since it has a "power series" representation.

Proof For any set X and $t : TX$, define $\kappa_X : TX \to \mathcal{P}X$ by

$$\kappa_X(t) \quad \equiv \quad \bigcap\{U \subset X \mid t \in TU\} \quad \equiv \quad \{x : X \mid \forall U \subset X.\, t \in TU \Longrightarrow x \in U\}.$$

Now $(t, V) \in TA \times \mathcal{P}U$ lies in the pullback iff $V = \kappa_X(t) \subset U$. If T preserves intersections, this happens iff $t \in TU$.

$$\begin{array}{ccccccc}
\mathcal{P}(\bigcap_i U_i) & = & \bigcap_i \mathcal{P}U_i & \rightarrowtail & \mathcal{P}U_i & \rightarrowtail & \mathcal{P}A \\
\uparrow \kappa_{\bigcap_i U_i} & & \uparrow & & \uparrow \kappa_{U_i} & & \uparrow \kappa_A \\
T(\bigcap_i U_i) & \rightarrowtail & \bigcap_i TU_i & \rightarrowtail & TU_i & \rightarrowtail & TA
\end{array}$$

The condition is necessary because $\mathcal{P}$ preserves arbitrary intersections and pullbacks commute with them [Tay96b, Prop 7.5]. $\square$

This paper is not restricted to **Set** or an elementary topos $\mathcal{S}$: in Section 4 we shall give a detailed statement of how general the category $\mathcal{C}$ may be on which the functor and coalgebras are defined. However, in the common situation when there is a "forgetful" functor $\mathcal{C} \to \mathcal{S}$ that provides a strong relationship between subobjects in the two categories, results like the two preceding ones will mean that well founded *coalgebras* in $\mathcal{C}$ are often given simply by well founded *relations* on the underlying set ($\mathcal{S}$-object) that interact appropriately with the structure in $\mathcal{C}$.

3 Fixed point theorems

I asked a MathOverflow question suggesting that the key results at the end of this section might have been identified and applied elsewhere. Arguably, this section (with the question folded in) should be moved to a "Prependix" at the top of the paper.

Although the main goal of this paper is a *categorical* result, we need to consider the *order*-theoretic analogue, which we will use to prove the main categorical theorem. The essential result is Pataraia's fixed point theorem at the end of this section, but the ideas are threaded through the whole of the twentieth century.

Notation 3.1 Throughout this section, let $s : X \to X$ be a monotone (order-preserving) endofunction of a partial order $(X, \leq)$. An element $x : X$ is called a ***pre-fixed point*** if $x \leq sx$ and a ***post-fixed point*** if $sx \leq x$; these are the order-theoretic forms of coalgebras and algebras respectively.

We will also assume that X has a least element $\perp$ and joins ($\bigvee$) of all subsets $I \subset X$ of one of the following forms:

(a) arbitrary ones, so X is a ***complete (join semi-)lattice***;

(b) ***totally ordered*** ones or ***chains***, for which $\forall xy \in I.\, (x \leq y) \vee (y \leq x)$; or

(c) ***directed ones***, written $\bigvee^{\uparrow}$, with $\exists z.\, z \in I$ and $\forall x, y \in I.\, \exists z \in I.\, x \leq z \geq y$.

A ***dcpo*** is a partial order that has all directed joins and an ***ipo*** also has a least element.

The simplest and best known order-theoretic fixed point theorem is usually called after Alfred Tarski, but it was apparently first announced by Bronisław Knaster at a meeting of the Polish Mathematical Society in Warsaw in 1927. The *Comptes Rendues* [Kna28] say just "$h(X)$ étant une fonction monotone d'ensembles et A un ensemble tel que $h(A) \subset A$, il existe un sous-ensemble D de A tel que $D = h(D)$".

Theorem 3.2 Any monotone endofunction $T : L \to L$ of a complete (meet semi-)lattice has a least (post-)fixed point.

Proof Firstly, a partial order has all meets iff it has all joins, the construction of which is implicit in the alternative names *greatest* lower bound and *least* upper bound respectively.

The subset $Z \equiv \{b : L \mid Tb \leq b\}$ is closed under meets because if $\forall i.\, Tb_i \leq b_i$ then $f \bigwedge b_i \leq \bigwedge Tb_i \leq \bigwedge b_i$. So $a \equiv \bigwedge Z$ has $a \leq Ta \leq a \leq Z$. □

Tarski's account [Tar55] gives many embellishments, such as that the fixed points themselves form a lattice. The applications are now so widespread in mathematics that it has become simply part of the narrative language and it seems pedantic to spell it out:

Lemma 3.3 Let $s : X \to X$ be as in Notation 3.1. Then there is a smallest subset $X_0 \subset X$ that contains $\perp$ and is closed under s and whichever joins we are considering; we say that X_0 is ***generated*** by these things.

Then $s : X_0 \to X_0$ is ***inflationary***:

$$\forall x. \quad x \in X_0 \Longrightarrow x \leq sx.$$

Also, if $x \in X_0$ and $sy \leq y$ then $x \leq y$.

Proof Apply the Knaster–Tarski Theorem to $T : L \equiv \mathcal{P}X \to L$, where, for $U \subset X$,

$$TU \quad \equiv \quad U \ \cup \ \{sx \mid u \in U\} \ \cup \ \{\bigvee V \mid V \subset U\},$$

the joins in the last uniend being restricted to those under consideration.

The other two parts are proved by *induction* on the construction of X_0. Consider the subsets

$$Y \ \equiv \ \{x : X \mid x \leq sx\} \qquad \text{and} \qquad Z \ \equiv \ \{x : X \mid x \leq sx \ \&\ x \leq y\}.$$

Clearly $\perp \in Y, Z$. If $x \leq sx$ then $sx \leq ssx$, whilst if also $x \leq y$ then $x \leq sx \leq sy \leq y$.

If $\forall i.\, x_i \in Y, Z$ and this family is totally ordered or directed as the case may be then

$$\bigvee x_i \ \leq \ \bigvee sx_i \ \leq \ s \bigvee x_i \qquad \text{and} \qquad \bigvee x_i \ \leq \ y.$$

Therefore Y and Z are closed under $\perp$, s and appropriate $\bigvee$. But X_0 was the least such subset, so $X_0 \subset Y, Z$, which is what the claims say. □

Corollary 3.4 If there is an element $x : X$ that is

(a) the greatest element of X_0,
(b) $x = sx \in X_0$,
(c) the least fixed point of s on X, or
(d) the least post-fixed point,

then it also has the other three properties and it's unique.

Proof The Lemma gives [b⇒c] and [d⇒a], whilst [a⇒b] and [c⇒d] use the fact that sx has the same property as x, but x is the greatest or least such. □

Each of the arguments below seeks to show that X_0 is itself a subset of the same kind as the joins that were used to define it in Notation 3.1. This is trivial in the first case, where X has *all* joins. However, all (and in particular binary) joins will not be available in the application to the more general recursion theorem that we shall prove.

One way to handle this problem would be to use recursion over the ordinals, but that would be begging the question (Remark 1.5). Anyway, I have found no account from the period 1928–49 when such a proof of the fixed point theorem would have merited a place in the literature, *i.e.* citing [Har15, vN28] but pre-dating the next result. In any case such a proof would not be "constructive".

The least fixed point of some construction is usually just a tool for finding the *unique* solution of some equation or for showing that some system that has a least member and directed unions but not necessarily binary ones in fact has a *greatest* element.

The classical ordinals are a sledgehammer to crack these nuts: we would really like to find some more direct characterisation of the subset X_0 of iterates. This requires some ingenuity — so there always remains scope for new ideas! The next result was found by the Bourbaki group [Bou49] and Ernst Witt [Wit51], although the idea had already been present in Zermelo's second proof that choice implies well-ordering [Zer08a].

Proposition 3.5 Let $X_0 \subset X$ be generated by $\bot$, $s : X \to X$ and joins of chains. Then, classically,

$$\forall x, y \in X_0. \qquad x < y \implies sx \leq y.$$

Therefore X_0 is itself a chain and so has a join. By Corollary 3.4, this is the greatest element of X_0, the sole fixed point of s on X_0 and the least one on X. Moreover, the subset X_0 is well ordered. ☐

Remark 3.6 The proof uses excluded middle because it involves the step

$$\forall i. \left(x_i \leq y \ \vee \ sy \leq x_i\right) \quad \implies \quad \forall i. \left(x_i \leq y\right) \ \vee \ \exists i. \left(sy \leq x_i \leq \bigvee x_j\right).$$

Andrej Bauer and Peter LeFanu Lumsdaine have investigated the difference between joins of chains and of directed subsets in the constructive setting of the effective topos [Bau09, BL12].

Even though the Bourbaki–Witt Theorem is not constructive, it does suggest a pattern for more lightweight idioms. It says that X_0 has a top element and this is the unique fixed point, but unfortunately it doesn't characterise this subset any better than by the original recursive problem (Lemma 3.3). We might also object to the use of the impredicative second order logic ("smallest subset") to define it.

The following provide alternatives; they are the order-theoretic analogues of our two principal definitions, well-foundedness (Definition 10.2) and the recursion scheme:

Lemma 3.7 The subsets

$$\begin{aligned} X_1 &\equiv \{x : X \mid x \leq sx \quad \& \quad \forall u.\, su \wedge x \leq u \Longrightarrow x \leq u\} \\ X_2 &\equiv \{x : X \mid x \leq sx \quad \& \quad \forall u.\, su \qquad \leq u \Longrightarrow x \leq u\}, \end{aligned}$$

satisfy

$$X_0 \ \subset \ X_1 \ \subset \ X_2.$$

The function s restricts to each of them and is inflationary. If any of the three has either a top element or a fixed point of s then it plays both roles in all three of them.

Proof For X_2, this follows easily from Lemma 3.3. The inclusion $X_0 \subset X_1$ is the order-theoretic form of results in Section 5, so we leave it as an exercise; of course $(X, \leq)$ must also have meets in order to define it. ☐

However, none of these properties of *elements* of X, whether using first, second or higher order logic, has managed to yield an intuitionistic proof of the fixed point theorem for ipos. The breakthrough came when Dito Pataraia considered *functions* instead. In keeping with tradition in this topic, he presented the result at an informal conference, the 65th Peripatetic Seminar on Sheaves and Logic, in Aarhus in 1997, but never wrote it up himself before his untimely death in 2011 at the age of 48.

Proposition 3.8 Any dcpo $(X, \leq)$ has a greatest inflationary monotone endofunction, $t : X \to X$. This is idempotent and its fixed points are exactly the points that are fixed by *all* inflationary monotone endofunctions.

Proof Consider the set

$$Y \quad \equiv \quad \{f : X \to X \mid (\forall x.\, x \leq fx) \wedge (\forall xy.\, x \leq y \Rightarrow fx \leq fy)\}$$

of inflationary monotone endofunctions of X. In the pointwise order, this inherits directed joins from (the values in) X, and id_X is the least element, so Y is an ipo.

For any $f, g \in Y$, the composites $f\,;g$ and $g\,;f$ both lie above both f and g in Y, because

$$\forall x. \quad x \quad \leq \quad fx,\ gx \quad \leq \quad f(gx),\ g(fx),$$

using both the inflationary and monotone properties. Hence the whole dcpo Y is directed.

Since Y is also directed-*complete*, it therefore has a greatest element, $t : X \to X$.

(For this, we are taking the join over the whole of Y, which is impredicative and is the order-theoretic form of the well powered requirement on categories.)

For any $f \in Y$, the composites $f\,;t$ and $t\,;f$ are in Y too, so $f\,;t \geq t \leq t\,;f$ by the previous argument, but also $f\,;t \leq t \geq t\,;f$ since t is the greatest element of Y. Hence $f\,;t = t = t\,;f$ and in particular $t = t\,;t$.

Finally, if $a = ta$ then $fa = f(ta) = ta = a$ for any $f \in Y$. □

Theorem 3.9 Any monotone endofunction $s : X \to X$ of an ipo has a least fixed point.

Proof Applying the Proposition to any of X_0, X_1 or X_2, we deduce that $a \equiv t\bot = tt\bot = ta$ is the top element and sole fixed point of s in these subsets. □

As we have hinted in our remarks about Tarski, Bourbaki and Witt, we can get a lot more milage from Pataraia's Theorem than just least fixed points. For uniqueness, we want to reduce Tarski's *lattice* of fixed points to just one. A simple extra condition does this, and it is one that will be convenient to verify in our applications. Indeed, the proof of the Theorem was actually an application of this Corollary, because X_0, X_1 and X_2 have this property.

Corollary 3.10 Let $s : X \to X$ be an inflationary monotone endofunction of an ipo such that $a = sa \leq b = sb \Longrightarrow a = b$. Then X has a top element and this is the unique fixed point of s.

Proof Let $a \equiv t\bot = tt\bot$, so $a = sa$ in the Proposition as before. For any $x : X$ we have $\bot \leq x$, so $t\bot = s(t\bot) \leq tx = s(tx)$. Hence $x \leq tx = t\bot$ by the hypothesis, which means that $t\bot$ is the top element. It is also the unique fixed point of s. □

We can also deduce properties of this solution, using a principle that we call ***Pataraia induction***:

Corollary 3.11 Let $s : X \to X$ be a monotone endofunction of an ipo and $U \subset X$ a subset containing $\bot$ and closed under s and directed joins. Then U also contains the (same) least fixed point of s in X.

Proof The set U has the same properties as X in the Theorem, so it contains a fixed point of s, but this must be the same as the one for the whole of X. □

Pataraia induction was also identified by Martín Escardó [Esc03, Thm 2.2] but it must surely lie hidden in many other places in the literature, given that it is the intuitionistic "drop-in" replacement of a classical result.

4 Categorical requirements

We have said that our theory applies to an endofunctor T that preserves monos, but we have said nothing about what we require of the category $\mathcal{C}$ on which it acts. Beyond that, as we generalise further and further away from **Set**, we find that they have many different kinds of "inclusions" that (have but) are not necessarily characterised by the standard cancellation property that defines "monos"in a category.

We address those questions in this section, but really this is a technical analysis of the proof to follow. Therefore, even if you are proficient in categorical logic, it would still be better to understand the proof in the next two sections *grosso modo* before reading this one, so that you can see why the following subtleties are needed.

So, on first reading, you should simply take $\mathcal{C} \equiv \mathcal{S} \equiv \mathbf{Set}$, read both arrowtails as injective functions and assume that the functor $T : \mathbf{Set} \to \mathbf{Set}$ preserves them. Then you may omit this section.

The simplest statement of more general but sufficient conditions is this:

Provisional assumption 4.1 The category $\mathcal{C}$
(a) has inverse images (pullbacks) of monos along coalgebra homomorphisms;
(b) has an initial object $\emptyset$ and all maps $\emptyset \to X$ are mono;
(c) has directed (Notation 3.1(c)) unions of subobjects, and
(d) is well powered, whilst
(e) the functor $T : \mathcal{C} \to \mathcal{C}$ preserves monos.

Besides defining "unions" and "well powered", we also need to examine all of these assumptions more carefully.

We use some other finite limits in $\mathcal{C}$ incidentally, *i.e.* not as part of the proof of our main theorem: Lemma 6.7 uses binary products to show how parametric recursion is handled. Lemma 6.4 uses equalisers to prove uniqueness of recursion, but we can deduce that in another way, without using them. The terminal object $\mathbf{1}$ is never used.

We will restrict the "monos" in (a,e) to some special class used for "predicates" and those in (b,c,d) to another possibly smaller class of "initial segments".

Remark 4.2 Any category of finitary algebras satisfies (a,c,d) but part (b) is more delicate. Recall from universal algebra that, in an appropriately constructed category of algebras, the ***initial object*** typically arises as the collection of terms *generated* by a given set of symbols (*cf.* Remark 1.6).

We can mimic this for any object I of any category: Working instead with the (coslice or cocomma) category whose objects are monos $I \hookrightarrow X$ and whose morphisms are commutative triangles, the initial object is id_I and all maps out of it are monos. This construction leaves the other provisional assumptions intact, because the subobjects, inverse images and directed unions in the coslice are essentially the same as those in the original category.

For example, the category of fields does not meet our requirements, but cutting it down to *those of a particular characteristic* does: the characteristic selects one of the components of the category and then $\mathbb{Q}$ or $\mathbb{F}_p$ is the initial object.

We also need to fix the characteristic if we want to work with rings (or commutative rings), because that ensures that all maps from the initial object ($\mathbb{Z}$ or $\mathbb{Z}_n$) are mono.

The opposite of the category of sets provides another example, so long as we omit the empty set (which would be the terminal object) and assume the axiom of choice (to obtain the union property).

The reason why maps out of the initial object need to be mono is to make the initial *object* serve as the least *subobject* of each object (which $\mathbb{Z}$ does not do for all rings).

More generally, in order to combine attempts (partial maps) as in Theorem 1.4, we need to make the colimits of monos in the category behave like unions of subobjects of each object. So we first need to be clear what *unions* are in general:

Definition 4.3 A ***union*** in a category is the colimit of a diagram such that
(a) the maps in the diagram are mono;
(b) the maps in the colimiting cocone are mono;
(c) for any other cocone consisting of monos, the colimit mediator is also mono,

Proposition 4.4 Set (or any topos $\mathcal{S}$) has directed unions.

Proof (Sketch) A colimit in **Set** is given by the quotient of a coproduct by an equivalence relation that is obtained from the diagram. The different components of a coproduct are disjoint.

Two elements are identified in the colimiting cocone iff they are linked by a finite zig-zag in the relation. Since the diagram is directed, it has some further stage that is a cocone (within the diagram) over the zig-zag. This cocone consists of monos, so the two elements were already equal.

Now consider the kernel (pullback against itself) of the mediator to any other cocone of monos. Since colimits are stable under pullback, this kernel is a doubly-indexed union. But since the diagram is directed, this is equivalent to a singly-indexed union, which is in fact the original diagram. Hence the projections from the kernel are isomorphisms and so the mediator is mono. □

We give the result for pushouts in a (pre)topos more formally in Lemmas 9.5 and 9.6. In other categories, the second part of the argument shows that mediators are mono (Definition 4.3(c)) if colimits are stable under pullback, but this is not sufficient for other kinds of inclusions.

Definition 4.5 A category is ***well powered*** if, for each object X, there is a "set" of isomorphism classes of monos $U \hookrightarrow X$.

On the face of it, the word "set" is an embarrassment, given that we aim to eliminate Set Theory from mathematical foundations. But as mathematicians we pay our words double to mean what *we* want them to mean. In general, we do this by listing the ways in which we intend to use them, *i.e.* the axioms.

A "set" of objects is not a chaotic jumble but a *single* object that is *dependent* on some *parameter*. In the geometric tradition, this arose as the object (such as a tangent space) *varied* from one place to another in a space. In type theory (and indeed longstanding symbolic usage in real analysis), it simply means a formula containing an unknown.

What we require of dependency is just to be able to *substitute* other formulae for the unknown parameter. This parameter has a certain *type*. These types and their formulae form a category $\mathcal{S}$, called the ***base***, which may be **Set**, an elementary topos or even something simpler. Then, for each type Γ in $\mathcal{S}$, the objects whose parameter is of type Γ together form the ***fibre*** over Γ.

Substitution of a formula for a parameter (or along a morphism f) is an operation f^* on dependent objects. There are two techniques for capturing how f^* takes one fibre to another:
(a) if we consider the fibres as *separate* structures, f^* is the morphism part of a *functor* that is contravariant in f, giving an ***indexed*** structure; but
(b) the fibres may be combined into a single structure, called a ***fibration***, in which f^* acts by *pullback*.

The account that develops well-poweredness in most detail, in the indexed style, is [PS78], although its goal is the adjoint functor theorem rather than our needs. The indexed approach has to contend with choices of isomorphic objects, which the fibred one avoids, but at greater learning cost. Unfortunately, both techniques have rather obscure notation and huge diagrams, so, since we have some very complicated ones already, we will just give a verbal description of how they work.

Definition 4.6 A ***generic*** object G is a parametric one that has the universal property that any *particular* object P (in fact, just one with other parameters) is obtained as $P \cong f^*G$, by substitution into the generic one, for some unique morphism f in the base category $\mathcal{S}$.

We call f the ***name*** of P. The type of names (parameters) belongs to the base category $\mathcal{S}$ and the generic object belongs to the fibre over this type. In particular, when the "objects" in question are monos $i : U \rightarrowtail X$ targeted at a particular object X of $\mathcal{C}$, the type of names is called $\mathsf{Sub}\,(X)$ and the generic subobject of X belongs to the fibre over this type.

Using the definition of genericity, any external structure that respects substitution induces an ***internal structure*** on the type of names in $\mathcal{S}$. For example, triangles of monos in $\mathcal{C}$ give rise to an internal order on $\mathsf{Sub}\,(X)$. In this sense we say that the external structure is ***equivalent*** to an internal one.

It's instructive to draw a few of these diagrams to show how, for example, pullbacks in $\mathcal{C}$ yield meets in $\mathsf{Sub}\,(X)$, making it an internal semilattice in $\mathcal{S}$. Then you will see that $\mathsf{Sub}\,(X)$ is like the handle of a marionette, with manoeuvres matching the actions of the doll. With practice, we can just describe what the doll does, so long as we remember *how* it does it: the diagram of strings conveys comparatively little information per cm^2 and is not really needed. In fact, the doll is *well powered* exactly when it is *impotent*, being able to do no more nor less than the puppeteer makes it do.

Nevertheless, there is perhaps a PhD in collecting and formalising the applications of well powered categories, analogous to those by Osius and others on the logic of a topos.

Corollary 4.7 Any construction on a generic object that respects substitution corresponds uniquely to a morphism of the base category. In particular, the construction of one subobject of X from another corresponds to an endomorphism of $\mathsf{Sub}\,(X)$.

Proof An operation on a parametric object yields another object with the same parameter, *i.e.* in the same fibre, whilst binary operations such as categorical products combine the parameters using pullbacks in $\mathcal{S}$. We then use the universal property of the *generic* object of the resulting kind to define the morphism of the base category. $\square$

So far we have only discussed *finitary* structure such as composition and pullback. The original reason for requiring a "small" set of subobjects was so that we could legitimately form their union.

Proposition 4.8 External $\mathcal{S}$-indexed unions in $\mathcal{C}$ correspond to joins in $\mathsf{Sub}\,(X)$.

Proof Any of the accounts of indexed and fibred categories explains how they handle colimits. Of course the "set" of objects of which we form the colimit is a single parametric one as before. In fact, the union operation is the left adjoint to the substitution functor and has an even simpler characterisation in that the opposite of the fibration functor is also a fibration.

The universal property of the generic subobject translates this into a join in the internal poset $\mathsf{Sub}\,(X)$. $\square$

Remark 4.9 Pataraia's Theorem 3.9 is for *internal* ipos in $\mathcal{S}$. The role of the union and well powered conditions that we have described is to provide an *equivalence* amongst external colimits and unions and internal joins. The same link also relates constructions in $\mathcal{C}$ to morphisms between objects of $\mathcal{S}$. In particular, the "relative successor" that we construct in the category in Constructions 5.6ff and 6.6 corresponds to a monotone inflationary endofunction of the internal ipo. This has a fixed point, which we translate back into the category as an object on which the construction yields an isomorphic object. □

Remark 4.10 There is yet another reason why we need a "set" of subobjects, namely to justify universal quantification over them as predicates. (In Set Theory this distinction is known as ***unbounded*** *versus* ***bounded quantification***.)

When we introduced well-foundedness in Definitions 1.1 and 2.4, we called it a *scheme*, which means a property that we assert for *each individual* predicate ϕ. We will develop the *general* theory of well-foundedness in this way.

On the other hand, when we come to *apply* well-foundedness in the proof of our main theorem, we need it to be a *single* legitimate property in the logic of an elementary topos. For this it cannot be a scheme but must be *quantified* over all predicates ϕ.

Once again, by a "set" of predicates we mean a single generic predicate with a parameter. Well-foundedness with respect to a particular predicate ϕ is expressed in $\mathsf{Sub}\,(X)$ as above, with a parameter ϕ. Universal quantification over ϕ is now the *right* adjoint to substitution for ϕ, as is amply explained in the topos literature. □

Remark 4.11 We now turn to investigating the classes of "inclusions" that we might use in place of ordinary categorical monos when applying our ideas to objects with richer structure than sets have. We will use inclusions for three purposes in this paper:

(a) as the extents of *predicates* that test well-foundedness;

(b) as the inclusions of subcoalgebras that are the *supports* of attempts; and

(c) as the structure maps of *extensional* coalgebras.

All supports must be predicates to prove totality of recursion (Lemma 5.8), whilst supports and extensionality are thoroughly mixed up in Construction 7.5, so we must treat these as the same thing. Therefore we potentially have *two* classes of inclusions, one contained in the other, and we write

$$\rightarrowtail \text{ for predicates} \quad \text{and} \quad \hookrightarrow \text{ for supports and extensionality.}$$

It is tempting (thinking in terms of so-called Descriptive Set Theory) to call $U \rightarrowtail X$ a *subspace* and $U \hookrightarrow X$ an *open* subspace of X, except that this need not be the same as an open subspace in whatever topology the object X might carry.

Beware that these two classes of monos are *additional structure* for the situation, along with the category $\mathcal{C}$ and functor T. Since our primary interest is likely to be in $\mathcal{C}$ and T, we are at liberty to choose the two classes of monos in whatever way yields the optimum results, although we may then want to show that these are independent of the choices.

As you will see in the next section, we have some conflict in the objectives for this paper between proving the central recursion theorem and developing the whole theory of well founded coalgebras. For the general theory, we might typically want

(a) a large class of predicates so that we can make liberal use of induction, but

(b) a small class of supports.

For the proof of the recursion theorem, it turns out that we only do induction over the supports, so the two classes are the same.

While studying the order-theoretic fixed point theorems, we introduced X_1 and X_2 in Lemma 3.7 in order to find some subset about which we could reason easily, but which was an approximation to the less tractable set X_0 of iterates of the endofunction s. Likewise, in the categorical theory, we would like to find a usable class of extensional well founded coalgebras that contains the iterates of the functor T applied to the initial object, as tightly as possible.

Therefore, in the particular application category, we would like to find some notion of inclusion that is both tractable and restrictive.

It is straightforward to substitute these chosen inclusions for the "monos" in the definitions above of unions and being well powered. However, Proposition 4.4 only works for plain monos and so needs to be replaced with some other argument, which is why we formulated Definition 4.3 instead of just asking that colimits be stable under pullback.

It may be possible to control the unions even further, such as by making the diagrams computable in some sense, using techniques from various forms of synthetic domain theory, but we leave that for another day.

For the general theory we do distinguish between the classes and so need to axiomatise them separately. In doing this, it is convenient to make an auxiliary definition for the closure conditions that are common to both classes:

Definition 4.12 A ***class of*** T***-monos*** $\mathcal{M}$ must
(a) contain all isomorphisms;
(b) contain all maps from the initial object (*cf.* Remark 4.2);
(c) be closed under composition;
(d) be preserved by the functor T;
(e) be preserved by pullback along coalgebra homomorphisms; and
(f) satisfy the cancellation property for plain monos, $\forall f g.\ \ f\,;i = g\,;i \Longrightarrow f = g$.

The reason why we need the cancellation property is this: For the initial object $A \equiv \emptyset$, there is a map $p : A \to U$ with $p\,; m = \mathsf{id}_A$. The ubiquitous idiom in using well-foundedness gives the same thing. We use the cancellation property to deduce that $m\,; p = \mathsf{id}_U$.

Another easy but useful property that is also known as ***cancellation*** may be deduced as a "warm-up" exercise in the kind of diagram-chasing that we shall use throughout this paper:

Lemma 4.13 For any class of T-monos $\mathcal{M}$,
(a) if $i\,; m \in \mathcal{M}$ and m is a (plain) mono then $i \in \mathcal{M}$ too; and
(b) if the predicate $(i : U \rightarrowtail X) \in \mathcal{M}$ in the broken pullback for the induction premise (Definition 2.4) then $(j : H \rightarrowtail U) \in \mathcal{M}$ too.

Proof Hint: The maps id, $(i\,; m)$, i and m form a pullback square. $\square$

We are now ready to state the conditions for the two classes:

Assumption 4.14 The maps $\rightarrowtail$ used for predicates form a class of T-monos $\mathcal{M}$ that must
(a) include all inclusions of initial segments $\hookrightarrow$; and
(b) every map $i \in \mathcal{M}$ must belong to some well-powered subclass $\mathcal{M}' \subset \mathcal{M}$ of T-monos.

For additional results beyond the main recursion theorem, the class could
(c) include all regular monos (equalisers, *cf.* Lemma 6.4); or
(d) admit inverse image operators f^* applied to predicates have right adjoints f_* (Section 9).

Recall that, in categorical logic, inverse images correspond to substitution, equalisers to equations, composition of monos to existential quantification and the right adjoint f_* to universal quantification, *cf.* Notation 2.2. The conditions above are therefore natural and very flexible for considering precise restrictions on the logical strength of the predicates over which we may perform induction. This is possible (contrary to what was said in [Tay96b, Prop 6.7]) because we are making a distinction between the roles of predicates and initial segments.

Assumption 4.15 The maps $\hookrightarrow$ used for inclusions of subcoalgebras and for structure maps of extensional coalgebras must form a class of T-monos (Definition 4.12) that

(a) is contained in the class of used for predicates;

(b) admits directed unions (Definition 4.3); and

(c) is well powered (Definition 4.5).

Remark 4.16 The monos in this class will often also be coalgebra homomorphisms and so will be written $\hookrightarrow\!\!\triangleright$. We will just call them ***initial segments***, to exploit the intuition from ordinals. However, the two ends of the arrow signify different things:

(a) the triangle arrow*head* ($\longrightarrow\!\!\triangleright$) says that the map is a coalgebra homomorphism, which captures the *traditional* order-theoretic ideas (*cf.* Remark 2.3); whilst

(b) the hook *tail* ($\hookrightarrow$) says that the underlying $\mathcal{C}$-map belongs to a special class of monos: this aspect is a *novelty* in this paper.

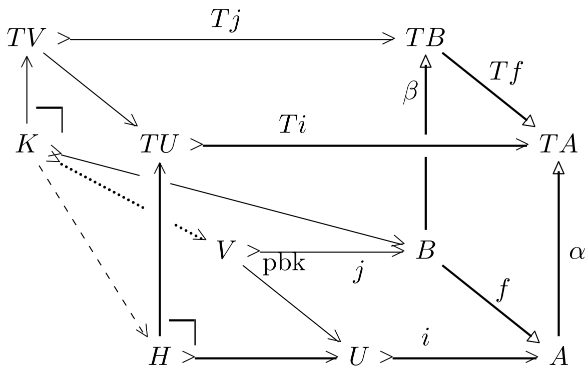
5 Generating well founded coalgebras

Before tackling the recursion theorem itself, we study how well founded coalgebras are built up. (Proposition 1.3 is another way of proving well-foundedness, but we postpone it to Section 9 because it depends on stronger hypotheses about the functor.)

As we explained in the previous section, the notion of well-foundedness depends on the class of *predicates* over which we choose to allow induction. If you skipped that section, you may also ignore the discussion of such things below and take both arrowtails to indicate ordinary monos.

The first lemma is the categorical proof of Lemma 2.6:

Lemma 5.1 The induction premise (broken pullback) is stable under pullback against coalgebra homomorphisms. (Beware that we are saying nothing about i being an isomorphism.)



Proof The thick lines show the homomorphism $f : B \longrightarrow\!\!\triangleright A$ and the given induction premise for the predicate $i : U \rightarrowtail A$.

Let $j : V \rightarrowtail B$ be the inverse image of i along f. Apply T to this pullback to give the parallelogram at the top, although we are not assuming that this is a pullback.

Form the inverse image $K \rightarrowtail B$ of Tj along β, so that K is the induction hypothesis for $V \rightarrowtail B$.

The top, back and right quadrilaterals commute (from K to TA), so there is a pullback mediator $K \to H$ that makes the left and bottom quadrilaterals commute, *i.e.* from K to TU and A. This map deduces the induction hypothesis for U from that for V.

Because of this, there is a pullback mediator $K \to V$ that makes everything commute, in particular from K to B, which is the required induction premise. $\square$

Von Neumann's proof of the recursion theorem for ordinals (Theorem 1.4(b,h)) forms the union of attempts, so we consider colimits next. (See Definition 4.3 for the relationship between colimits and unions.) Note, however, that we are merely *enhancing* the properties of those that *already* exist in the category $\mathcal{C}$, not asking for extra ones. Although we state the Proposition for general colimits, we only use directed unions in our main proof of the recursion theorem, but we will consider pushouts in Section 9.

Lemma 5.2 The initial object $\emptyset$ of $\mathcal{C}$ carries a unique T-coalgebra structure, which is well founded and is the least subcoalgebra of any coalgebra.

Proof Easy, but *cf.* Theorem 1.4(a), Remark 4.2 and Definition 4.12(b,f). $\square$

Proposition 5.3 The forgetful functors **WfCoAlg** $\longrightarrow$ **CoAlg** $\longrightarrow \mathcal{C}$ create colimits. That is, the colimit of any diagram of coalgebras and homomorphisms is given by the colimit of their carriers, if this exists, and then the structure map is uniquely determined. If the individual coalgebras are well founded then so is their colimit (*cf.* Theorem 1.4(b)).

Proof The structure map α on the colimit is the colimit mediator, as shown in the diagram on the left, where the colimiting cocone consists of coalgebra homomorphisms, *i.e.* the parallelograms from A_i to TA commute.

$$\begin{tikzcd}
 & & TA \\
TA_i \arrow[r] \arrow[urr] & TA_j \arrow[ur] & \\
 & & A \arrow[uu, dotted, "\alpha"'] \\
A_i \arrow[uu, "\alpha_i"] \arrow[r] \arrow[urr] & A_j \arrow[uu, "\alpha_j"] \arrow[ur] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & TU \arrow[rrr, tail, "Ti"] & & & TA \\
TU_i \arrow[ur, dotted] \arrow[rrr, dotted, tail, "Tk_i"] & & & TA_i \arrow[ur] & \\
 & H \arrow[uu] \arrow[r, tail] & U \arrow[rr, tail, "k"] & & A \arrow[uu, "\alpha"'] \\
K_i \arrow[uu, dotted] \arrow[ur, dotted] \arrow[r, dotted, tail] & U_i \arrow[ur, dotted, "\text{pbk}"'] \arrow[rr, dotted, tail, "k_i"'] & & A_i \arrow[uu, "\alpha_i"] \arrow[ur] &
\end{tikzcd}$$

Now suppose that the α_i are well founded and let $k : U \rightarrowtail A$ be a predicate satisfying the induction premise for the colimit α (the upper rectangle, from H to TA).

Form the inverse images K_i of this induction premise against the homomorphisms $A_i \longrightarrow A$ of the colimiting cocone, using Lemma 5.1.

Since each A_i is well founded, $k_i : U_i \cong A_i$.

Now U is the vertex of a cocone over the diagram A_i, so it has a mediator from the colimit A, and $i : U \cong A$ as required [Tay96b, Prop 6.6]. $\square$

Corollary 5.4 The category of subcoalgebras of any coalgebra (A, α) and inclusions between them is equivalent to an $\mathcal{S}$-internal ipo $\mathsf{Seg}\,(A, \alpha)$. The well founded subcoalgebras form a subipo

$$\mathsf{WfSeg}\,(A, \alpha) \;\subset\; \mathsf{Seg}\,(A, \alpha)$$

of this, *i.e.* a subset ($\mathcal{S}$-subobject) that contains the least element and is closed under directed joins.

Proof Assumption 4.15, Lemma 5.2 and Proposition 5.3 provide the colimits in $\mathcal{C}$, **CoAlg** and **WfCoAlg**. However, we need Definition 4.3 to make these colimits agree with unions of subcoalgebras and then the well powered condition (Proposition 4.8) to link the external unions with the internal joins.

Finally, the well powered condition is used again to justify quantification over the class of predicates in the definition of well-foundedness (Remark 4.10); note here that, for the main recursion theorem, we will only use initial segments for these predicates. $\square$

The next four results study notions of "successor" for (sub)coalgebras. We leave the proof of the first as an exercise because it is a special case of Lemma 5.9 with $c \equiv \mathsf{id}$ and we don't actually use it. It would be instructive to work out Construction 5.6 in the case of $T \equiv \mathcal{P}$ and compare it with Theorem 1.4(f).

Lemma 5.5 The functor T preserves well founded coalgebras. $\square$

Construction 5.6 Let $i : (B, \beta) \hookrightarrow (A, \alpha)$ be a subcoalgebra. Then its ***relative successor*** $k : (C, \gamma) \hookrightarrow (A, \alpha)$ is given by pullback of α and Ti.

$$\begin{tikzcd}
& A \arrow[r, "\alpha"] & TA & \\
& C \arrow[u, hook, "k"'] \arrow[r, "c"] \arrow[drr, "\gamma"'] & TB \arrow[u, hook, "Ti"] \arrow[dr, hook, "Tj"] & \\
B \arrow[uur, hook, "i"] \arrow[ur, "j"] \arrow[urr, "\beta"] & & & TC \arrow[uul, "Tk"']
\end{tikzcd}$$

The pullback mediator $j : B \to C$ makes $(B, \beta) \hookrightarrow (C, \gamma) \hookrightarrow (A, \alpha)$ as subcoalgebras (initial segments) when we define $\gamma \equiv c\,; Tj$.

We will write sB for the relative successor (C); it is inflationary because $j : B \hookrightarrow sB$.

Proof Since the functor T and inverse images preserve initial segments (Definition 4.12) and the latter obey the cancellation property (Lemma 4.13), if i is an initial segment then so successively are Ti, k, j, Tk and Tj. Finally,

$$k\,; \alpha = c\,; Ti = c\,; Tj\,; Tk = \gamma\,; Tk \qquad \text{and} \qquad j\,; \gamma = j\,; c\,; Tj = \beta\,; Tj,$$

so j and k are coalgebra homomorphisms. $\square$

Lemma 5.7 The relative successor construction is monotone (functorial) in B.

$$
\begin{tikzcd}
 & A \arrow[r, "\alpha"] & TA & \\
B \arrow[ur, hook, "i"] \arrow[r, hook, "j"] & sB \arrow[u, hook, "k"'] \arrow[r, "c"] & TB \arrow[u, hook, "Ti"] \arrow[r, hook, "Tj"] & TsB \arrow[ul, hook', "Tk"'] \\
B' \arrow[u, hook, "\ell"] \arrow[r, hook, "j'"] & sB' \arrow[u, hook] \arrow[r, "c'"] & TB' \arrow[u, hook] \arrow[r, hook, "Tj'"] & TsB' \arrow[u, hook, "T\ell"']
\end{tikzcd}
$$

Proof Given initial segments $B' \hookrightarrow B \hookrightarrow A$, apply T and then pullback; the one giving sB' factors uniquely through the one for sB. □

The next result is actually the sole place in our proof of the recursion theorem where we use well-foundedness. Indeed, we only use the definition and none of the theory above. The predicate is the initial segment $i : B \hookrightarrow A$, *cf.* Theorem 1.4(g).

Lemma 5.8 In the previous diagram, if (B, β) is well founded and both it and (B', β') are fixed by the relative successor ($j : B \cong sB$ and $j' : B' \cong sB'$) then $\ell : B' \cong B$.

Proof B, TB, TB' and B' form a pullback. It is the one in Definition 2.4 of well-foundedness, except that $K = U = B'$. Therefore $B' \cong B$. □

In the case of the covariant powerset, any subcoalgebra of a well founded coalgebra is again well founded, by Proposition 1.3. Using this, we could deduce well-foundedness of C from that of TB and hence from that of B by Lemma 5.5. Since we have chosen to use weaker conditions in our account, we need a slightly more complicated result at this point, which we call the ***sandwich lemma***.

Lemma 5.9 Let (B, β) be a well founded coalgebra and $j : B \to C$ and $c : C \to TB$ maps such that $\beta = j\,;c$. Put $\gamma \equiv c\,;Tj$. Then (C, γ) is also a well founded coalgebra and j and c are homomorphisms.

Proof They are homomorphisms because

$$j\,;\gamma \;\equiv\; j\,;c\,;Tj \;=\; \beta\,;Tj \qquad \text{and} \qquad \gamma\,;Tc \;\equiv\; c\,;Tj\,;Tc \;=\; c\,;T\beta.$$

Now let $k : W \rightarrowtail C$ satisfy the induction premise given by the pullback H and form the inverse image of this along j, using Lemma 5.1. This gives the induction premise K for the predicate

$\ell : V \rightarrowtail B$:

$$\begin{tikzcd}
& TW \arrow[rrr, tail, "Tk"] & & & TC \\
TV \arrow[ur, dotted] \arrow[rrr, tail, dotted, "T\ell"] & & & TB \arrow[ull, dotted, "Th"'] \arrow[ur, "Tj"] & \\
& H \arrow[uu] \arrow[r, tail] & W \arrow[rr, tail, "k"] & & C \arrow[uu, "\gamma"'] \arrow[ul, "c"'] \\
K \arrow[uu, dotted] \arrow[ur, dotted] \arrow[r, tail, dotted] & V \arrow[ur, dotted] \arrow[rr, tail, dotted, "\ell"] & & B \arrow[uu, "\beta"] \arrow[ull, dotted, "h"'] \arrow[ur, "j"'] &
\end{tikzcd}$$

Since B is well founded, $\ell : V \cong B$ and so there is a map $h : B \to W$ making the triangle with C commute. The one with TB, TW and TC also commutes.

The top right triangle ($\gamma = c;Tj$) commutes too, so the maps $C \to TB \to TW$ and $\mathsf{id} : C \to C$ form a commutative square at TC. This factors through the pullback H, splitting the inclusion $H \rightarrowtail W \rightarrowtail C$ as required [Tay96b, Lemma 8.2]. $\square$

We sum up this section in two different ways. The simpler one provides exactly what we will require in the next section and only uses the *definition* of well-foundedness with respect to initial segments, not any of the theory that we have developed.

Proposition 5.10 For any well founded coalgebra (A, α), the relative successor defines an endofunction of the ipo $\mathsf{Seg}(A, \alpha)$ whose unique fixed point is the top element, A itself.

Proof The well powered requirement that we used to define the ipo in Corollary 5.4 also says that categorical constructions correspond to endofunctions of it (Corollary 4.7) By Lemmas 5.6 and 5.7, the relative successor therefore defines a monotone inflationary function $s : \mathsf{Seg}(A, \alpha) \to \mathsf{Seg}(A, \alpha)$.

By construction, the ipo has a top element (A) and this is a fixed point of successor. Since A is well founded, Lemma 5.8 says directly that it is the only fixed point. Note that this statement makes a quantification over subcoalgebras, which also requires the well powered condition (Remark 4.10). $\square$

The second version applies to *general* coalgebras and does exploit the theory of well-foundedness that we have developed.

Proposition 5.11 Any coalgebra $A \xrightarrow{\alpha} TA$ has a greatest well founded subcoalgebra, which is independent of the choice of classes of predicates and initial segments.

Proof Corollary 5.4 also defined the subipo $\mathsf{WfSeg}(A, \alpha) \subset \mathsf{Seg}(A, \alpha)$ of well founded subcoalgebras. By Lemma 5.9, the relative successor restricts to an endofunction of the smaller ipo, where it is inflationary and monotone.

By Lemma 5.8, if the subcoalgebra B is well founded with respect to initial segments and $B' \subset B$ are subcoalgebras that are each fixed by the successor then $B' = B$. This is the extra condition in Corollary 3.10 to Pataraia's theorem, so $\mathsf{WfSeg}(A, \alpha)$ has a top element, say (B, β).

As an element of the larger ipo $\mathsf{Seg}(A, \alpha)$, B is characterised as the *least* fixed point of the successor. The statement of this property is independent of the notion of well-foundedness. If we re-define the latter for a larger class of predicates with the appropriate closure conditions (Definition 4.12), even though there may be fewer well founded subcoalgebras, B is still one of them and the proof that it is the largest one also remains valid. $\square$

In Section 8 we improve this greatest subcoalgebra to an adjoint, on an additional assumption.

6 The recursion theorem

The proof of the recursion theorem has similar components to the constructions in the previous section. However, the weaker hypotheses that we have allowed in this paper mean that several steps in (the categorical analogue of) the traditional proof in Theorem 1.4 no longer work quite as before. Nevertheless, the solution to these problems is not to discard the old proof, but to gain a more precise understanding of how it works, particularly at the successor stage.

Remark 6.1 An *attempt* from a coalgebra $\alpha : A \longrightarrow\!\!\triangleright\, TA$ to an algebra $\theta : T\Theta \to \Theta$ is intended to be a partial map $f : A \rightharpoonup \Theta$ that is a subhomomorphism in the sense that

$$\begin{tikzcd} TA \arrow[r, harpoon, "Tf"] & T\Theta \arrow[d, "\theta"] \\ A \arrow[u, "\alpha"] \arrow[r, harpoon, "f"'] \arrow[ur, phantom, "\sqsubseteq"] & \Theta \end{tikzcd}$$

i.e. if the composite *via* TA is defined then so is that *via* Θ and then they are equal, *cf.* the definition in Theorem 1.4.

Composition of partial functions in a category uses inverse images. In order to define a category of coalgebras and *partial* homomorphisms, the functor T should therefore *preserve* inverse image diagrams, as the powerset and term algebra functors do.

However, the structure maps α and θ are total and we never need to compose partial maps. The notion of attempt therefore has a simple equivalent form that is sufficient to carry out the proof of the theorem:

Definition 6.2 An ***attempt*** from A to Θ is a diagram of the form

$$\begin{tikzcd} TA & TB \arrow[l, hook', "Ti"'] \arrow[r, "Tf"] & T\Theta \arrow[d, "\theta"] \\ A \arrow[u, "\alpha"] & B \arrow[l, hook', "i"] \arrow[u, "\beta"'] \arrow[r, "f"'] & \Theta \end{tikzcd}$$

That is, a subcoalgebra inclusion (initial segment) $i : B \hookrightarrow A$ together with coalgebra-to-algebra homomorphism $f : B \longrightarrow\!\!\triangleright\, \Theta$. A map f satisfies the recursion scheme (Definition 2.8) exactly when it is a ***total attempt***, with $i : B \cong A$. We call the attempt ***well founded*** if the ***support*** B is.

We need a well powered assumption for attempts, which is easily adapted from that for initial segments (Definition 4.5ff). Alternatively we may consider them as subobjects of $A \times \Theta$ instead of those of A. Then, for any given coalgebra and algebra, there is a set or $\mathcal{S}$-object $\mathsf{Att}\,(A, \alpha, \Theta, \theta)$ of attempts from A to Θ, *cf.* $\mathsf{Seg}\,(A, \alpha)$ in Corollary 5.4.

Lemma 6.3 There is a "support" function (morphism of $\mathcal{S}$)

$$\mathsf{supp} : \mathsf{Att}\,(A, \alpha, \Theta, \theta) \longrightarrow \mathsf{Seg}\,(A, \alpha) \quad \text{by} \quad (A \xleftarrow{\ i\ } B \xrightarrow{\ f\ } \Theta) \longmapsto (B \xhookrightarrow{\ i\ } A).$$

Proof Corollary 4.7. $\square$

One way to show that attempts are *unique* is by an easy application of well-foundedness. This shows that the function `supp` is mono when restricted to well founded subcoalgebras.

Lemma 6.4 Let A be a well founded coalgebra, Θ an algebra and $f, g : A \rightrightarrows \Theta$ be total attempts. Then $f = g$ (*cf.* Theorem 1.4(d)).

Proof The two parallel squares on the right commute since f and g are total attempts. Let $i : E \rightarrowtail A \rightrightarrows V$ be the equaliser in $\mathcal{C}$.

$$\begin{tikzcd}
TE \arrow[rr, tail, "Ti"] & & TA \arrow[r, shift left, "Tg"] \arrow[r, shift right, "Tf"'] & T\Theta \arrow[dd, "\theta"] \\
& E \arrow[dr, tail, "i"] & & \\
H \arrow[uu] \arrow[ur, dotted] \arrow[ur, phantom, "\lrcorner", very near start] \arrow[rr, tail] & & A \arrow[uu, "\alpha"] \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & \Theta
\end{tikzcd}$$

Form the pullback H of $A \to TA \leftarrow TE$; the composites $H \rightrightarrows T\Theta$ are equal by construction, so those $H \rightarrowtail A \rightrightarrows \Theta$ are also equal. Then $H \rightarrowtail A$ factors through the equaliser, so $H \rightarrowtail E \rightarrowtail A$. Hence $i : E \cong A$ by well-foundedness of A and so $f = g$. [Mik76, page 99] [Osi74, Prop 6.5] [Osi75, Prop 6.3] [Tay96a, 2.5] [Tay96b, Prop 6.5] [Tay99, Prop 6.3.9]. $\square$

Remark 6.5 You will object that we did not ask for equalisers in Section 4, either in the category itself or in the class of predicates over which we may perform induction. Plenty of familiar categories have all finite limits and are well powered with respect to *all* monos and so the Lemma is valid for those.

However, it transpires that there is a more subtle proof by induction on the structure of subcoalgebras that does not need equalisers.

Another problem is that the traditional proof of the *existence* of a solution to the recursion equation involves taking the union of all of them (Theorem 1.4(h)). For this we must be able to amalgamate pairs of attempts, but we cannot do this in the setting that we have chosen, where we only have *directed* unions, although we will re-consider binary ones in Section 9.

For totality we form the successor using Lemma 5.6 but prove that it adds nothing more (Lemma 5.8). This part does remain the same, because it only required well-foundedness with respect to initial segments.

In fact these are not obstacles. By a more careful analysis of the building blocks of the proof, we find that the successor provides uniqueness as well as existence, *without* expanding the class of predicates. Everything is driven by recursive principles that we obtained in Sections 3 and 5, where we have already seen that Pataraia's fixed point theorem allows us to work with directed joins instead of arbitrary ones.

The two corollaries of Pataraia's Theorem 3.9 offer different strategies for the proof. Here we will use Pataraia *induction* (Corollary 3.11), but the similar construction in the next section is based on Corollary 3.10 instead.

Lemma 6.6 There is a bijection between attempts

$$A \overset{i}{\hookleftarrow} B \xrightarrow{f} \Theta \qquad \text{and} \qquad A \overset{j}{\hookleftarrow} sB \xrightarrow{g} \Theta,$$

where sB is the relative successor of B (Lemma 5.6).

Hence the successor lifts not only the existence but also the uniqueness of an attempt.

$$\begin{tikzcd}
& A \arrow[r, "\alpha"] & TA & \\
& sB \arrow[u, hook, "k"'] \arrow[ur, phantom, very near start, "\urcorner"] \arrow[r, "c"] \arrow[d, "g"] \arrow[rrd, "\gamma"] & TB \arrow[u, hook, "Ti"'] \arrow[d, "Tf"'] \arrow[rd, hook, "Tj"] & \\
B \arrow[ruu, hook, "i"] \arrow[ru, "j"] \arrow[rru, "\beta"] \arrow[r, "f"] & \Theta & T\Theta \arrow[l, "\theta"'] & TsB \arrow[l, "Tg"'] \arrow[luu, "Tk"']
\end{tikzcd}$$

Proof Let $(A, \alpha) \xleftarrow{i} (B, \beta) \xrightarrow{f} (\Theta, \theta)$ be an attempt, so

$$i\,;\alpha \;=\; \beta\,;T\alpha \qquad \text{and} \qquad f \;=\; \beta\,;Tf\,\theta$$

Then the relative successor attempt is defined by

$$\gamma \;\equiv\; c\,;Tj \qquad \text{and} \qquad g \;\equiv\; c\,;Tf\,;\theta$$

and satisfies

$$\begin{aligned}
f \;&=\; \beta : Tf\,;\theta \;=\; j\,;c\,;Tf\,;\theta \;=\; j\,;g \\
g \;&\equiv\; c\,;Tf\,;\theta \;=\; c\,;Tj\,;Tc\,;TTf\,;T\theta\,;\theta \;=\; c\,;Tj\,;Tg\,;\theta \;\equiv\; \gamma\,;Tg\,;\theta.
\end{aligned}$$

So $(A, \alpha) \xleftarrow{j} (sB, \gamma) \xrightarrow{g} (\Theta, \theta)$ is also an attempt, extending f.

Conversely, $f \equiv i\,;g$ satisfies

$$\begin{aligned}
f \;&\equiv\; j\,;g \;=\; j'\gamma\,;Tg\,;\theta \;=\; j\,;c\,;Tj\,;Tg\,;\theta \;=\; \beta\,;Tf\,;\theta \\
g' \;&\equiv\; c\,;Tf\,;\theta \;=\; c\,;Tj\,;Tg\,;\theta \;=\; \gamma\,;Tg\,;\theta \;=\; g,
\end{aligned}$$

establishing the bijection. □

The parametric version is similar and is the only place where we use binary products:

Lemma 6.7 There is a bijection between parametric attempts given by

$$g \;\equiv\; \langle k, (c\,;Tf)\rangle\,;\theta \qquad \text{and} \qquad f \;=\; j\,;g. \qquad \square$$

$$\begin{tikzcd}
B \arrow[r, "{\langle i, \beta\rangle}"] \arrow[d, hook, "j"'] \arrow[dd, bend right=60, "f"'] & A \times TB \arrow[d, "A \times Tj"] \arrow[dd, bend left=60, "A \times Tf"] \\
sB \arrow[ru, "{\langle k, c\rangle}"] \arrow[r, "{\langle k, \gamma\rangle}"'] \arrow[d, dotted, "g"'] & A \times TsB \arrow[d, dotted, "A \times Tg"] \\
\Theta & A \times T\Theta \arrow[l, "\theta"']
\end{tikzcd}$$

Lemma 6.8 The initial object is the support of a unique attempt. For any directed diagram of subcoalgebras, if each member is the support of a unique attempt then so is the union of the diagram (*cf.* Theorem 1.4(a,b)).

Proof The statements are the universal properties of the initial object and filtered colimits, but we need to say that they are unions (Definition 4.3). Also *cf.* Remark 4.2, Definition 4.3 , Lemma 5.2 and Proposition 5.3. ☐

We can now achieve our principal goal, the ***Recursion Theorem*** (*cf.* Theorem 1.4(h)).

Theorem 6.9 From any well founded coalgebra to any algebra there is a unique total attempt.

Proof By Proposition 5.10, $\mathsf{Seg}\,(A,\alpha)$ is an ipo on which the relative successor defines an endofunction, whose unique fixed point is the top element, A itself. The uses of the well powered condition that we make here were explained there.

Lemma 6.6 defined an endomorphism of $\mathsf{Att}\,(A,\alpha,\Theta,\theta)$ and $\mathsf{supp} : \mathsf{Att} \to \mathsf{Seg}$ commutes with the two successors. Hence this situation is wholly about objects and morphisms of the topos $\mathcal{S}$.

Consider the subset $U \subset \mathsf{Seg}\,(A,\alpha)$ consisting of those initial segments $i : B \hookrightarrow A$ such that there is a unique attempt with support B. That is,

$$U \quad \equiv \quad \{B \in \mathsf{Seg}\,(A,\alpha) \mid \exists! a \in \mathsf{Att}\,(A,\alpha,\Theta,\theta).\, \mathsf{supp}\,(a) = i\}.$$

Then $\emptyset \in U$ and it is closed under directed unions by Lemma 6.8, whilst $s : U \to U$ by Lemma 6.6.

Therefore, by Pataraia induction (Corollary 3.11) U contains the least fixed point of the successor, which is A itself. This means that there is a unique attempt with support A, *i.e.* a total one or solution to the recursion equation.

The *statement* of the Theorem is independent of the notion of initial segment that we choose. Also, if we enlarge the class of predicates then there are just fewer well founded coalgebras and the result remains the same [Mik76, pp 101–4] [Osi75, Prop 6.5] [Tay99, Thm 6.3.13] ☐

From this we can deduce the categorical analogue of Corollary 3.4. We only have equivalence and not existence, because the functor T need not have any fixed point at all, for example in the case of the powerset. Two of the steps in the circular equivalence below are based on observations by Joachim Lambek [Lam68] and Daniel Lehmann and Michael Smyth [LS81]:

Proposition 6.10 The structure maps of the initial algebra, final coalgebra and final well founded coalgebra, if they exist, are isomorphisms.

$$\begin{tikzcd}
T\Theta \arrow[r, dotted, shift right, "T\alpha"'] \arrow[d, "\theta"'] & TT\Theta \arrow[l, shift right, "T\theta"'] \arrow[d, "T\theta"] \\
\Theta \arrow[r, dotted, shift right, "\alpha"'] & T\Theta \arrow[l, shift right, "\theta"']
\end{tikzcd}
\qquad
\begin{tikzcd}
TA \arrow[r, shift left, "T\alpha"] & TTA \arrow[l, dotted, shift left, "T\theta"] \\
A \arrow[u, "\alpha"] \arrow[r, shift left, "\alpha"] & TA \arrow[l, dotted, shift left, "\theta"] \arrow[u, "T\alpha"']
\end{tikzcd}$$

These objects are therefore both algebras and coalgebras and we call them ***fixed points*** of the functor. Coalgebra-to-algebra homomorphisms from or to them are respectively the same as plain algebra or coalgebra homomorphisms.

The successor relative (Lemma 5.6) to the initial algebra is just the functor T.

Proof This is illustrated by the diagrams. It also applies to the final *well founded* coalgebra because the functor T preserves well-foundedness by Lemma 5.5. In Lemma 5.6, since $A \cong TA$ also $C \cong TB$. $\square$

Proposition 6.11 The initial algebra A is well founded *quâ* coalgebra.

$$\begin{tikzcd}
TU \arrow[rr, tail, shift left, "Ti"] & & TA \arrow[ll, dotted, shift left, "Tp"] \\
H \arrow[u, "\cong"] \arrow[r, tail, "j"] \arrow[urr, phantom, "\lrcorner", very near start] & U \arrow[r, tail, shift left, "i"] & A \arrow[u, "\alpha"', "\cong"] \arrow[l, dotted, shift left, "p"]
\end{tikzcd}$$

Proof Since the structure map is invertible, so is its (broken) pullback, so $TU \cong H \overset{j}{\rightarrowtail} U$ makes U an algebra and $i : U \rightarrowtail A$ an algebra monomorphism. But this is split since A is initial *quâ* algebra. Hence A is well founded *quâ* coalgebra. $\square$

Corollary 6.12 If any of the following exists then it satisfies the other properties too:
(a) a final well founded coalgebra;
(b) a well founded coalgebra whose structure map is an isomorphism;
(c) an initial fixed point;
(d) an initial algebra.
Moreover, it is unique up to unique isomorphism.

Proof The Recursion Theorem 6.9 says that the final well founded coalgebra has the universal property of the initial algebra, so b⇒c. Proposition 6.11 is almost the converse, d⇒c, where Proposition 6.10 fills in the gaps, a⇒b and c⇔d. $\square$

7 Extensionality

Gerhard Osius used extensional well founded coalgebras for the powerset functor as the basis of his reconstruction of set theory within an elementary topos [Osi74, §6], but we shall not concern ourselves in this paper with modelling Zermelo's other axioms for set theory.

Instead we pick upon the idea that a "set" is a *fragment* of the unattainable universe that would be the free algebra for the powerset functor. We generalise this to other functors that may or may not have free algebras, characterising the initial segments of the free algebra for themselves, *i.e.* without assuming that the free algebra exists.

In doing this, we will make powerful use of the corollaries of Pataraia's theorem.

Definition 7.1 A coalgebra $A \xrightarrow{\alpha} TA$ is ***extensional*** if α is an initial segment, *i.e.* it belongs to the same class of monos that we used for subcoalgebras in our proof of the recursion theorem (Assumption 4.15). If you skipped Section 4, you should simply take α to be a mono, as in Definition 1.7 when $T \equiv \mathcal{P}$.

An ***ensemble*** is an extensional well founded coalgebra. If need be, the name could be qualified by stating the category, functor and two classes of monos that are used in the definition.

Remark 7.2 Zermelo's generalisation from well *ordered* to well *founded* systems introduced "noise" in the form of repetition, but extensionality removes this so thoroughly that there are no isomorphisms or even automorphisms aside from the identity. Extensional well founded coalgebras are fragments of the initial algebra (if there is one) and behave very much like set theory, justifying the name *ensemble*. By restricting the kinds of maps that we call "monos" (initial segments), we more closely approximate the iterates of the functor applied to $\emptyset$ and their colimits, *cf.* the results in Section 3.

Strictly speaking, the "sets" that we are mimicking here are those that are called ***transitive*** in set theory, by which is meant those X for which

$$y \in x \in X \Longrightarrow y \in X, \qquad \text{but not necessarily} \qquad z \in y \in x \in X \Longrightarrow z \in x.$$

From the point of view of category theory (or any *ordinary* mathematical methodology), this *must* be our starting point, in the absence of the "universal" set, because, as non-believers, we cannot understand how an "element" of a structure can have any meaning independently of that structure. A transitive set is thus a partial model of set theory.

Osius therefore used the name ***transitive set object*** for our extensional well founded coalgebras for the covariant powerset functor in an elementary topos $\mathcal{S}$. He defined a general "set" to be an $\mathcal{S}$-subobject of some transitive set object and developed set theory in the style of Zermelo [Zer08b], in fact giving a logical subtopos of $\mathcal{S}$ [Osi74, §7]. See [Tay96a, §3] for another account of this.

We will show that these features of set theory are shared by ensembles for more general functors. We need first to adapt the recursion theorem from the previous section, but this time we use Corollary 3.10 to Pataraia's theorem.

Definition 7.3 An ***attempt*** from one coalgebra (A, α) to another (D, δ) is a pair of coalgebra homomorphisms (also known as a ***span***),

$$\begin{tikzcd} TA & TB \arrow[l, hook', "Ti"'] \arrow[r, "Tf"] & TD \\ A \arrow[u, "\alpha"] & B \arrow[l, hook', "i"'] \arrow[r, "f"] \arrow[u, "\beta"] & TD \arrow[u, hook, "\delta"'] \end{tikzcd}$$

We call the attempt ***well founded*** if the ***support*** B is. Of course, this is the same as Definition 6.2, apart from reversing the arrow δ; the relationship is that we think of (D, δ) as a *partial* algebra whose evaluation part is id_D.

By the time we get to the Theorem, all of the arrows will be initial segments, but we need slightly more generality at first: We assume that δ and i are initial segments, so D is extensional, but *â priori* α and j need not be.

Lemma 7.4 The category whose objects are attempts from A to D and whose morphisms are coalgebra homomorphisms $B' \longrightarrow B$ that make commutative triangles is equivalent to an $\mathcal{S}$-ipo. This has a subipo of well founded attempts,

$$\mathsf{WfAtt}\,(A, \alpha, D, \delta) \quad \subset \quad \mathsf{Att}\,(A, \alpha, D, \delta).$$

Proof For the same reasons as in Corollary 5.4, relying on the assumptions about unions and being well powered. $\square$

The next diagram may be daunting, but it is just the adaptation of Construction 6.6 for successor attempts to the situation where the target is a partial algebra or extensional coalgebra. It is more complicated because we have to trim the support according to the partial target.

Construction 7.5 The ***relative successor*** of an attempt from a coalgebra (A, α) to an extensional coalgebra (D, δ).

Write sB for C.

Proof Let $(A, \alpha) \xleftarrow{i} (B, \beta) \xrightarrow{f} (D, \delta)$ be an attempt, where i and δ are initial segments, as shown in the bold lines in the diagram (which is rotated relative to the one in Definition 7.3):

$$\begin{tikzcd}
 & & A \arrow[r, "\alpha"] & TA & TC_1 \arrow[l, hook', "Tj"'] \\
 & & C_1 \arrow[u, hook, "j"] \arrow[dr, "a"] & & \\
B \arrow[uurr, hook, "i"] \arrow[rrr, bend left, "\beta"] \arrow[r, hook, dotted, "\ell"] \arrow[ddrr, "f"'] & C \arrow[ur, hook, "k"] \arrow[dr, "h"'] \arrow[rrr, bend right, "\gamma"'] \arrow[r, phantom, "\text{pbk}"] & & TB \arrow[uu, hook, "Ti"'] \arrow[dd, "Tf"] \arrow[r, hook, dotted, "T\ell"] & TC \arrow[uu, hook, "Tk"'] \arrow[dd, "Th"] \\
 & & C_2 \arrow[d, "g"] \arrow[ur, hook, "d"'] & & \\
 & & D \arrow[r, hook, "\delta"] & TD & TC_2 \arrow[l, "Tg"']
\end{tikzcd}$$

Let C_1, C_2 and C be the pullbacks shown. Then Ti, j, d, k and $k\,;j$ are initial segments because T, pullback and composition preserve them. Notice that we have used both the subcoalgebra inclusion $B \xhookrightarrow{i} A$ and the extensional structure map $D \xhookrightarrow{\delta} TD$ to do this, so it is not possible to separate these two uses of "monos", *cf.* Remark 4.11.

This construction makes C the limit of the W-diagram

$$A \xrightarrow{\alpha} TA \xhookleftarrow{Ti} TB \xrightarrow{Tf} TD \xhookleftarrow{\delta} D,$$

over which B is the vertex of another cone, with arrows i, β and f. Hence there is a unique mediator ℓ with

$$i \;=\; \ell\,;k\,;j, \qquad \beta \;=\; \ell\,;k\,;a \;=\; \ell\,;h\,;d \qquad \text{and} \qquad f \;=\; \ell\,;h\,;g.$$

Then ℓ is an initial segment by the Cancellation Lemma 4.13 since i and $k\,;j$ are.

Now we make C a coalgebra by defining $\gamma \equiv k\,;a\,;T\ell$ and then ℓ is a homomorphism because

$$\ell\,;\gamma \;\equiv\; \ell\,;k\,;a\,;T\ell \;=\; \beta\,;T\ell.$$

The new attempt with support C is given by the composites $k\,;j : C \to A$ and $h\,;g : C \to D$, whose composites with ℓ are i and f. Then $k\,;j$ and $h\,;g$ are homomorphisms because

$$(k\,;j)\,;\alpha \;=\; k\,;a\,;Ti \;=\; k\,;a\,;T\ell\,;T(k\,;j) \;=\; \gamma\,;T(k\,;j)$$

and

$$(h\,;g)\,;\delta \;=\; h\,;d\,;Tf \;=\; h\,;d\,;T\ell\,;T(h\,;g) \;=\; \gamma\,;T(h\,;g).$$

The map $\ell : B \hookrightarrow sB \equiv C$ makes the successor inflationary. $\square$

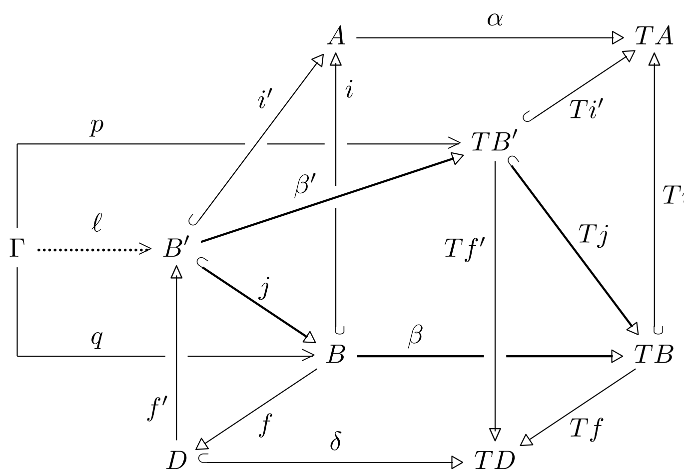
Lemma 7.6 This construction is monotone (functorial) in B.

Proof The proof amounts to the mediator between two W-limits, using a diagram similar to the preceding and following ones or that in Lemma 5.7. $\square$

Lemma 7.7 If B is well founded then so is $C \equiv sB$.

Proof By Lemma 5.9, since C is sandwiched between B and TB. $\square$

Lemma 7.8 If B is well founded and $B' \cong sB' \xhookrightarrow{j} B \cong sB$ then $j : B' \cong B$.



Proof That B' is a fixed point means that it is already the limit of the W that defines its successor:

$$A \xrightarrow{\alpha} TA \xleftarrow{Ti'} TB' \xrightarrow{Tf'} TD \xleftarrow{\delta} D.$$

We claim that the homomorphism quadrilateral for $B' \xhookrightarrow{j} B$ (shown in bold) is a pullback, so let Γ be the vertex of a cone, with $p\,;Tj = q\,;\beta$. Then

$$q\,;i : \Gamma \to A, \qquad p : \Gamma \to TB' \qquad \text{and} \qquad q\,;f : \Gamma \to D$$

define a cone over the W for B' because

$$q\,;i\,;\alpha \;=\; q\,;\beta\,;Ti \;=\; p\,;Tj\,;Ti \;=\; p\,;Ti'$$

and

$$q\,;f\,;\delta \;=\; q\,;\beta\,;Tf \;=\; p\,;Tj\,;Tf \;=\; p\,;Tf'.$$

Since B' is the limit, there is a unique mediator $\ell : \Gamma \to B'$ with

$$\ell\,;i' \;=\; \ell\,;j\,;i \;=\; q\,;i, \qquad \ell\,;\beta' \;=\; p \qquad \text{and} \qquad \ell\,;f' \;=\; q\,;f$$

whence $\ell\,;j = q$ since i is mono. Thus ℓ provides the mediator that is required for B' to be the pullback. However, B is well founded by hypothesis, so any such pullback degenerates, making $j : B' \cong B$. $\square$

Corollary 7.9 There is a greatest well founded attempt from (A, α) to (D, δ).

Proof $\mathsf{WfAtt}\,(A, \alpha, D, \delta)$ has a top element by Corollary 3.10 to Pataraia's Theorem. $\square$

Lemma 7.10 If (A, α) and (D, δ) are both extensional then the greatest attempt with well founded support consists of a pair of initial segments.

Proof We use Pataraia induction (Corollary 3.11) for the property that the evaluation part of the attempt is an initial segment (mono).

The least attempt, whose support is the initial object, satisfies this property.

The successor Construction 7.5 preserves it: if α and f are initial segments then so too are Tf, g, a, h and $h\,;g$.

Directed unions also preserve it.

Therefore the greatest attempt has it too. $\square$

Lemma 7.11 Any coalgebra homomorphism $f : A \longrightarrow\!\!\!\triangleright\, D$ between ensembles is an initial segment. There is at most one such homomorphism. If there are homomorphisms in both directions then they are inverse.

Proof Without using the hypothesis that A is extensional, the homomorphism f provides an attempt $(\mathsf{id}\,, f)$, in the sense of Definition 7.3.

This is fixed by the successor (Construction 7.5): i, Ti and j are isomorphisms, so the initial segments ℓ and k are too since initial segments are plain monos.

Hence the attempt $(\mathsf{id}\,, f)$ coincides with the greatest one (Corollaries 3.10 and 7.9), which is a pair of initial segments by Lemma 7.10. Moreover this greatest attempt is unique.

In particular, the only endomorphism of an ensemble is the identity, so if there are homomorphisms both ways then they must be inverse. $\square$

Corollary 8.11 strengthens this to say that *any* outgoing homomorphism from an extensional well founded coalgebra is an initial segment, *whatever* the codomain, but on much stronger assumptions. An ensemble is therefore a very ***rigid*** structure.

Theorem 7.12 The category **Ens** of ensembles and coalgebra homomorphisms

(a) is a preorder with

(b) a least (isomorphism class of) object(s),

(c) directed unions,

(d) binary meets and

(e) an inflationary monotone successor, namely the functor T.

Moreover,

(f) the greatest ensemble is the initial algebra (Corollary 6.12), if either of these exists, and is the unique fixed point of T.

Proof Regarding meets, any pair of homomorphisms $A \triangleleft\!\!\!\longleftarrow C \hookrightarrow\!\!\!\triangleright\, D$ where C is an ensemble defines an attempt, which lies below (factors through) the greatest attempt $A \triangleleft\!\!\!\longleftarrow B \hookrightarrow\!\!\!\triangleright\, D$. In the last part, the successor coalgebra relative to the initial algebra is just given by the functor T. $\square$

Remark 7.13 The meet therefore has the property that we would normally call a *product* in a category. We avoid that word because the construction looks like set-theoretic intersection and nothing like the Cartesian or Kuratowski–Wiener product. Set-theoretically, the maps $A \leftarrow C \rightarrow D$ do have to be subset inclusions and not arbitrary functions. Categorically, $A \leftarrow C \rightarrow D$ must be homomorphisms and not just $\mathcal{C}$-maps.

This corollary was a bonus that we should not have expected unless T preserves inverse images (Lemma 9.4). However, we can't take it any further unless that is the case; in particular, we do not yet have binary *joins* of ensembles, but we will study them in Section 9.

For two classical ordinals, this is the result that says that one must be an initial segment of the other, in a unique way [Can97, §13 Thms N&E]. $\square$

Proposition 7.14 Suppose that the category $\mathcal{C}$ has set-indexed filtered colimits. Then the functor T has an initial algebra iff there is a set rather than a proper class of isomorphism classes of ensembles.

Proof Since an ensemble is an initial segment of the initial algebra the forward direction follows from the well powered assumption. Conversely, another way of stating the "size" condition is that the preorder **Ens** is equivalent to an *internal* poset in $\mathcal{S}$. The initial object, endofunctor and filtered colimits in **Ens** become the least element, directed joins and endofunction of the poset. By Pataraia's Theorem 3.9, there is a fixed point, which is the top element, and this corresponds to the initial algebra. □

Corollary 7.15 If there is an initial algebra, it satisfies any property of coalgebras that holds of the initial object and is preserved by isomorphism, the functor and filtered colimits.

Proof By Pataraia induction, Corollary 3.11. □

Remark 7.16 The way that we envisage these results being used is this: Suppose that we have a functor that does have a free algebra, but it is very complicated. The ensembles or initial segments are potentially simpler, so we use well-foundedness and extensionality to build a characterisation of them. Then the condition that the structure map be an isomorphism completes the identification of the free algebra.

The traditional language for this situation is that the functor ***has rank***. Also, strong principles of recursion that are often required to demonstrate theorems in proof theory such as conservativity, completeness and consistency. These are standardly expressed as *ordinals*. These ordinals may be obtained from the structures that we are studying here.

We feel that the recursive structure of free algebras for a functor, or of the a ***term model*** of some logic should be retained in its natural state.

8 Imposing the properties

In this section we show how to turn a general coalgebra into a well founded one and then make it extensional too. That is, we will find adjoints to the inclusions $\mathbf{Ens} \rightarrowtail \mathbf{WfCoAlg} \rightarrowtail \mathbf{CoAlg}$.

The key idea in doing this is (the categorical abstraction of) the fact that any function can be expressed as the composite of a surjection and the inclusion of its image. One of the earliest achievements of category theory, or rather of Modern or Universal Algebra, was to bring together the various "isomorphism theorems" relating these for groups, rings, vector spaces, *etc.* The abstract formulation was given by Peter Freyd and Max Kelly [FK72]:

Definition 8.1 Two maps $e : X \twoheadrightarrow Q$ and $m : V \hookrightarrow Y$ in any category are ***orthogonal***, written $e \perp m$, if, for any two maps f and g such that the square commutes, there is a unique morphism $p : Q \to V$ making the two triangles commute:

$$\begin{tikzcd} X \arrow[r, two heads, "e"] \arrow[d, "f"'] & Q \arrow[d, "g"] \arrow[dl, dotted, "p"'] \\ V \arrow[r, hook, "m"'] & Y \end{tikzcd}$$

A ***factorisation system*** is a pair of classes of morphisms $(\mathcal{E}, \mathcal{M})$ such that
(a) the classes $\mathcal{E}$ and $\mathcal{M}$ each contain all isomorphisms;
(b) they are each closed under composition;
(c) $e \perp m$ for every $e \in \mathcal{E}$ and $m \in \mathcal{M}$ and
(d) every morphism $f : X \to Y$ can be expressed as $f = e\,;m$ with $e \in \mathcal{E}$ and $m \in \mathcal{M}$.

Lemma 8.2 The $\mathcal{E}$-class of a factorisation system satisfies:
(a) if $e \perp m$ for all $m \in \mathcal{M}$ then $e \in \mathcal{E}$;
(b) the cancellation property that if $f,\ (f\,;e) \in \mathcal{E}$ then $e \in \mathcal{E}$, *cf.* Lemma 4.13;
(c) if the maps in a directed or pushout diagram are all in $\mathcal{E}$ then so are those in the colimiting cocone; and
(d) the mediator from such a colimit to a cocone consisting of $\mathcal{E}$-maps is also in $\mathcal{E}$.

$$\begin{tikzcd}
X_0 \arrow[r, two heads] \arrow[d] \arrow[rrr, bend left] & X_i \arrow[r] \arrow[dl, dotted] & \mathsf{colim} \arrow[r] \arrow[d] & \Theta \\
U \arrow[rr, tail] & & Y
\end{tikzcd}$$

Proof (a,b) Easy, but see *e.g.* Lemma 5.7.6(e) and Proposition 5.7.7 in [Tay99]. (c) Any pushout has a *root* X_0 (with maps to all of the other vertices of the diagram), and any directed diagram is equivalent to one with a root. Using $(X_0 \twoheadrightarrow X_i) \perp (U \rightarrowtail Y)$, there is a unique mediator $X_i \to U$. These maps form a cocone, with mediator $\mathsf{colim} \to U$. Finally, (d) follows from (b,c). ☐

Examples 8.3
(a) Inclusions (1–1 maps, monomorphisms) and surjections (onto maps, epimorphisms) in **Set** or a topos, where surjections are quotients by equivalence relations and this class is stable under pullbacks.
(b) More generally in type theories, if the factorisation is stable under pullback then the $\mathcal{E}$ class is associated with an existential quantifier
(c) In a general category with inverse images, a maps e that is orthogonal to all monos is called an ***extremal epi*** and is characterised by $\forall m \in \mathcal{M}.\, e = m\,;f \Longrightarrow m = \mathsf{id}$.

Assumption 8.4 In this section we require that the category $\mathcal{C}$
(a) have a factorisation system $(\mathcal{E}, \mathcal{M})$ in which $\mathcal{M}$ is the class of initial segment maps that we have been using,
(b) be well *co*powered with respect to $\mathcal{E}$-maps, and
(c) have filtered colimits of $\mathcal{E}$-homomorphisms.

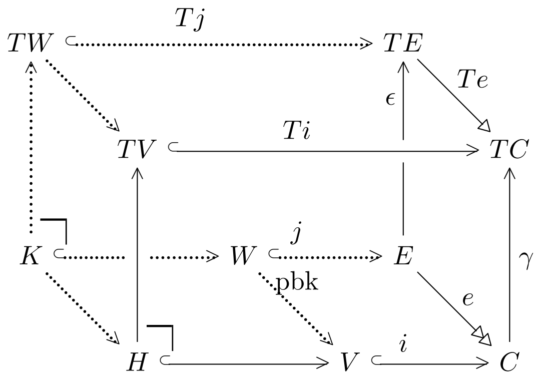
Theorem 8.7 only requires the first of these, but we use the others for Theorem 8.9.

The notion of being well *co*powered is the obvious analogue of being well powered (Definition 4.5): that the (pre-ordered) external category of outgoing $\mathcal{E}$-maps is equivalent to an internal poset in the base topos $\mathcal{E}$.

Remark 8.5 We will call $\mathcal{E}$-maps ***cofinal***. As in Remark 4.16 for initial segments, this term is intended to hint at certain intuitions, which are linked to the fact that all of the maps that we call cofinal are coalgebra homomorphisms. However, the notion is not necessarily the same as the traditional order-theoretic one. It is a novelty of this work that $\mathcal{E}$ can be a special class of morphisms in a category that need not be a topos.

The following is the categorical version of Corollary 2.7:

Lemma 8.6 Let E be a well founded coalgebra and $e : E \twoheadrightarrow C$ a cofinal homomorphism. Then C is also well founded.



Proof Let $i : V \hookrightarrow C$ be an initial segment that satisfies the induction premise given by the broken pullback from H to TC (at the front).

Pull this back along the homomorphism $e : E \twoheadrightarrow C$, using Lemma 5.1.

By well-foundedness of E, we have $j : W \cong E$.

Since $e : E \twoheadrightarrow C$ is cofinal and it factors through the initial segment $i : V \hookrightarrow C$, the latter is also an isomorphism [Tay96b, Prop 7.8]. $\square$

Theorem 8.7 The inclusion **WfCoAlg** $\rightarrowtail$ **CoAlg** has a right adjoint (coreflection), which is independent of the choices of classes of predicates, initial segments and cofinal maps.

$$\begin{tikzcd}
E \arrow[r, "f"] \arrow[d, "e"', two heads] & A \\
C \arrow[ur, hook, "j"] \arrow[r, hook, "k"'] & D \arrow[u, hook, "i"']
\end{tikzcd}$$

Proof We claim that the largest well founded subcoalgebra $i : D \hookrightarrow A$ (Proposition 5.11) provides the adjoint. That is, any coalgebra homomorphism $f : E \longrightarrow A$ with E well founded factors uniquely through i.

Let $E \stackrel{e}{\twoheadrightarrow} C \stackrel{j}{\hookrightarrow} A$ be the factorisation in $\mathcal{C}$ of f as a cofinal map followed by an initial segment. Since Tj is also an initial segment, the orthogonality mediator provides a coalgebra structure on C such that e and j are homomorphisms. Then C is well founded by Lemma 8.6 and it is a subcoalgebra of A by construction.

It is therefore a subcoalgebra of D, since D was the largest such. The map $E \longrightarrow D$ is unique since $i : D \hookrightarrow A$ is mono.

Proposition 5.11 said that the largest well founded subcoalgebra D is independent of the classes. This is also true of the factorisation, because a stronger notion of well-foundedness just replaces **WfCoAlg** with a full subcategory. $\square$

Now we turn to imposing extensionality, which is our categorical version of Mostowski's Theorem (Remark 1.8). This is achieved by means of a *co*recursive construction, which was done symbolically in [Tay96a, Thm 2.11].

Construction 8.8 The ***successor quotient*** (C, γ) of any coalgebra (B, β) is given by factorising β as a cofinal homomorphism followed by an initial segment, as shown below. Then $e : B \cong C$ iff (B, β) is extensional. If B is well founded then so is C. Any homomorphism $f : (B, \beta) \longrightarrow (E, \epsilon)$ to an extensional coalgebra factors uniquely through C.

$$
\begin{tikzcd}
TTB & & \\
& TC \arrow[ul, hook, "Ti"'] \arrow[dr, dotted, "Tg"] & \\
TB \arrow[uu, "T\beta"] \arrow[ur, "Te"] \arrow[rr, "Tf"'] & & TE \\
& C \arrow[uu, "\gamma"] \arrow[ul, hook, "i"'] \arrow[dr, dotted, "g"] & \\
B \arrow[uu, "\beta"] \arrow[ur, "e"] \arrow[rr, "f"] & & E \arrow[uu, hook, "\epsilon"']
\end{tikzcd}
$$

Proof Let $\beta = e\,;i$ be the factorisation, *via* C, and put $\gamma \equiv i\,;Te$. Then the three triangles on the left commute, so $e : B \longrightarrow C$ and $i : C \longrightarrow TB$ are coalgebra homomorphisms.

If $e : B \cong C$ then $\beta \cong i \in \mathcal{M}$, so B is extensional, and conversely.

If B is well founded then so is C by either Lemma 5.9 or 8.6.

Since $e : B \longrightarrow\!\!\!\!\!\rhd C$ is orthogonal to $\epsilon : E \hookrightarrow TE$, there is a unique map $g : C \to E$ such that $e\,;g = f$ and $i\,;Tf = g\,;\epsilon$. Hence g is a homomorphism:

$$\gamma\,;Tg \;\equiv\; i\,;Te\,;Tg \;=\; i\,;Tf \;=\; g\,;\epsilon. \qquad \square$$

Theorem 8.9 The inclusion $\mathbf{Ens} \longrightarrow \mathbf{WfCoAlg}$ has a left adjoint, called the ***extensional reflection***.

Proof Let (A, α) be a well founded coalgebra. Since the category is well copowered, we may consider the poset of isomorphism classes of cofinal maps $A \longrightarrow\!\!\!\!\!\rhd B$ (and commutative triangles). The identity $\mathsf{id} : A \to A$ provides the least element.

Since the category has filtered colimits and Lemma 8.2 says that these are joins, this poset is directed complete, so it is an ipo.

The successor quotient (Construction 8.8) defines an inflationary monotone endofunction.

An object B is fixed by the successor iff it is extensional.

Any coalgebra homomorphism $B' \longrightarrow B$ between extensional well founded coalgebras is an initial segment by Lemma 7.11. But the present construction only uses cofinal maps, so $B' \cong B$.

By Corollary 3.10 to Pataraia's Theorem, the ipo has a greatest element. This is the unique fixed point of the successor, so it is an extensional well founded coalgebra.

Now let $f : A \longrightarrow E$ be a homomorphism to an extensional coalgebra. We repeat the construction, but now using factorisations $A \longrightarrow\!\!\!\!\!\rhd B \hookrightarrow E$ of f. This contains $\perp \equiv (\mathsf{id}_A, f)$ and is closed under successor quotient and filtered colimits. It embeds in the simpler version and so contains $A \twoheadrightarrow D$ by Pataraia induction (Corollary 3.11). The corresponding $A \twoheadrightarrow D \longrightarrow B$ is the required factorisation that shows that D is the extensional reflection of A. □

Corollary 8.10 If $\mathcal{C}$ has pushouts then so does **Ens**.

Proof **WfCoAlg** $\to \mathcal{C}$ creates them (Proposition 5.3) and any left adjoint preserves them. That is, the pushout in **Ens** is the extensional reflection of that in **WfCoAlg**, which is actually calculated in $\mathcal{C}$. However, the result could be vastly larger than the given objects or $\mathcal{C}$-pushout; Proposition 9.8 looks at when the pushout in **WfCoAlg** is already extensional. □

When we put these two results together, we deduce an even stronger rigidity property:

Corollary 8.11 Any coalgebra homomorphism $E \longrightarrow C$ from an ensemble to any coalgebra is an initial segment. However, there could be multiple maps $E \rightrightarrows C$.

$$\begin{tikzcd} E \arrow[r] \arrow[d, hook, dotted] \arrow[dr, dotted] & C \\ R & W \arrow[l, two heads, dotted] \arrow[u, hook, dotted] \end{tikzcd}$$

Proof Let W be the largest well founded subcoalgebra of C, so $W \hookrightarrow C$ is an initial segment and $E \longrightarrow C$ factors through it by Theorem 8.7. Then let R be the extensional reflection of W, so by Lemma 7.11 the composite $E \longrightarrow W \longrightarrow R$ is also an initial segment. By cancellation (Lemma 4.13), so is $E \hookrightarrow W$ and by composition $E \hookrightarrow C$ is too. Finally, consider $C \equiv \mathbf{2} \times E$. □

Warning 8.12 Assumption 8.4 that the category be well copowered with respect to a class $\mathcal{E}$ of cofinal maps that are "surjective" in only the most tenuous of senses must not be taken lightly. It is a candidate for the categorical form of the axiom-scheme of replacement.

9 When the functor preserves pullbacks

In von Neumann's original recursion Theorem 1.4, we took a simple union of all attempts, because initial segments inherited well-foundedness and admitted binary unions. The same idea carried through to the categorical formulations, so long as the functor preserves inverse images. However, in this paper we have only asked that the functor preserve monos. We therefore had to develop a more delicate proof using directed joins and Pataraia's fixed point theorem.

In this section we restore the stronger condition and give the categorical proof of the result, but still paying attention to our analysis of special classes of monos in different categories and their role in the proof.

We observed that Proposition 1.3 is a very important result for the way that well founded relations are used across mathematics, but this depends on the stronger condition.

Another essential requirement in this proof is the *universal quantifier.* In the categorical formulation this quantifier appears in the form of the adjunction $f^* \dashv f_*$. Gerhard Osius noted this in his version of the result [Osi74, Prop 6.3(a)]. Any topos has this (Notation 2.2), but since we are considering more general categories, we state it as a further assumption on the subobjects.

Assumption 9.1 In addition to the assumptions in Section 4,

(a) the functor $T : \mathcal{C} \to \mathcal{C}$ must preserve inverse image diagrams of predicates along coalgebra homomorphisms;

(b) each inverse image operation f^* must have a right adjoint f_* on predicates; and

(c) pushouts that are pullbacks must be unions, in the sense explained below;.

We begin by giving the proof for well founded *relations* in a form that is as close as possible to the generalisation to follow. See [Tay99, Prop 2.6.2] for a box-style proof in natural deduction for well founded relations.

Proposition 9.2 Let $(A, \prec)$ be a well founded relation and $f : (B, <) \to (A, \prec)$ a ***strictly monotone*** function in the sense that

$$\forall b b' {:} B. \quad b' < b \implies f b' \prec f b$$

then $(B, <)$ is also well founded.

Proof Let ψ be a predicate on B satisfying the induction premise

$$\forall b. \qquad (\forall b'.\, b' < b \Rightarrow \psi b') \Longrightarrow \psi b.$$

For comparison with the categorical proof below, *cf.* Lemma 2.5, this is $K \subset V$, where

$$V \;\equiv\; \{b : B \mid \psi b\} \subset B \qquad \text{and} \qquad K \;\equiv\; \{b : B \mid \forall b'.\, b' < b \Rightarrow \psi b'\} \subset B.$$

The key step is to define $f_* V \equiv \{a : A \mid \phi a\} \subset A$, where $\phi a \;\equiv\; (\forall b'.\, f b' = a \Rightarrow \psi b)$, and

$$H \;\equiv\; \{a : A \mid \forall a'.\, a' \prec a \Rightarrow \phi a'\} \;\equiv\; \{a : A \mid \forall b'.\, b' \prec a \Longrightarrow \psi b'\} \;\subset\; A.$$

Strict monotonicity and the induction premise give $f^* H \subset K \subset V$, which is

$$\forall b. \qquad (\forall b'.\, f b' \prec f b \Rightarrow \psi b') \Longrightarrow (\forall b'.\, b' < b \Rightarrow \psi b') \Longrightarrow \psi b.$$

Quantifying over $\{b' \mid f b' = a\}$, we obtain $H \subset f_* V$, which is

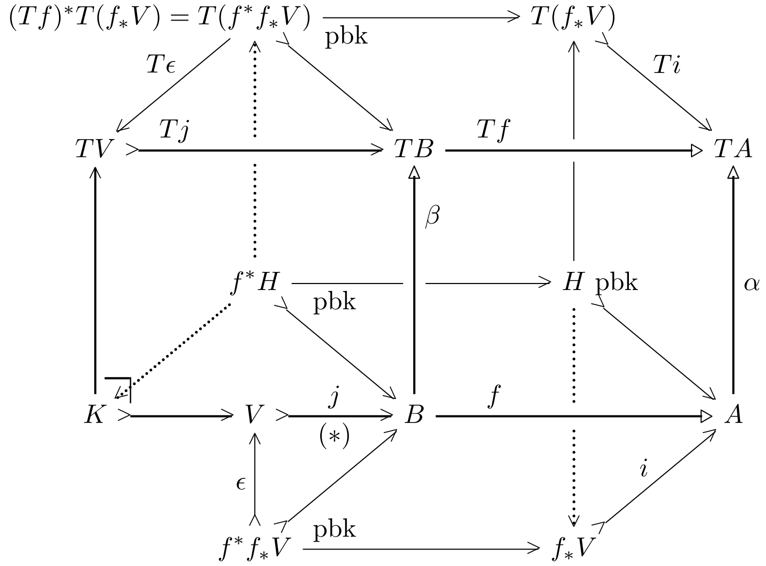
$$\forall a. \qquad (\forall a'.\, a' \prec a \Rightarrow \phi a') \iff (\forall b'.\, f b' \prec a \Rightarrow \psi b') \Longrightarrow (\forall b'.\, f b' = a \Rightarrow \psi b') \;\equiv\; \phi a.$$

Then $\forall a.\, \phi a$ since $(A, \prec)$ is well founded, whence $\forall b.\, \psi b$ as required. $\square$

We are ready to prove the result for general functors that preserve inverse images and coalgebra homomorphisms that are equipped with f_*.

Theorem 9.3 Let $f : (B, \beta) \to (A, \alpha)$ be a coalgebra homomorphism with f_*, where (A, α) is well founded. Then (B, β) is also well founded.

Proof Given the diagram marked in thick lines, apply the right adjoint f_* to $j : V \rightarrowtail B$, to get $i : f_* V \rightarrowtail A$. The counit of this adjunction is $\epsilon : f^* f_* V \to V$ and makes the little triangle $(*)$ commute, where f^* is given by pullback (inverse image) along f. The upper part of the diagram is the T-image of the lower part, including the pullback but not K. Let $H \equiv \alpha^* T(f_* V)$ be the pullback of Ti and α and $f^* H$ its pullback along f.

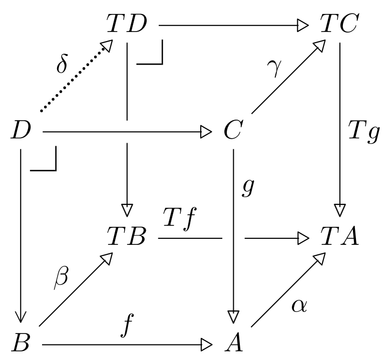


By construction, the whole diagram of solid lines commutes from $f^* H$ to TA. In particular, $f^* H \rightarrowtail B \to TB$ and $f^* H \to H \to T(f_* V)$ agree at TA, so there is a pullback mediator $f^* H \to T(f^* f_* V)$. Then $f^* H \to T(f^* f_* V) \to TV$ agrees with $f^* H \rightarrowtail B$ at TB, so there is also a pullback mediator $f^* H \to K$.

This shows that $f^* H \subset V$ as $\mathcal{C}$-subobjects of B. Therefore, by the adjunction $f^* \dashv f_*$, we have $H \subset f_* V$ as subobjects of A.

That is, there is a map $H \to f_* V$ that makes the right-hand part of the diagram into a broken pullback. Now, since A is well founded, $i : f_* V \cong A$, so $f^* f_* V \cong B$ and $j : V \cong B$ [Tay96b, Prop 7.3]. $\square$

Lemma 9.4 The functors **Ens** $\longrightarrow$ **WfCoAlg** $\longrightarrow$ **CoAlg** $\longrightarrow \mathcal{C}$ create pullbacks.



Proof Computing pullbacks of coalgebras for a functor that preserves them is an easy standard exercise.

If the given coalgebras are well founded then so is the pullback, by Proposition 9.3, which used the right adjoints f_*.

If they are extensional then all of structure maps and homomorphisms in the cube are initial segments (by Lemma 7.11, composition and cancellation), so the pullback is extensional too. $\square$

We now turn to *binary* unions (*cf.* Theorem 1.4(e)). For this we need the union property for pushouts, *cf.* Definition 4.3. As we did in Proposition 4.4 for directed unions, we prove this for **Set**, indeed for any *pre*topos.

$$\begin{tikzcd}
& B \arrow[dr, "i"] \arrow[drr, tail, "f"] & & \\
A \arrow[ur, tail, "m"] \arrow[dr, tail, "n"'] \arrow[rr, phantom, "\text{pbk} \qquad\qquad \text{psh}"] & & D \arrow[r, "d"] & \Theta \\
& C \arrow[ur, "j"'] \arrow[urr, tail, "g"'] & &
\end{tikzcd}$$

The first result is known as the ***Amalgamation Lemma***.

Lemma 9.5 In **Set** or any pretopos, the pushout of a pair of monos $B \overset{m}{\leftarrowtail} A \overset{n}{\rightarrowtail} C$ is another pair of monos and is also a pullback.

Proof The following is a congruence:

$$(A + B) + (A + C) \underset{[m;\nu_1,\ \nu_0,\ n;\nu_0,\ \nu_1]}{\overset{[m;\nu_0,\ \nu_0,\ n;\nu_1,\ \nu_1]}{\rightrightarrows}} B + C.$$

If $f : B \to \Theta$ and $g : C \to \Theta$ make a commutative square then $[f, g] : B + C \to \Theta$ coequalises the congruence. Since the quotient is effective, to verify monos and equalisers, it suffices to inspect the congruence [FS90, 1.651] [Tay99, 5.8.10]. $\square$

Lemma 9.6 In **Set** or any [what kind of category?], if A, B, Θ and C form a pullback and A, B, D and C form a pushout, with all these maps mono, then the mediator $d : D \to \Theta$ is also mono.

Proof We consider the ***kernel*** of d (the pullback of d against itself), $K \subset D \times D$.

Since D is the union of its subobjects B and C and the pullback d^* preserves unions, $D \times D$ is the union of four parts, $B \times B$, $B \times C$, $C \times B$ and $C \times C$, and K is the union of their intersections with it. Putting these parts together,

$$\left.\begin{array}{lclcl}
\mathsf{ker}\,(i\,;d) & = & K \cap B \times B & = & \Delta_B \\
\mathsf{pbk}\,(i\,;d,\ j\,;d) & = & K \cap B \times C & = & \Delta_A \\
\mathsf{pbk}\,(j\,;d,\ i\,;d) & = & K \cap C \times B & = & \Delta_A \\
\mathsf{ker}\,(j\,;d) & = & K \cap C \times C & = & \Delta_C
\end{array}\right\} \longrightarrow\!\!\!\!\to K \subset D \times D$$

so the kernel $K \subset D \times D$ is $\Delta_B \cup \Delta_C$, which is the diagonal Δ_D. Hence d is mono, as required. $\square$

Lemma 9.7 Well founded subcoalgebras and attempts admit binary joins.

$$\begin{tikzcd}
& & B \arrow[dll] \arrow[dl, hook] \arrow[drr] & & \\
E & B \cup C \arrow[l, hook, dotted] \arrow[rrr, bend left=60] \arrow[rrr, bend right=60] & & B \cap C \arrow[ul, hook] \arrow[dl, hook] \arrow[r, two heads] & \Theta \\
& & C \arrow[ull] \arrow[ul, hook] \arrow[urr] & &
\end{tikzcd}$$

Proof Suppose that the outer diamond defines two attempts with well founded supports B and C. Let $B \cap C$ be the intersection (pullback) of these subobjects of E, so $B \cap C$ is a well founded coalgebra by Lemma 9.4. By Lemma 6.4, the restrictions $B \cap C \to B \to \Theta$ and $B \cap C \to C \to \Theta$ agree. By the union property we have $B \cup C \to \Theta$. □

Recall that Lemma 9.4 used equalisers and so assumed that regular monos are predicates admitting induction, whereas in that section we went on to prove (uniqueness in) the recursion theorem by another argument, without using this. In other words, we could avoid using equalisers here by relying instead on the main Theorem, for which this was intended to be a lemma.

The other remaining issue with pushouts or binary joins is to show that **Ens** has them. This is the generalisation of the strange "overlapping union" in Set Theory: putting B and C together does not yield a *coproduct* $B + C$ but their *pushout* rooted at their meet $A \equiv B \cap C$ from Theorem 7.12(d).

We already know from Proposition 5.3 that the functors $\mathbf{WfCoAlg} \longrightarrow \mathbf{CoAlg} \longrightarrow \mathcal{C}$ create colimits, whilst by Theorem 7.12, $\mathbf{Ens} \to \mathbf{WfCoAlg}$ creates *filtered* colimits and the initial object.

Proposition 9.8 The preorder **Ens** has binary joins.

$$\begin{tikzcd}
 & B \arrow[rr, hook, "\beta"] \arrow[dr, hook, "i"] & & TB \arrow[dr, hook, "Ti"] & \\
A \equiv B \cap C \arrow[ur, hook] \arrow[dr, hook] & & D \arrow[rr, "\delta"] & & TD \\
 & C \arrow[rr, hook, "\gamma"] \arrow[ur, hook, "j"] & & TC \arrow[ur, hook, "Tj"'] &
\end{tikzcd}$$

Proof Let (B, β) and (C, γ) be ensembles, so β and γ are initial segments.

By Theorem 7.12, they have a meet $A \equiv B \cap C$, and the maps $B \leftarrow A \to C$ are initial segments.

Let D be the pushout in $\mathcal{C}$; it is well founded by Proposition 5.3. By the union assumption (*cf.* Lemma 9.5), i and j are initial segments, as are Ti, Tj, $\beta\,; Ti$ and $\gamma\,; Tj$.

The key point is that $B \cap C$ is the pullback rooted at either D or TD, because it was constructed in Theorem 7.12(d) as the greatest span (attempt) $B \hookleftarrow A \hookrightarrow D$, not with reference to any particular pullback cospan $B \to D \leftarrow C$.

Therefore $\delta : D \to TD$ is an initial segment by the union assumption (*cf.* Lemma 9.6), making D extensional. □

The argument that Osius gave for this [Osi74, Thm 6.6] is rather more complicated. Throughout his paper he used recursion instead of well-foundedness (*cf.* Proposition 2.10) and of course $T \equiv \mathcal{P}$, but for this particular result he used the partial map classifier $\tilde{C}$ (nowadays written $C_\perp$) in a topos.

10 Well founded elements

I don't know whether to include this section in the paper, or where to put it.

Lemma 10.1 The relative successor can be defined on subobjects of the carrier of a coalgebra.

Then such a subobject is a well founded element iff it is a well founded subcoalgebra.

$$\begin{tikzcd}
TU \arrow[rr, hook, "Tj"] & & TB \arrow[r, hook, "Ti"] & TA \\
sU \arrow[u] \arrow[ur, phantom, "\urcorner", very near start] \arrow[rr, hook] & & sB \arrow[u] \arrow[ur, phantom, "\urcorner", very near start] \arrow[r, hook] & A \arrow[u, "\alpha"'] \\
H \arrow[u, hook] \arrow[ur, phantom, "\urcorner", very near start] \arrow[r] & U \arrow[r, hook] & B \arrow[u, hook] \arrow[ur, phantom, "\urcorner", very near start] \arrow[r, "\mathsf{id}"] & B \arrow[u, hook, "i"'] \arrow[ul, "\beta"', to path={-- (\tikztotarget)}]
\end{tikzcd}$$

Definition 10.2 Let X have binary meets and $s : X \to X$ be a monotone endofunction. We say that $x : X$ is a ***well founded element*** if

$$x \ \leq \ sx \qquad \text{and} \qquad \forall u{:}X.\ \ (su \wedge x \ \leq \ u) \implies x \ \leq \ u.$$

Lemma 10.3 Joins (such as they exist) and s preserve well-foundedness.

Proof If there is a least element $\bot$ then it is trivially well founded.

(a) Suppose that all x_i are well founded, $x = \bigvee x_i$ and $sv \wedge x \leq v$. Put $u_i \equiv v \wedge x_i \leq v$. Then

$$su_i \wedge x_i \ \leq \ sv \wedge x_i \ = \ sv \wedge x \wedge x_i \ \leq \ v \wedge x_i \ \equiv \ u_i,$$

so $x_i \leq u_i \leq u$ by well-foundedness of x_i and $x \equiv \bigvee x_i \leq u$. Also $x_i \leq sx_i \leq sx$ so $\bigvee s_i x \leq sx$ and therefore x is well founded.

(b) We prove the slightly more general result that if $x \leq y \leq sx$ with x well founded then y is too. Suppose that $sv \wedge y \leq v$ and put $u \equiv v \wedge x \leq v$. Then

$$su \wedge x \ \leq \ sv \wedge y \wedge x \ \leq \ v \wedge x \ \equiv \ u,$$

so $x \leq u \leq v$ by well-foundedness of x, whence $y \leq sx \leq sv$ and $y = sv \wedge y \leq v$. Also $y \leq sx \leq sy$ since $x \leq y$, so y is well founded. □

Lemma 10.4 If there is an element x that is

(a) the greatest well founded element,

(b) well founded and fixed by s,

(c) the least post-fixed point, or

(d) the least fixed point,

then it also has the other three properties.

Proof As in Corollary 3.4, [a⇒b] and [c⇒d] use the fact that sx has the same property as x, but x is the greatest or least such.

[b⇒c]: If x is well founded and $sy \leq y$ then $sy \wedge x \leq sy \leq y$, so $x \leq y$.

[c⇒a]: If $su \wedge x \leq u$ and $sx \leq x$ then

$$s(u \wedge x) \ \leq \ su \wedge sx \ \leq \ su \wedge x \ \leq \ u \wedge x,$$

so since x was least we have $x \leq u \wedge x \leq u$. Hence x is well founded.

If w is also well founded then, since $sx \wedge w \leq sx \leq x$, we have $w \leq x$. □

Proposition 10.5 If X has $\bot$, $\wedge$ and directed joins then any $s : X \to X$ does have an element with these properties.

Proof By the same argument as for Theorem 3.9, except that we use the subset of well founded elements instead of X_0, which is now redundant. $\square$

Definition 10.6 A ***Heyting semilattice*** $(X, \leq, \wedge, \to)$ is a poset with meets and another binary operation, written $\to$ and called ***implication***, that satisfies

$$(a \wedge b) \leq c \iff a \leq (b \to c), \qquad \text{so} \qquad b \wedge (b \to c) \leq c.$$

Lemma 10.7 In a Heyting semilattice, $(-) \wedge b$ preserves (distributes over) all joins that exist. If this holds in a complete lattice then $b \to (-)$ exists. $\square$

Proposition 10.8 Let X be a Heyting semilattice, $s : X \to X$ preserve binary meets and $x, y : X$. If x is well founded and $x \geq y \leq sy$ then y is well founded too.

Proof Suppose that $v : X$ satisfies $sv \wedge y \leq v$ and put $u \equiv (y \to v)$. Then

$$su \wedge y = s(y \to v) \wedge sy \wedge y = s\big((y \to v) \wedge y\big) \wedge y \leq sv \wedge y \leq v,$$

since $y = sy \wedge y$, s preserves meets and $(y \to v) \wedge y \leq v$. Hence, by definition of $(y \to v)$,

$$su \wedge x \leq su \leq (y \to v) \equiv u.$$

Then $y \leq x \leq u \equiv (y \to v)$ by well-foundedness of x, and $y \leq (y \to v) \wedge y \leq v$. $\square$

Examples 10.9 The additional hypotheses are necessary.

$$\begin{array}{ccc} y & \leq & sy = ss\bot \\ \text{\rotatebox{90}{$\leq$}} & & \text{\rotatebox{90}{$\leq$}} \\ \bot & \leq & s\bot \end{array} \qquad\qquad \begin{array}{ccccccccccc} y & \leq & sy & \leq & ssy & \leq & sssy & \leq & \cdots & \leq & s^\omega y \\ \text{\rotatebox{90}{$\leq$}} & & \text{\rotatebox{90}{$\leq$}} & & \text{\rotatebox{90}{$\leq$}} & & \text{\rotatebox{90}{$\leq$}} & & & & \| \\ \bot & \leq & s\bot & \leq & ss\bot & \leq & sss\bot & \leq & \cdots & \leq & s^\omega \bot \end{array}$$

In both cases, the elements $s^n\bot$ and $s^\omega\bot$ are well founded by Lemma 10.3, but y is not, because $s\bot \wedge y \leq \bot$ but $y \not\leq \bot$.

The first is a Heyting semilattice, but s does not preserve the meet $y \wedge s\bot = \bot$.

The second is also distributive but it is not a Heyting semilattice, since $y \wedge (-)$ does not preserve the directed join $\bigvee^{\uparrow} s^n\bot$. However, s preserves meets because, for $n < \omega$ and $m \leq \omega$,

$$s^n\bot \wedge s^m y = s^{\mathsf{min}(n,m)}\bot. \qquad \square$$

Remark 10.10 You may perhaps feel that lattices do not provide "real" counterexamples. However,

(a) any poset is a category, so if there is a poset counterexample to a statement then it's not true for categories either; whilst

(b) we have seen that the real business of well-foundedness, in particular their role as "fragments" of the initial algebra, goes on amongst the *extensional* ones, which form a preorder, so anything else is noise.

One question that is not settled by reflecting the discussion into extensional coalgebras is what that reflection itself does regarding well-foundedness. That is, does the opposite direction of

Lemma 8.6 hold? This comes down to whether there are coalgebras W, E and C and a triangle of homomorphisms

$$\begin{tikzcd} W \arrow[rr, tail, "j"] \arrow[dr, twoheadrightarrow, "p"'] & & E \arrow[dl, twoheadrightarrow, "f"] \\ & C & \end{tikzcd}$$

in which C is well founded (and maybe extensional or even an initial algebra), p and f are extremal epi and j is a mono but not an isomorphism, so E is not well founded. I don't know of such an example at the moment. $\square$

11 Ordinals for monads

Since I don't have much to add to what Joyal and Moerdijk did, this section will probably not go in a journal paper.

As with our generalisations of well founded relations and transitive sets, we want to extend the idea of *ordinals* to many different kinds of functors besides the powerset. Since the same generalisation also treats general sets as ordinals, we do not expect them to look like transfinite numbers or be linearly ordered or transitive relations.

The features that we regard as characteristic of ordinals are *successors*, *limits* and *recursion*, where the last uses operations for successors and limits. Again, the familiar case of general sets shows that there is no *case analysis* of ordinals into successors and limits. Sufficiency of the successor and limit data for recursion derives instead from the fact that every ordinal is the limit of the successors of its elements.

These ideas were introduced for total algebras by André Joyal and Ieke Moerdijk [JM95, Appx A]. In this paper we have shown how well founded coalgebras and ensembles are *fragments* of initial algebras that need not themselves exist, so we will adapt their results to partial algebras or coalgebras.

In particular, they showed how "limits" or joins of ordinals come from additional structure on the *functor*:

Definition 11.1 Recall that a ***monad*** (T, η, μ) on a category $\mathcal{C}$ consists of a functor $T : \mathcal{C} \to \mathcal{C}$ and natural transformations $\eta : \mathsf{id}_{\mathcal{C}} \to T$ and $\mu : TT \to T$ such that

$$\eta_{TX} \,;\, \mu_X \;=\; \mathsf{id}_{TX} \;=\; T\eta_X \,;\, \mu_X \qquad \text{and} \qquad \mu_{TX} \,;\, \mu_X \;=\; T\mu_X \,;\, \mu_X.$$

Then an ***algebra*** for (T, η, μ) is an object A of $\mathcal{C}$ together with a map $m_A : TA \to A$ such that

$$\eta_A \,;\, m_A \;=\; \mathsf{id}_A \qquad \text{and} \qquad \mu_A \,;\, m_A \;=\; Tm_A \,;\, m_A.$$

Throughout this section we suppose given a particular monad, where the category and functor obey the conditions from earlier in the paper.

Remark 11.2 Joyal and Moerdijk characterised the universes of sets and ordinals as free complete join-semilattices equipped with a successor operation s satisfying varying conditions. Together with the corresponding treatments in [Tay96a], these are:

(a) with no condition on $sx \equiv \{x\}$, general sets;

(b) if $x \leq sx \equiv x \cup \{x\}$, *thin* ordinals;

(c) if $x \leq y \Rightarrow sx \leq sy$, *plump* ordinals, where sx is the set of plump ordinal subsets of x; and

(d) if $s(x \vee y) = sx \vee sy$, *directed* plump ordinals, where $\leq$ is the order associated with the join structure.

If we look at this in comparison to other kinds of algebra (again trying to put aside set-theoretic expectations), it seems very strange to have an extra operation s with no or very weak connection to the rest of the structure. Yet, this actually comes out of another strange but general observation, due to Jean Bénabou and Mamuka Jibladze, that "less is more":

Proposition 11.3 Any fixed point $\theta : TA \cong A$ of the functor T *alone* carries the structure of an algebra for the monad with an *additional* unary operation:

$$m \equiv T\theta^{-1} ; \mu_A ; \theta : TA \to A \qquad \text{and} \qquad s \equiv \eta_A ; \theta : A \to A,$$

from which we recover $\theta = Ts ; m$.

Proof

$$\begin{array}{rcl} \eta_A ; m & \equiv & \eta_A ; T\theta^{-1} ; \mu_A ; \theta \;=\; \theta^{-1} ; \eta_{TA} ; \mu_A ; \theta \;=\; \theta^{-1} ; \theta \;=\; \mathsf{id}_A \\ Tm ; m & \equiv & TT\theta^{-1} ; T\mu_A ; T\theta ; T\theta^{-1} ; \mu_A ; \theta \;=\; TT\theta^{-1} ; T\mu_A ; \mu_A ; \theta \\ & = & TT\theta^{-1} ; \mu_{TA} ; \mu_A ; \theta \;=\; \mu_A ; T\theta^{-1} ; \mu_A ; \theta \;\equiv\; \mu_A ; m \\ Ts ; m & \equiv & T\eta_A ; T\theta ; T\theta^{-1} ; \mu_A ; \theta \;=\; T\eta_A ; \mu_A ; \theta \;=\; \theta. \end{array}$$ □

Example 11.4 Recall that the monad defining complete join-semilattices over **Set** has $T \equiv \mathcal{P}$ with

$$\eta x \equiv \{x\} \qquad \text{and} \qquad \mu\mathcal{U} \equiv \bigcup \mathcal{U} \equiv \bigcup \{U \mid U \in \mathcal{U}\}.$$

In this example, $Ts ; m = \mathsf{id}$, which says that any set is the union of the singletons of its elements.

Example 11.5 Lists represented as binary trees.

Lemma 11.6 The functor from (T, η, μ, s)-algebras to T-algebras given by

$$m_A : TA \to A,\ s_A : A \to A \longmapsto \theta_A \equiv Ts_A ; m_A : TA \to A$$

is well defined and faithful and every θ-homomorphism $f : A \to B$ is an s-homomorphism. If $T\theta_A$ is epi then f is an m-homomorphism too.

Proof If $f : A \to B$ is a (s, m)-homomorphism then it is also a $(\theta \equiv Ts ; m)$-homomorphism (easily).

Conversely, let $f : A \to B$ be a θ-homomorphism. Then it is also an s-homomorphism since $s = \eta ; \theta$ and η is natural. Also,

$$\begin{array}{rcl} T\theta_A ; m_A & \equiv & TTs_A ; Tm_A ; m_A \;=\; TTs_A ; \mu_A ; m_A \;=\; \mu_A ; Ts_A ; m_A \;\equiv\; \mu_A ; \theta_A \\ T\theta_A ; m_A ; f & = & \mu_A ; \theta_A ; f \;=\; \mu_A ; Tf ; \theta_B \;\equiv\; \mu_A ; Tf ; Ts_B ; m_B \\ & = & TTf ; TTs_B ; \mu_B ; m_B \;=\; TTf ; TTs_B ; Tm_B ; m_B \\ & \equiv & TTf ; T\theta_B ; m_B \;=\; T\theta_A ; Tf ; m_B \end{array}$$

and so $m_A ; f = Tf ; m_B$ if $T\theta_A$ is epi. □

Proposition 11.7 (Joyal and Moerdijk) The initial (T, η, μ, s)-algebra (A, m, s) has $\theta \equiv Ts ; m$ invertible and is the initial T-algebra.

Proof From the usual theory of monads, (TA, μ_A) is an algebra and θ is a homomorphism for the monad structure. We define a successor operation

$$t \equiv Ts ; m ; \eta_A : TA \to TA,$$

for which θ is a homomorphism because

$$t\,;\theta \;\equiv\; Ts\,;m\,;\eta_A\,;Ts\,;m \;=\; Ts\,;m\,;s \;\equiv\; \theta\,;s.$$

Since (A, m, s) was the initial (T, η, μ, s)-algebra, there is a unique homomorphism $f : A \to TA$, so

$$f\,;t \;\equiv\; f\,;Ts\,;m\,;\eta_A \;=\; s\,;f \qquad \text{and} \qquad Tf\,;\mu_A \;=\; m\,;f,$$

whilst

$$f\,;\theta \;\equiv\; f\,;Ts\,;m \;=\; \mathsf{id}_A$$

by uniqueness of $A \to A$. Hence

$$Ts\,;m\,;f \;=\; Ts\,;Tf\,;\mu_A \;=\; Tf\,;TTs\,;Tm\,;T\eta_A\,;\mu_A \;=\; T(f\,;Ts\,;m) \;=\; \mathsf{id}_{TA},$$

so f and θ are inverse. $\square$

Corollary 11.8 If either the initial T-algebra $\theta_A : TA \cong A$ or the initial (T, η, μ, s)-algebra exists then it serves as the other.

Proof Let (B, m_B, s_B) a (T, η, μ, s)-algebra and define m_A, s_A and θ_B as in the Lemma. Since (A, θ) is initial, there is a unique θ-homomorphism $f : A \to B$. This is also an (m, s)-homomorphism since $T\theta_A$ is epi. It is unique because any (m, s)-homomorphism is also a θ-homomorphism. $\square$

Now we adapt these results to coalgebras.

The operations s and m are to be *endofunctions* of a *particular* ordinal. To obtain these we use θ to encode a subset as an element and its inverse α to do the reverse, but these need not be total operations. The situation is a bit like doing fixed precision multiplication in a CPU: we can either do it accurately with overflow, or approximately with truncation.

Definition 11.9 For an extensional coalgebra $\alpha : A \hookrightarrow TA$, the ***truncated successor*** and ***join*** are the partial functions $s_A : A \rightharpoonup A$ and $m_A : TA \rightharpoonup A$ defined by the inverse images

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, hook, "\alpha"] \\ A \arrow[r, "\eta_A"'] & TA \end{tikzcd} \qquad \begin{tikzcd} \bullet \arrow[rr, two heads] \arrow[d, hook] \arrow[dr, phantom, "\lrcorner", very near start] & & A \arrow[ll, dotted, bend right] \arrow[d, hook, "\alpha"] \\ TA \arrow[r, hook, "T\alpha"] & T^2A \arrow[r, "\mu_A"] & TA \end{tikzcd}$$

where the top right map is split epi because we have $\eta_A : A \to TA$ with $\eta_A\,;T\alpha\,;\mu_A = \alpha$.

However, these are not strong enough to obtain uniqueness in the recursion theorem below.

Definition 11.10 The ***overflow successor*** and ***join*** are the total functions

$$S_A \equiv \eta_A : A \to TA \qquad \text{and} \qquad M_A \equiv T\alpha\,;\mu_A : TA \to TA.$$

For any map $f : A \to \Theta$ to a T-algebra $\theta : T\Theta \to \Theta$ the ***extension*** of f is

$$\hat{f} \equiv Tf\,;\theta : TA \to \Theta.$$

Lemma 11.11 Any map f together with its extension $\hat{f}$ define an overflow homomorphism for s,

$$S_A\,;\hat{f} \;\equiv\; \eta_A : Tf\,;\theta \;=\; f\,;\eta_\Theta\,;\theta \;=\; f\,;s_\Theta.$$

If $f : A \to \Theta$ satisfies the recursion equation

$$f \;=\; \alpha \,;\, Tf \,;\, \theta$$

then f and $\hat{f}$ define an overflow homomorphism for m too,

$$M_A \,;\, \hat{f} \;\equiv\; T\alpha \,;\, \mu_A \,;\, Tf \,;\, \theta \;=\; Tf \,;\, m_\Theta.$$

Conversely, any overflow homomorphism for m also satisfies the recursion equation.

Any overflow homomorphism restricts to a homomorphism of the truncated operations.

Proof If $f : A \to \Theta$ satisfies the recursion equation then

$$\begin{aligned}
M_A \,;\, \hat{f} &\equiv T\alpha \,;\, \mu_A \,;\, Tf \,;\, \theta \\
&\equiv T\alpha \,;\, \mu_A \,;\, Tf \,;\, Ts_\Theta \,;\, m_\Theta \\
&= T\alpha \,;\, TTf \,;\, TTs_\Theta \,;\, \mu_\Theta \,;\, m_\Theta \\
&= T\alpha \,;\, TTf \,;\, TTs_\Theta \,;\, Tm_\Theta \,;\, m_\Theta \\
&\equiv T\alpha \,;\, TTf \,;\, T\theta \,;\, m_\Theta \\
&\equiv Tf \,;\, m_\Theta
\end{aligned}$$

Conversely,

$$\begin{aligned}
\alpha \,;\, Tf \,;\, \theta &= \alpha \,;\, \eta_{TA} \,;\, \mu_A \,;\, Tf \,;\, \theta \\
&= \eta_A \,;\, T\alpha \,;\, \mu_A \,;\, Tf \,;\, \theta \\
&= \eta_A \,;\, Tf \,;\, m_\Theta \\
&= f \,;\, \eta_\Theta \,;\, m_\Theta \quad = \quad f.
\end{aligned}$$

□

Then Corollary 11.8 becomes the ***Transfinite Recursion Theorem***:

Theorem 11.12 Let A be a well founded T-coalgebra and $(\Theta, m_\Theta, s_\Theta)$ any (T, η, μ, s)-algebra. Then there is a unique overflow homomorphism $f : A \to \Theta$.

Proof Considering Θ as a T-algebra with structure $Ts_\Theta \,;\, m_\Theta : T\Theta \to \Theta$, the Recursion Theorem 6.9 provides $f : A \to \Theta$ satisfying the recursion equation. By the Lemma this is an overflow homomorphism. It is unique because the Lemma also shows that any overflow homomorphism satisfies the recursion equation. □

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