The Dedekind Reals in ASD

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www.cs.man.ac.uk/~pt/ASD

Axioms for the real line

An object $\mathbb{R}$ is a Dedekind real line if

- it is overt, with $\exists$;
- it is Hausdorff, with $\neq$;
- it has a total order, *i.e.* $(x \neq y) \Leftrightarrow (x < y) \vee (y < x)$;
- it is a field, where x^{-1} is defined iff $x \neq 0$;
- it is Dedekind complete;
- it is Archimedean:

$$p, q : \mathbb{R} \vdash q > 0 \Rightarrow \exists n : \mathbb{Z}.\ q(n-1) < p < q(n+1);$$

- and the closed interval is compact, with $\forall$.

Is any of these axioms optional?

We don't try to axiomatise arithmetic on a Pentium, Cray or abacus as different kinds of "fields".

Compactness of the closed interval shouldn't depend on foundations.

But compactness fails in theories in which $\mathbb{R}$ is the set of computable numbers:

- Type-One Effectivity (See Andrej Bauer's notes.)
- *Synthetic Topology* with the internal view of data
- (Russian) Recursive Analysis (See, *e.g.*, *Varieties of Constructive Mathematics* by Douglas Bridges and Fred Richman, LMS Lecture Notes 97, 1987.)

Abstract Stone Duality is a *recursive* theory of topology in which the closed real interval *is* compact.

Objectives

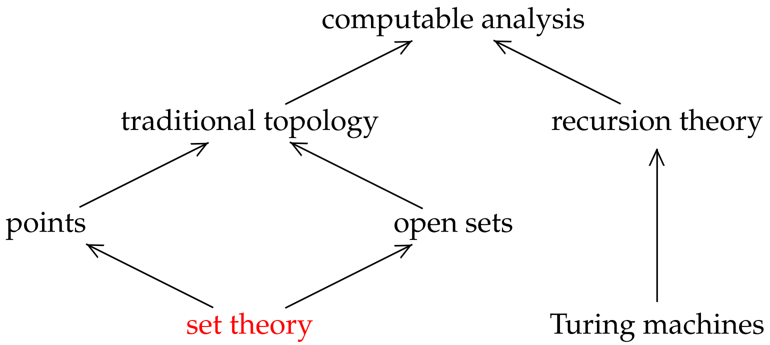
This lecture:

- How to fix models of analysis to make the interval compact.
- The computational interpretation of the axioms for the Dedekind real line.

Second lecture:

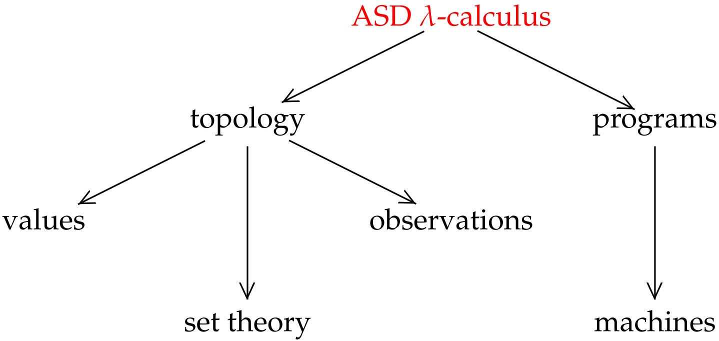
- Learning to use the ASD language for analysis.
- Overt subspaces and the Intermediate Value Theorem.

The traditional picture



From amongst general set-theoretic functions, continuous and/or computable ones are selected by means of extra conditions.

Direct axiomatisation of computable topology



Abstract Stone Duality only introduces computably continuous functions.

From amongst general spaces,
it selects the overt discrete ones to play the role of sets.

The methodology of ASD

Use the experience of proof theory (Gentzen, 1935) and categorical logic (Lawvere, 1963, 1970).

- Identify the key properties of topology and analysis as universal properties,
- translate the universal properties into proof rules (introduction, elimination, β- and η-rules),
- develop topology and analysis in the new language,
- and use the proof rules for computation.

No pre-conceived ideas from set theory or recursion theory.

Topology as λ-calculus — the classical ideas

- $\Sigma \equiv \binom{\odot}{\bullet}$ is the Sierpiński space
  (\$ in Martín Escardó's lecture)
- its points are ("geometric") truth values $\top$ and $\bot$
- continuous functions $X \to \Sigma$ correspond to open subspaces of X (inverse images of $\top$)
- continuous functions $X \to \Sigma$ correspond to closed subspaces of X (inverse images of $\bot$)
- Σ^X is the topology on X
- Σ^X itself has the Scott (= compact–open) topology
- this works fine when X is locally compact
- in this case, Σ^X is a continuous lattice
- Scott continuous $(\top, \wedge)$-homomorphisms $\Sigma^X \to \Sigma$ correspond to compact subspaces of X.

Bibliography — topology as λ-calculus

- 1945 Fox *On topologies for function spaces*
- 1972 Scott *Continuous lattices*
- 1981 Hofmann & Mislove *Local Compactness and Continuous Lattices*
- 2000 Taylor [C] *Geometric and higher order logic in terms of ASD*
- 2002 Taylor [A] *Sober spaces and continuations*
- 2002 Taylor [B] *Subspaces in ASD*
- 2003 Taylor [G–] *Local compactness and the Baire category theorem in ASD*
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Bibliography — Σ-split subspaces

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An abstract λ-calculus for *Synthetic Topology*

The axioms consist of

- the simply typed λ-calculus,
  but with restricted type-formation for Σ^X
- distributive lattice structure $\top, \bot, \wedge, \vee$ on Σ (not $\Rightarrow, \neg$)
- the Phoa principle $F\sigma \Leftrightarrow F\bot \vee \sigma \wedge F\top$
  (this captures the extensional correspondence amongst terms of type Σ^X, open subspaces and closed subspaces of X — see *Geometric & Higher Order Logic* [C])
- the natural numbers $\mathbb{N}$ with zero, successor, recursion, description and existential quantification
- Scott continuity.

See *The Dedekind Reals in ASD*, Section 4 for details.

In models of this system, $[0, 1]$ need not be compact.

The λ-calculus for Abstract Stone Duality

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- the natural numbers $\mathbb{N}$ with zero, successor, recursion, description and existential quantification
- Scott continuity.
- Σ-split subspaces.

See *The Dedekind Reals in ASD*, Sections 4–5 for details.

In this system, $[0, 1]$ is provably compact.

Dedekind cuts

A (Dedekind) cut (δ, υ) is a pair of predicates on $\mathbb{Q}$ or $\mathbb{R}$

$$\Gamma,\ q \ \vdash\ \delta q,\ \upsilon q : \Sigma,$$

such that

$$\begin{array}{lcll}
\upsilon u & \Leftrightarrow & \exists t.\, \upsilon t \ \wedge\ (t < u) & \upsilon \text{ rounded upper} \\
\delta d & \Leftrightarrow & \exists e.\, (d < e) \ \wedge\ \delta e & \delta \text{ rounded lower} \\
\top & \Leftrightarrow & \exists u.\, \upsilon u & \text{bounded above} \\
\top & \Leftrightarrow & \exists d.\, \delta d & \text{bounded below} \\
\delta d \ \wedge\ \upsilon u & \Rightarrow & (d < u) & \text{disjoint} \\
(d < u) & \Rightarrow & (\delta d \ \vee\ \upsilon u) & \text{located}
\end{array}$$

Both halves of the cut are needed since there is no negation.

Legitimate and illegitimate cuts

Let $a : \mathbb{R}$ and $e < t$.

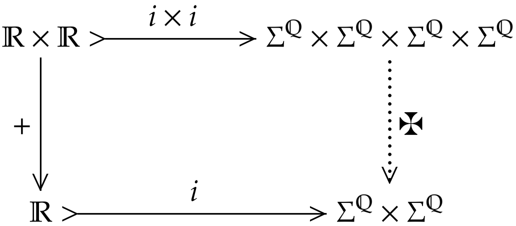
		δd	υu	
real	a	$d < a$	$a < u$	legitimate
	$-\infty$	$\bot$	$\top$	unbounded below
	$+\infty$	$\top$	$\bot$	unbounded above
interval	$[e, t]$	$d < e$	$t < u$	not located
	$[t, e]$	$d < t$	$e < u$	not disjoint

Rounded disjoint pseudo-cuts form the interval domain.

Constructively, they need not have endpoints $[e, t]$.

Extending functions from reals to cuts

In order to use Dedekind cuts for real computation,
we must extend the definitions of the operations.

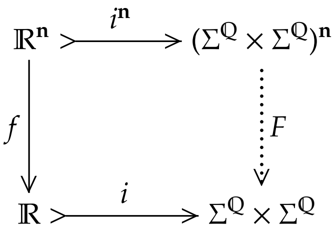


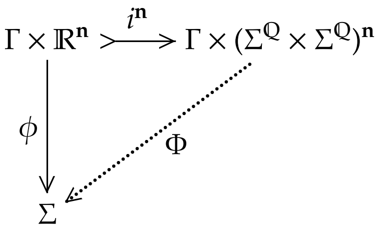
For the arithmetic operations, this was done classically by Ramon Moore, Interval Analysis, 1966.

How is this generalised to other continuous functions?

Extending functions from reals to cuts

The extension of functions can be obtained from that for predicates with parameters.





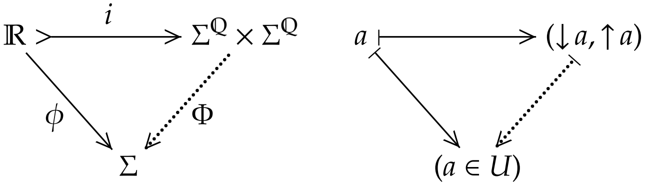
Let Φ_d and Ψ_u be the extensions of

$$\phi_d \equiv \lambda\vec{x}.\, d < f\vec{x} \quad \text{and} \quad \psi_u \equiv \lambda\vec{x}.\, f\vec{x} < u$$

Then $F(\vec{\delta},\vec{v}) \;\equiv\; \left(\lambda d.\, \Phi_d(\vec{\delta},\vec{v}),\; \lambda u.\, \Psi_u(\vec{\delta},\vec{v})\right)$ extends f.

Extending open subspaces classically

Recall that ϕ defines an open subspace $U \subset \mathbb{R}$.



We require $(a \in U) \equiv \phi a \iff \Phi(ia) \equiv \Phi(\downarrow a, \uparrow a)$.

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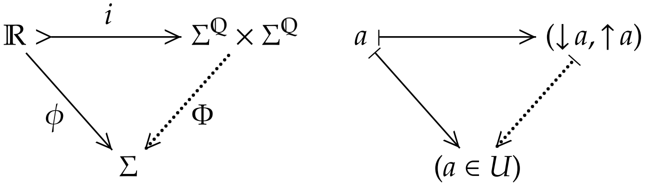

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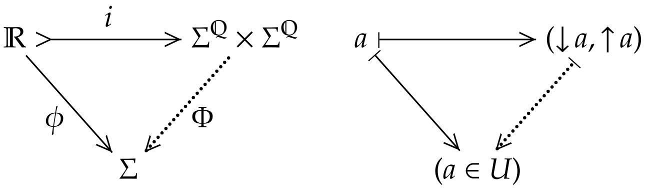

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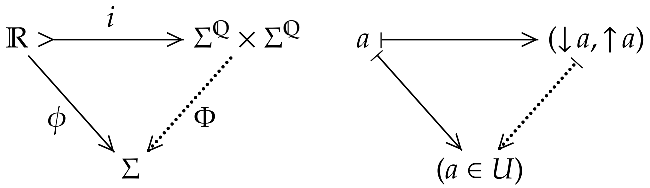

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Since $\mathbb{R}$ is locally compact,

$(a \in U) \iff \exists du.\, (d < a < u) \wedge \big([d, u] \subset U\big) \iff \Phi(\downarrow a, \uparrow a)$

The extension of predicates is uniform

Since $[d, u]$ is compact, $U \mapsto ([d, u] \subset U)$ is Scott continuous.

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So $\mathcal{E}$ determines $\mathbb{R}$ amongst sober spaces from $\Sigma^{\mathbb{Q}} \times \Sigma^{\mathbb{Q}}$.

It can all be expressed rationally

We have defined the idempotent $\mathcal{E} \equiv I \cdot \Sigma^i$ on $\Sigma^{\Sigma^{\mathbb{Q}} \times \Sigma^{\mathbb{Q}}}$ by

$$\begin{aligned} \mathcal{E}\Phi(\delta, \upsilon) &\equiv I\Big(\lambda x.\, \Phi(ix)\Big)(\delta, \upsilon) \\ &\Leftrightarrow \exists du : \mathbb{R}.\ \delta d \wedge \upsilon u \wedge \forall x : [d, u].\ \Phi(\delta_x, \upsilon_x) : \Sigma^{\Sigma^{\mathbb{Q}} \times \Sigma^{\mathbb{Q}}}. \end{aligned}$$

Since Φ is Scott continuous and $[d, u]$ is compact, this is

$$\begin{aligned} \exists q_0 < \cdots < q_{2n+1} : \mathbb{Q}. \quad & \delta q_1 \wedge \upsilon q_{2n} \ \wedge \\ & \bigwedge_{k=0}^{n-1} \Phi(\lambda e.\, e < q_{2k},\ \lambda t.\, q_{2k+3} < t) \end{aligned}$$

(See *Dedekind Reals in ASD*, Section 3.)

This only depends on rational numbers and predicates.

The classical *versus* other situations

Let $\mathcal{E}$ be the rationally defined idempotent on $\Sigma^{\Sigma^{\mathbb{Q}} \times \Sigma^{\mathbb{Q}}}$.
This is the same in all foundational situations.

In each situation, let $i : R \rightarrowtail \Sigma^{\mathbb{Q}} \times \Sigma^{\mathbb{Q}}$ be the subspace of Dedekind cuts.

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When it exists, we say that the subspace is Σ-split by I.

It all depends on the category

The subspace R of Dedekind cuts is an equaliser.

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If there are not enough objects, the wrong one may be the equaliser.

If we add new objects, one of them may become the equaliser. (The old one is "relieved of its duties".)

The new object may have the right properties.

Type theory for Σ-split subspaces

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The bad news: all of this takes over 200 journal pages [A,B,G].

Instead, we *specialise* to $\mathbb{R}^n$

To complete the construction of $\mathbb{R}$:

- (δ, υ) is a cut iff it's $\mathcal{E}$-admissible
- $\mathbb{R}$ is overt, Hausdorff and totally ordered
- $\mathbb{R}$ is Dedekind complete: $\quad \mathbb{R}\text{-cuts} \;\cong\; \mathbb{Q}\text{-cuts}$
- define the arithmetic operations
- the closed interval is compact
- any model of the axioms is uniquely isomorphic to the construction.

A project for you!

In your favourite model of analysis (TTE, Bishop, Brouwer, realisability, ...)

- Does $I : \Sigma^{\mathbb{R}} \to \Sigma^{\Sigma^{\mathbb{Q}} \times \Sigma^{\mathbb{Q}}}$ exist already?
- Follow the details of the construction in *Subspaces in ASD* [B].
- Is the other interesting structure of your model preserved?
- How do the theorems in ASD compare with the original ones?

This could be an exercise for a PhD student, or a major piece of research.

Please ask me for help.

Computation using interval notation

Now use the notation of Interval Analysis (but not its semantics).

- Instead of Dedekind cuts $(\delta, \upsilon) = (\lambda d.\, d < a, \lambda u.\, a < u) : \Sigma^{\mathbb{Q}} \times \Sigma^{Q}$
- we use families of intervals (connected neighbourhoods) $\theta = (\lambda du.\, d < a < u) : \Sigma^{Q \times Q}$,

where Q is the type of dyadic rationals.

$$\theta du \equiv \delta d \wedge \upsilon u \qquad \delta d \equiv \exists u.\, \theta du \qquad \upsilon u \equiv \exists d.\, \theta du.$$

Note: $[d, u]$ is a pair of rational numbers, not a subspace of $\mathbb{R}$.

In n dimensions, a system of close-packed spheres could be used, instead of cubes.

Use some proof theory

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Normalisation theorems say that the last rule
or outermost connective is an introduction rule.
Apply the η-rule corresponding to the type of the term.

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Normalisation theorems say that the last rule
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So we just consider propositions (terms of type Σ).

Propositions with real-valued sub-expressions

Remember that $[d, u]$ is a pair of rationals — not a subset of $\mathbb{R}$.

It's like an open interval in $a \in [d, u] \;\equiv\; d < a < u$
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Using

$$[d, u] < [e, t] \equiv (u < e) \quad \text{and} \quad [d, u] \,\#\, [e, t] \equiv (u < e) \vee (t < d),$$

we reduce $<$ and $\neq$ for numbers to $\in$,

$$\begin{aligned} a < b &\Leftrightarrow \exists duet.\ a \in [d, u] \wedge [d, u] < [e, t] \wedge b \in [e, t] \\ a \neq b &\Leftrightarrow \exists duet.\ a \in [d, u] \wedge [d, u] \,\#\, [e, t] \wedge b \in [e, t] \end{aligned}$$

So just consider propositions of the form $a \in [e, t]$

Proof theory again

Recall that, since $\mathbb{R}$ is locally compact,

$$\phi x \iff \exists du.\,(d < x < u) \land \forall_o d \le x' \le u.\,\phi x'$$

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Let $\mathbf{R}$ be the type of rational pairs $[d, u]$ with $d < u$,
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In particular, when ϕx is of the form $x \in [e, t]$,

$$\forall_o x \in \mathbf{x}.\, (x \in [e, t]) \iff \mathbf{x} \Subset [e, t].$$

So, in the normalisation of $a \in [e, t] \equiv \mathbf{z}$ under $\forall_o x \in \mathbf{x}$,

$$x \in \mathbf{z} \quad \text{becomes} \quad \mathbf{x} \Subset \mathbf{z}.$$

Hence real variables become interval variables, $\in$ becomes $\Subset$.

Real arithmetic becomes interval arithmetic

When a is an arithmetic expression $b \star c$,

$$b \star c \in \mathbf{z} \;\Leftrightarrow\; \exists \mathbf{xy}.\, b \in \mathbf{x} \wedge c \in \mathbf{y} \wedge (\mathbf{x} \star \mathbf{y} \Subset \mathbf{z})$$

where $\mathbf{x} \star \mathbf{y}$ has Moore's interpretation:

$$[d,u] + [e,t] \;\equiv\; [d+e, u+t] \qquad -[d,u] \;\equiv\; [-u,-d]$$

$$[d,u] \times [e,t] \;\equiv\; [\mathsf{min}(de,dt,ue,ut),\ \mathsf{max}(de,dt,ue,ut)]$$

For division and roots we must say separately that

$$[d,u]^{-1} \Subset [e,t] \;\equiv\; (0 < d \wedge ue < 1 < dt) \vee (u < 0 \wedge dt < 1 < ue)$$
$$\sqrt{[d,u]} \Subset [e,t] \;\equiv\; (e < 0 \vee e^2 < d) \wedge (0 < t \wedge u < t^2).$$

If a is a Dedekind cut or family of intervals,

$$\mathsf{admit}(\delta, \upsilon) \in [d,u] \;\Leftrightarrow\; \delta d \wedge \upsilon u \qquad \mathsf{admit}(\theta) \in [d,u] \;\Leftrightarrow\; \theta du.$$

Compactness, quantification and optimisation

The one non-trivial case is the universal quantifier arising from the compact interval $[d, u]$.

For any $d < \mathfrak{m} < u$ we have

$$\forall x \in [d, u].\, \phi x \quad \Leftrightarrow \quad \forall x \in [d, m].\, \phi x \wedge \forall x \in [m, u].\, \phi x$$

but this recursion has no base case.

Compactness, quantification and optimisation

The one non-trivial case is the universal quantifier arising from the compact interval $[d, u]$.

For $\frac{2}{3}d + \frac{1}{3}u < m < \frac{1}{3}d + \frac{2}{3}u$ we have

$$\begin{aligned} \forall x \in [d, u].\, \phi x \quad &\Leftrightarrow \quad \forall_o x \in [d, u].\, \phi x \\ &\vee \quad \forall x \in [d, m].\, \phi x \wedge \forall x \in [m, u].\, \phi x \end{aligned}$$

The first disjunct provides the base of the recursion.

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The restriction on m causes the intervals to get arbitrarily small.

The recursion is well founded because $[d, u]$ is compact.

In fact, $\forall x : [d, u].\ \Phi(\downarrow x, \uparrow x) \Leftrightarrow \mathcal{E}\Phi(\downarrow d, \uparrow u)$.

The result of the normalisation

- λ-abstraction and application normalise using essentially Peter Landin's SECD machine,
- equality of integer terms normalises by unification,
- excluding disjunction and recursion, this leaves a Prolog clause,
- in which real arithmetic becomes a system of polynomial constraints.
- With disjunction or recursion we have a non-deterministic parallel Prolog program.
- The universal quantifier becomes an optimisation problem.

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Interval analysis without intervals!

Conclusion

We axiomatised $\mathbb{R}$ according to analyst's intuition.

We rejected foundational prejudice
from both set theory and recursion theory.

Instead we relied on categorical intuition.

Using techniques of proof theory,
we turned this into computation.

We still have to show that this is useful for analysis.