A lambda calculus for real analysis

Paul Taylor

Senior Research Fellow,

Department of Computer Science,

University of Manchester

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www.cs.man.ac.uk/∼pt/ASD

pt@cs.man.ac.uk

077 604 625 87

This lecture

It takes me more than 20 mins to introduce my research to my **own** community,
so this lecture will be a **classical translation** of some recent results.
It will require **First Year Undergraduate Real Analysis**.

Please contact me this week by mobile (**077 604 625 87**)
or later by email (**pt@cs.man.ac.uk**) to learn more.

(Maybe even invite me to your university to give a full seminar.)

Intellectual pedigree

mine:
Mathematical Tripos 1979–83
PhD in Category Theory 1983–6
A mathematician in exile in Computer Science

of this general area of research:
(compact–open) topologies on function-spaces, topological lattice theory,
semantics of programming languages, formal correctness of programs.

of the Abstract Stone Duality programme:
locale theory ("point-less topology", *i.e.* only using open sets)
category theory, domain theory.

The journey that Abstract Stone Duality has made so far:
from an abstract hypothesis from category theory
to computably based locally compact spaces (not just $\mathbb{R}^n$)
to **constructive analysis**.

Constructive analysis

The classics, although I don't myself belong to this tradition.

Errett Bishop, *Foundations of Constructive Analysis*, 1967
A "can do" attitude to constructivity,
entirely compatible with the classical results:
we just have to be a lot more careful.

Errett Bishop and Douglas Bridges, *Constructive Analysis*, Springer, 1985

Douglas Bridges and Fred Richman, *Varieties of Constructive Mathematics*, CUP, 1987

The subject is based on **metrical** (ϵ–δ) methods.
$S \subset \mathbb{R}$ is **totally bounded** if it has an ϵ-net — needed to define its supremum,
$S \subset \mathbb{R}$ is **located** if $d(x, S) \equiv \inf \{d(x, y) \mid y \in S\}$ is definable.

Recursive analysis — the bad news

Cantor space $\mathbf{2}^{\mathbb{N}}$ and the closed real interval $\mathbb{I} \equiv [0,1] \subset \mathbb{R}$ **are not compact**.

Basic problem: definable/computable/recursive values can be **enumerated**
(like the rationals — it's just a bit more complicated).
Richard's Paradox 1900, Turing's Computable Numbers 1937, Specker Sequences 1949.

Let (u_n) be such an enumeration of the definable elements of $[0,1]$.
Cover each u_n with the open interval $(u_n \pm \epsilon \cdot 2^{-n})$.
These intervals have total length 2ϵ.
With $\epsilon \equiv \frac{1}{2}$, no finite sub-collection can cover.

Another way: **König's Lemma fails**:
there is an infinite binary (**Kleene**) tree with no infinite **computable** path.

In addition to the metrical (ϵ–δ) methods,
everything has to be coded using Gödel numbers.

Recursive analysis — the good news

This doesn't happen in Abstract Stone Duality.

Cantor space $\mathbf{2}^{\mathbb{N}}$ and the closed real interval $\mathbb{I} \equiv [0,1] \subset \mathbb{R}$ **are compact**.

$\forall$ in ASD doesn't mean "for every definable element" —
it's defined to satisfy the **formal rules** of predicate calculus.

A categorical construction ensures that subspaces always have the **subspace topology**.
(But I'm not going to talk about this in this lecture.)

The mathematical arguments are **topological**, not metrical.

Programming languages can be translated naturally **into** ASD (denotational semantics).

Conversely, every ASD term has a natural **computational interpretation**
(as a parallel, non-deterministic, higher-order logic program).

The Classical Intermediate Value Theorem

Any continuous $f : [0, 1] \to \mathbb{R}$ with $f(0) \leq 0 \leq f(1)$ has a zero.

Indeed, $f(x_0) = 0$ where $x_0 \equiv \sup\{x \mid f(x) \leq 0\}$.

A so-called "closed formula".

A program: interval halving

Let $a_0 \equiv 0$ and $e_0 \equiv 1$.

By recursion, consider $c_n \equiv \frac{1}{2}(a_n + e_n)$ and

$$a_{n+1},\ e_{n+1} \ \equiv \ \begin{cases} a_n,\ c_n & \text{if } f(c_n) > 0 \\ c_n,\ e_n & \text{if } f(c_n) \le 0, \end{cases}$$

so by induction $f(a_n) \le 0 \le f(e_n)$.

But a_n and e_n are respectively (non-strictly) increasing and decreasing sequences, whose differences tend to 0.

So they converge to a common value c.

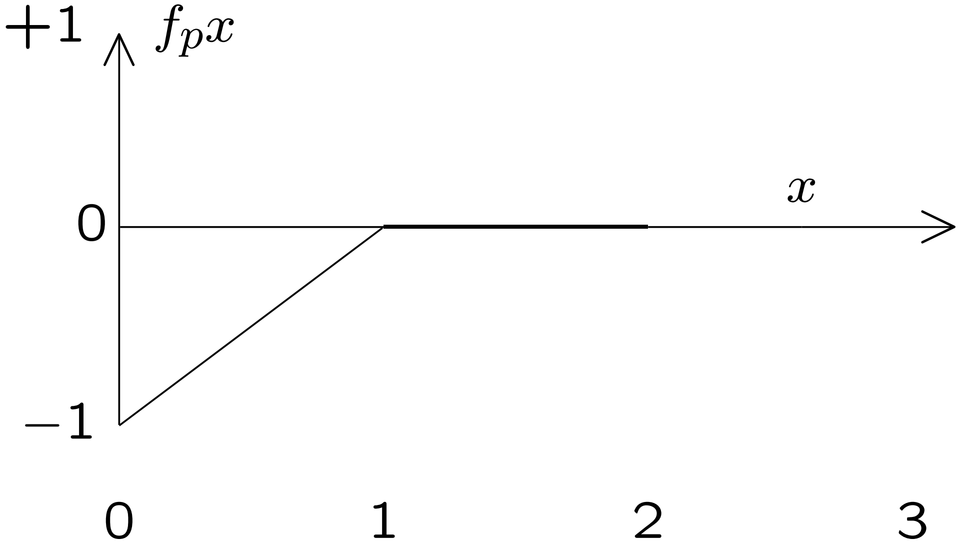
By continuity, $f(c) = 0$.

Where is the zero?

For $-1 \leq p \leq +1$ and $0 \leq x \leq 3$ consider

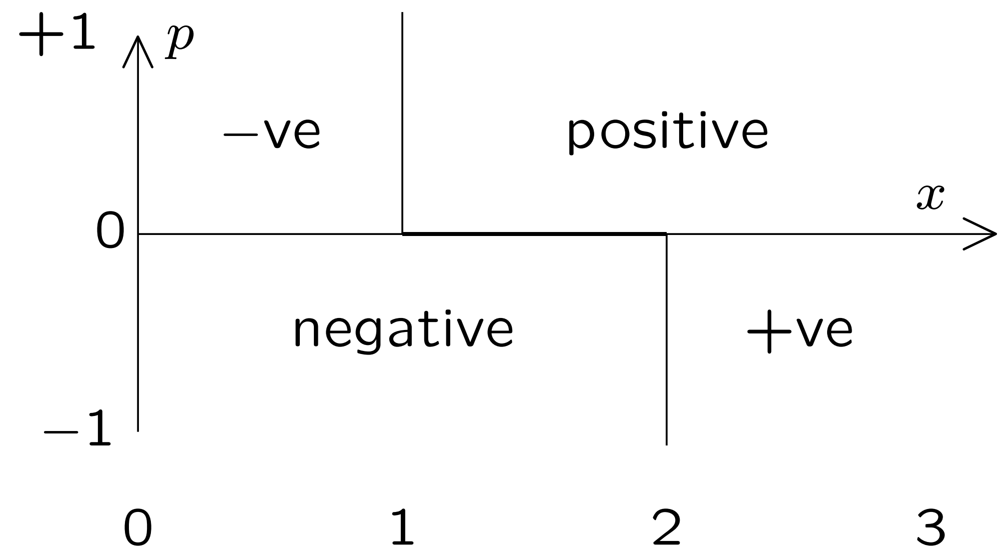
$$f_p x \;\equiv\; \mathsf{min}\,(x-1,\, \mathsf{max}\,(p, x-2))$$

Here is the graph of $f_p(x)$ against x for $p \approx 0$.



Where is the zero?

The behaviour of $f_p(x)$ depends qualitatively on p and x like this:



$$
\begin{aligned}
f(1) = 0 &\iff p \geq 0 \\
f(2) = 0 &\iff p \leq 0 \\
f(\tfrac{3}{2}) = 0 &\iff p = 0
\end{aligned}
$$

If there is some way of finding a zero of f_p,
as a side-effect it will decide how p stands in relation to 0.

Let's bar the monster

$f : \mathbb{R} \to \mathbb{R}$ **doesn't hover** if,

for any $e < t$, $\quad \exists x. \, (e < x < t) \ \wedge \ (fx \neq 0)$.

Any nonzero polynomial doesn't hover.

Interval halving again

Suppose that f doesn't hover.

Let $a_0 \equiv 0$ and $e_0 \equiv 1$.

By recursion, consider

$$b_n \equiv \tfrac{1}{3}(2a_n + e_n) \quad \text{and} \quad d_n \equiv \tfrac{1}{3}(a_n + 2e_n).$$

Then $f(c_n) \neq 0$ for some $b_n < c_n < d_n$, so put

$$a_{n+1},\ e_{n+1} \equiv \begin{cases} a_n,\ c_n & \text{if } f(c_n) > 0 \\ c_n,\ e_n & \text{if } f(c_n) < 0, \end{cases}$$

so by induction $f(a_n) < 0 < f(e_n)$.

But a_n and e_n are respectively (non-strictly) increasing and decreasing sequences, whose differences tend to 0.

So they converge to a common value c.

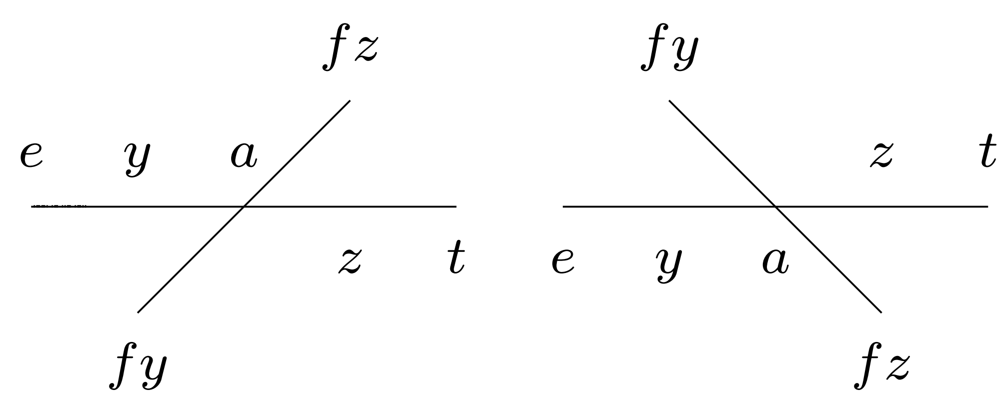
By continuity, $f(c) = 0$.

Stable zeroes

The revised interval halving algorithm finds zeroes with this property:

$a \in \mathbb{R}$ is a **stable zero** of f
if, for all $e < a < t$,

$$\exists yz.\, (e < y < a < z < t) \;\wedge\; (fy < 0 < fz \;\vee fy > 0 > fz).$$



Check that a stable zero of a continuous function really is a zero.

Classically, a zero is stable iff
every nearby function (in the sup or ℓ_∞ norm) has a nearby zero.

Straddling intervals

An open subspace $U \subset \mathbb{R}$ **touches** S, *i.e.* contains a stable zero, $a \in U \cap S$,
iff U contains a **straddling interval**,

$$[e, t] \subset U \quad \text{with} \quad fe < 0 < ft \quad \text{or} \quad fe > 0 > ft.$$

Proof [$\Leftarrow$] The straddling interval is an intermediate value problem in miniature.

If an interval $[e, t]$ straddles with respect to f
then it also does so with respect to any nearby function.

The possibility operator

Write $\Diamond U$ if U contains a straddling interval.

By hypothesis, $\Diamond I \Leftrightarrow \top$ (where I is some open interval containing $\mathbb{I}$).

Trivially, $\Diamond \emptyset \Leftrightarrow \bot$.

$$\Diamond \bigcup_{i \in I} U_i \iff \exists i.\ \Diamond U_i.$$

Consider

$$V^{\pm} \equiv \{x \mid \exists y{:}\mathbb{R}.\ \exists i{:}I.\ (fy \gtrless 0) \wedge [x,y] \subset U_i\}$$

so $\mathbb{I} \subset V^{+} \cup V^{-}$.

Then $x \in (a,c) \subset V^{+} \cap V^{-}$ by connectedness, with $fx \neq 0$ and $[x,y] \subset U_i$.

The Possibility Operator as a Program

Let $\Diamond$ be a property of open subspaces of $\mathbb{R}$
that preserves unions and satisfies $\Diamond\, U_0$ for some open interval U_0.

Then $\Diamond$ has an "accumulation point" $c \in U_0$,
i.e. one of which every open neighbourhood $c \in U \subset \mathbb{R}$ satisfies $\Diamond\, U$.
In the example of the intermediate value theorem, any such c is a stable zero.

Interval halving again: let $a_0 \equiv 0$, $e_0 \equiv 1$
and, by recursion, $b_n \equiv \frac{1}{3}(2a_n + e_n)$ and $d_n \equiv \frac{1}{3}(a_n + 2e_n)$, so

$$\Diamond(a_n, e_n) \;\equiv\; \Diamond\,((a_n, d_n) \cup (b_n, e_n)) \;\Leftrightarrow\; \Diamond(a_n, d_n) \vee \Diamond(b_n, e_n).$$

Then at least one of the disjuncts is true,
so let (a_{n+1}, e_{n+1}) be either (a_n, d_n) or (b_n, e_n).

Hence a_n and e_n converge from above and below respectively to c.

If $c \in U$ then $c \in (a_n, e_n) \subset (c \pm \epsilon) \subset U$ for some $\epsilon > 0$ and n,
but $\Diamond(a_n, e_n)$ is true by construction,
so $\Diamond\, U$ also holds, since $\Diamond$ takes $\subset$ to $\Rightarrow$.

Enclosing cells of higher dimensions

Straddling intervals can be generalised.

Let $f : \mathbb{R}^{\mathbf{n}} \to \mathbb{R}^{\mathbf{m}}$ with $n \geq m$.

Let $C \subset \mathbb{R}^{\mathbf{n}}$ be a sphere, cube, *etc*.

C is an **enclosing cell** if
0 $\in \mathbb{R}^{\mathbf{m}}$ lies in the interior of the image $f(C) \subset \mathbb{R}^{\mathbf{m}}$.

(There is a definition for locally compact spaces too.)

Write $\Diamond U$ if $U \subset \mathbb{R}^{\mathbf{n}}$ contains an enclosing cell.

If $\Diamond \left(\bigcup_{i \in I} U_i \right) \Leftrightarrow \exists i.\ \Diamond U_i$ then
cell halving finds stable zeroes of f.

Modal operators, separately

$Z \equiv \{x \in \mathbb{I} \mid fx = 0\}$ is closed and compact.
$W \equiv \{x \mid fx \neq 0\}$ is open.
S is the subspace of stable zeroes.

For $U \subset \mathbb{R}$ open, write $\Box U$ if $Z \subset U$ (or $U \cup W = \mathbb{R}$).

$\Box X$ is true and $\Box U \wedge \Box V \Rightarrow \Box(U \cap V)$

$\Diamond \emptyset$ is false and $\Diamond(U \cup V) \Rightarrow \Diamond U \vee \Diamond V$.

$(Z \neq \emptyset)$ iff $\Box \emptyset$ is false

$(S \neq \emptyset)$ iff $\Diamond \mathbb{R}$ is true

Both operators are Scott continuous.

Modal operators, together

The modal operators $\Diamond$ and $\Box$ for the subspaces $S \subset Z$ are related in general by:

$$\Box U \wedge \Diamond V \;\Rightarrow\; \Diamond(U \cap V)$$

$$\Box U \;\Longleftrightarrow\; (U \cup W = X)$$

$$\Diamond V \;\Rightarrow\; (V \not\subset W)$$

S is dense in Z iff

$$\Box(U \cup V) \;\Rightarrow\; \Box U \vee \Diamond V$$

$$\Diamond V \;\Leftarrow\; (V \not\subset W)$$

In the intermediate value theorem for functions that don't hover (*e.g.* polynomials):
$S = Z$ in the **non-singular** case
$S \subset Z$ in the **singular** case (*e.g.* double zeroes).

Open maps

For continuous $f : X \to Y$,
if $V \subset Y$ is open, so is $f^{-1}(V) \subset X$
if $V \subset Y$ is closed, so is $f^{-1}(V) \subset X$
if $U \subset X$ is compact, so is $f(U) \subset Y$
(if $U \subset X$ is overt, so is $f(U) \subset Y$)

$f : X \to Y$ is **open** if,
whenever $U \subset X$ is open, so is $f(U) \subset Y$.

If $f : X \to Y$ is open then
if $V \subset Y$ is overt, so is $f^{-1}(V) \subset X$.

If $f : X \to Y$ is open then all zeroes are stable.

Examples of open maps

If $f : \mathbb{R}^{\mathbf{n}} \to \mathbb{R}^{\mathbf{n}}$ is continuously differentiable, and det $\left(\frac{\partial f_j}{\partial x_i}\right) \neq 0$.

If $f : \mathbb{C} \to \mathbb{C}$ is analytic and not constant — even if it has coincident zeroes.

Cauchy's integral formula:
a disc $C \subset \mathbb{C}$ is enclosing iff $\oint_{\partial C} \frac{dz}{f(z)} \neq 0$.

Stokes's theorem!

Possibility operators classically

Define $\lozenge U$ as $U \cap S \neq \emptyset$,
for *any subset* $S \subset \mathbb{R}$ *whatever*.

Then $\lozenge (\bigcup_{i \in I} U_i)$ iff $\exists i.\ \lozenge U_i$.

Conversely, if $\lozenge$ has this property, let

$$S \ \equiv\ \{a \in \mathbb{R} \mid \text{for all open } U \subset \mathbb{R}, \quad a \in U \Rightarrow \lozenge U\}.$$

$$W \ \equiv\ \mathbb{R} \setminus S \ =\ \bigcup \{U \text{ open} \mid \neg \lozenge U\}$$

Then W is open and S is closed.
$\neg \lozenge W$ by preservation of unions.
Hence $\lozenge U$ holds iff $U \not\subset W$, *i.e.* $U \cap S \neq \emptyset$.

If $\lozenge$ had been derived from some S'
then $S = \overline{S'}$, its closure.

Possibility operators: summary

$\Diamond$ is defined, like compactness, in terms of unions of open subspaces, so it is a concept of **general topology**

The proof that $\Diamond$ preserves joins uses ideas from **geometric topology**, like connectedness and sub-division of cells.

$\Diamond$ is like a bounded existential quantifier, so it's **logic**.

A very general **algorithm** uses $\Diamond$ to find solutions of problems.

But classical point-set topology is too clumsy to take advantage of this.

A lambda calculus for topology — predicates

Only use **predicates** (ϕ, ψ) that denote **open subspaces**,
equivalently, which are **computably observable**.

On $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$: $n = m$, $n \neq m$, $n < m$, $n \leq m$, $n > m$ and $n \geq m$.

On $\mathbb{R}$: $a \neq b$, $a < b$ and $a > b$, but not $a = b$, $a \leq b$ or $a \geq b$.
(This is entirely familiar in numerical computation.)

Logically: $\top$ (true), $\bot$ (false), $\phi \wedge \psi$, $\phi \vee \psi$ and $\exists n : \mathbb{N}.\phi n$
but not $\neg\phi$ (not), $\phi \Rightarrow \psi$ or $\forall n : \mathbb{N}.\phi n$.

We shall **also** allow $\exists x : \mathbb{R}.\phi x$ and $\forall x : \mathbb{I}.\phi x$,
but **not** $\forall \epsilon > 0.\phi\epsilon$.

Statements — comparing predicates

You can't say very much in the language of predicates.

A **statement** is an **equality** $\phi \Leftrightarrow \psi$ or **inequality** $\phi \Rightarrow \psi$
where ϕ and ψ are predicates (technical distinction).

Predicates can be existentially quantified $(\exists n : \mathbb{N}.\phi n)$,
statements cannot.

Nested implication $(\phi \Rightarrow \psi) \Rightarrow \theta$ is not allowed (in the current version).

Examples: $\phi \Rightarrow \bot$ (not ϕ), $(a < b) \Rightarrow \bot$ $(a \geq b)$, $(a \neq b) \Rightarrow \bot$ $(a = b)$.

Open and closed subspaces

If $\phi(x)$ is a predicate with a free variable (argument) $x : \mathbb{R}$ then
$\{x \mid \phi(x)\} \subset \mathbb{R}$ is an open subspace and $\{x \mid \phi(x) \Leftrightarrow \bot\} \subset \mathbb{R}$ is a closed subspace.

We can think of $\phi : \mathbb{R} \to \binom{\odot}{\bullet}$ as a **continuous function**
whose target is the **Sierpiński space**.

$\{x \mid \phi(x)\} \subset \mathbb{R}$ is $\phi^{-1}(\top)$ and
$\{x \mid \phi(x) \Leftrightarrow \bot\} \subset \mathbb{R}$ is $\phi^{-1}(\bot)$.

The **Sierpiński space** $\binom{\odot}{\bullet}$ has two points (classically)
one (called $\odot$ or $\top$) is **open**, the other ($\bullet$ or $\bot$) is **closed**.

It is not Hausdorff.
It appears in many textbooks as a pathetic counterexample.

It is the **key** to understanding:
topologies of function-spaces,
semantics of programming languages,
Abstract Stone Duality.

Compact subspaces

The **neighbourhoods** of a compact subspace are more important than its **points**.

This had emerged by about 1970 in the study of topologies of function-spaces.

A compact subspace K (at least, of a Hausdorff space H)
is determined by which open subspaces $U \subset H$ **cover** it — $K \subset U$.

We write $\Box U$ or $\Box \phi$ when this happens.

$\Box$ satisfies the modal laws, in particular $\Box H \Leftrightarrow \top$ and $\Box(U \cap V) \Leftrightarrow \Box U \wedge \Box V$.

Directed unions

By the "**finite open sub-cover**" definition of compactness,
if $\Box \bigcup_{i \in I} U_i$ then $\Box \bigcup_{i \in F} U_i$ for some finite $F \subset I$.

This definition can be simplified by assuming something about the system $\{U_i \mid i \in I\}$.

We call it **directed** if I is **non-empty** (better, **inhabited**)
and, for each U_i, U_j there's some U_k with $U_i \cup U_j \subset U_k$.

Then the "**finite open sub-cover**" definition becomes:
$\Box$ preserves **directed unions**.

Scott continuity

A function between complete lattices that preserves directed unions is called **Scott continuous**.

Dana Scott (1972) defined the corresponding topology on function-spaces $X \to \binom{\odot}{\bullet}$.

It is a special case (in fact, the critical one) of Ralph Fox's **compact–open topology** (1945).

The function-space $X \to \binom{\odot}{\bullet}$ is the topology (lattice of opens) of X, itself equipped with the Scott topology.

If X is locally compact, so is $X \to \binom{\odot}{\bullet}$.

In our language, **all functions are continuous**
in the traditional Weierstrass "ϵ–δ" sense for $f : X \to \mathbb{R}$
in Scott's "directed joins" sense for function-spaces.

Any compact subspace $K \subset H$ of a Hausdorff space is closed.

In the ambient Hausdorff space H,
$x \neq y$ is an open predicate (since $H \subset H \times H$ is closed).

The compact subspace $K \subset H$ has a $\Box$ operator.

The closed subspace is defined by its open/observable **non**-membership predicate.

This is $\omega x \;\equiv\; \Box(\lambda y.\, x \neq y)$.

It says that $x \notin C$ iff $C \subset \{y \mid x \neq y\} \;\equiv\; \overline{\{x\}}$.

Any closed subspace $C \subset K$ of a compact space is compact.

The ambient compact space H has a $\Box$ operator that we call $\forall_K$
with $\forall_K \phi \Leftrightarrow \top$ iff $\phi \Leftrightarrow \top$.

The closed subspace has an open non-membership predicate ω.

As a compact subspace, it has a $\Box$ operator given by
$$\Box\phi \equiv \forall_K(\omega \vee \phi)$$
which says that ϕ and the complement ω of C together cover K.

The Intermediate Value Theorem in ASD

Develop $\Box$ as a **finitary** theory of compact subspaces using the modal laws.

Develop $\Diamond$ as a theory of **overt** subspaces using the modal laws in an entirely lattice ("de Morgan") dual way.

Use the modal laws for compact overt $K \subset \mathbb{R}$ to define a Dedekind cut, which is $\max K$. (Bishop-style constructive analysis uses total boundedness to do this.)

$[0, 1]$ is connected — by the usual argument.

$\Diamond$ (defined using straddling intervals) preserves joins.

Use interval-halving with $\Diamond$ to find stable zeroes.

The subspaces S and Z again

In the non-singular case, $\Box$ and $\Diamond$ make the zero-set compact overt.
It therefore has a maximum element!

The operators $\Box$ and $\Diamond$ are Scott-continuous
throughout the parameter space (eg for $x^3 - 3px - 2q = 0$),
unlike Z and S considered as sets.

In the possibly singular case (eg double zeroes)
Z (all zeroes) is closed and compact, but not necessarily overt
S (stable zeroes) is overt, but not necessarily closed.

Midlands Graduate School

Next week at Birmingham University.

`www.cs.bham.ac.uk/~pbl/mgs`

Four (hour) lectures on Abstract Stone Duality.

Others on Category Theory, Lambda Calculus, Denotational Semantics, Functional Programming, Quantum Programming, Game Semantics, *etc.*

Accommodation still available: email `A.Jung@cs.bham.ac.uk`

My contact details again:

`www.cs.man.ac.uk/~pt/ASD`

`pt@cs.man.ac.uk`

077 604 625 87