Midlands Graduate School 2005

Abstract Stone Duality

Paul Taylor

University of Manchester

Funded by UK EPSRC GR/S58522

www.cs.man.ac.uk/~pt/ASD

pt@cs.man.ac.uk

077 604 625 87

Programme of Lectures

Monday	Methodology	
5pm	Euclidean Principle	[C] *Geometric and Higher Order L*
	Underlying Set Functor	[H] *An Elementary Theory of Vario*
		Categories of Spaces and Locales
Tuesday	Recursive compactness	
4pm	Monadic λ-calculus	[B] *Subspaces in ASD*
	Dedekind reals	[I] *Dedekind Reals in ASD*
	Cantor Space	notes available privately
Wednesday	Intermediate value theorem	[J] *A λ-calculus for real analysis*
4pm		
Thursday	ASD for locales and beyond	[H]
4pm	The extended calculus	

What is the relationship between ASD and Escardó's *Synthetic Topology*?

Martín Escardó uses λ-calculus to **describe** proofs
that are **founded** in traditional point-set topology or locale theory.

His quantifiers mean "for every" and "there exists", referring to **points**.
His recursive theory therefore runs into well known problems
with compactness of Cantor Space $\mathbf{2}^{\mathbb{N}}$ and $\mathbb{I} \equiv [0,1] \subset \mathbb{R}$.

ASD is a **direct, complete axiomatisation** of topology (**not** *via* set theory).

The quantifiers satisfy the **formal rules**
of predicate calculus and categorical logic.

(For reasons in addition to this)
Cantor Space $\mathbf{2}^{\mathbb{N}}$ and $\mathbb{I} \equiv [0,1] \subset \mathbb{R}$ are compact
in both the topos-based and recursive versions of the theory.

Another Disclaimer

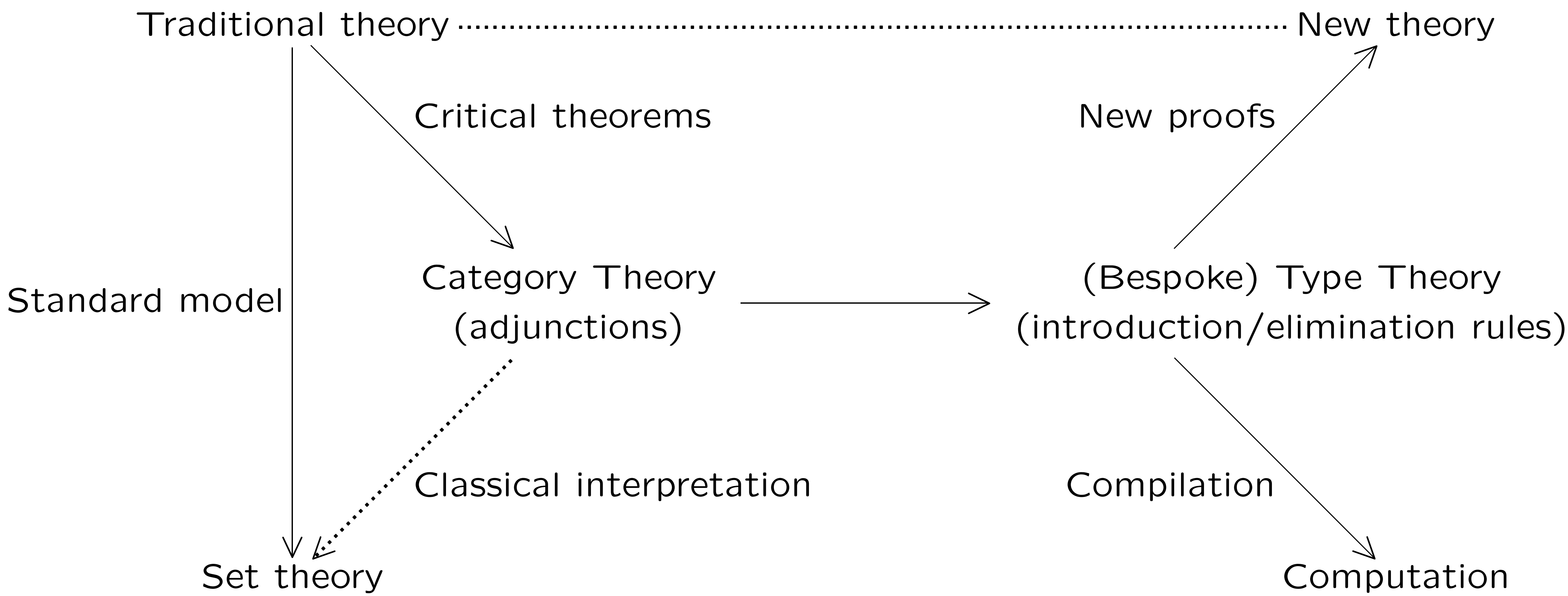
The lectures will be about **ideas** and **arguments**.

For **axioms** see the handout.

For **theorems** and **proofs** see the papers.

ASD is about **topology**
it's not about sets with collections of subsets (so called "topological spaces")
it's not about infinitary lattices (locales)

The underlying methodology



Why Category Theory?

It doesn't **pre-judge** foundations or notation.

(If you start with set theory, you're stuck with it.)

It distills decades of abstract mathematical experience.

It allows ideas from one mathematical discipline
to be **compared** with those of another.

It **translates** ideas from one mathematical discipline
into the language of another.

It can express **normal forms**,
or, equally easily, **generators and relations**.

It can be its own **meta-language**.

It's good for stating **foundational principles** (axioms).

Why Type Theory?

It is much closer to the way in which mathematicians write mathematics (at least since René Descartes' time).

It's fluent (for some things),
whereas diagrams are clumsy (for some things).

Its **transformation rules** (β)
have a natural "**direction**" (forwards/backwards), which
(fortuitously)
has a useful computational interpretation.

(Universal properties, pullbacks, *etc.*, have no such natural direction.)

It's good for **mathematical arguments** (theorems).

It's good for **computation** (programs).

Category Theory **and** Type Theory

The methodology depends on a **fluent** translation, in particular:

Adjunctions (universal properties)	=	Introduction and elimination rules.
$$\frac{\Gamma \times X \xrightarrow{p} Y}{\Gamma \xrightarrow{\tilde{p}} Y^X}$$	=	$$\frac{\Gamma,\ x : X \ \vdash\ p(x) : Y}{\Gamma \ \vdash\ \lambda x{:}X.\ p(x) : X \to Y}$$
$\mathsf{ev} : Y^X \times X \to Y$	=	$f : X \to Y,\ x : X \ \vdash\ fx : Y$
naturality	=	substitution

See my book for the details of the translation.

The Critical Axioms of Topology

The Sierpiński space Σ as a "space of truth values"

Extensional correspondence between open $U \subset X$ and $\phi : X \to \Sigma$.

(Classical spaces have underlying sets of points.)

Subspaces have the subspace topology.

Compact subspaces are determined by their neighbourhoods, not their points.

Subsets and predicates — symbolically

The **Axiom of Comprehension**:
form the **subset** $U \equiv \{x \mid \phi(x)\}$ from the **predicate** $\phi(x) \equiv (x \in U)$
where $\phi(x)$ is expressed in some logical language.

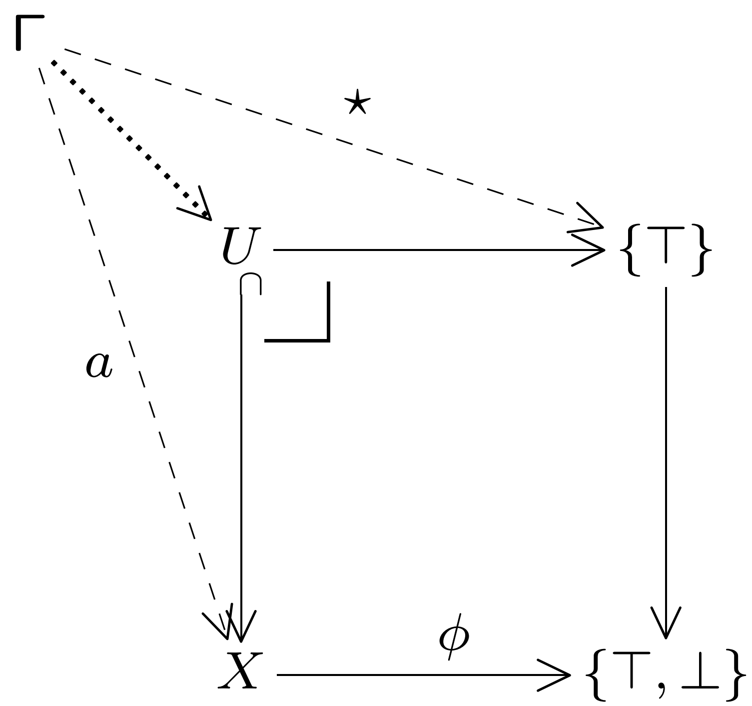
This is really just notation.

U and ϕ are **the same thing**:
Membership $(a \in U)$ is application (ϕa)
Set formation $\{x \mid \phi(x)\}$ is λ-abstraction $\lambda x.\, \phi(x)$.

U and ϕ are **not** the same thing:
Set theory carries historical Platonist baggage — "collections"
λ-calculus carries historical Formalist baggage — **computation**.

Subsets and predicates — diagrammatically

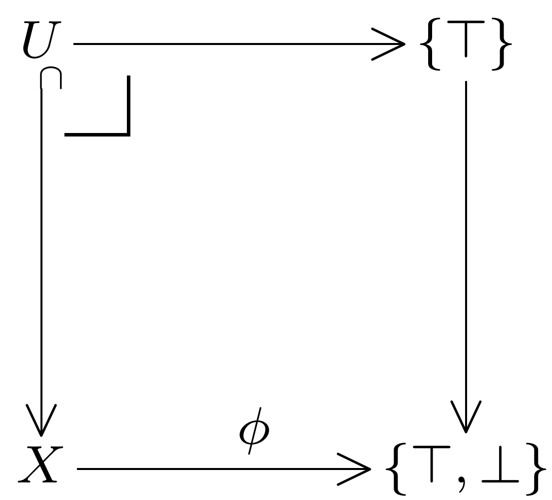
The correspondence between $U \equiv \{x \mid \phi(x)\} \subset X$ and $\phi(x) \equiv (x \in U)$
is given by a **pullback diagram**.



To test the pullback, consider $a : X$.
This may have parameters (free variables)
$u_1 : U_1, \ldots, u_k : U_k$.
Type-theoretically, we write $\Gamma \vdash a : X$.
Categorically, we write
$\Gamma \equiv U_1 \times \cdots \times U_k \xrightarrow{a} X$.
If a satisfies ϕ then $\phi a = \top$,
so the kite commutes.
Then a belongs to U,
so there's a map $a : \Gamma \to U$.
It's unique — there's only one element $a \in U$
that's the same as $a \in X$.

Subsets and predicates — diagrammatically

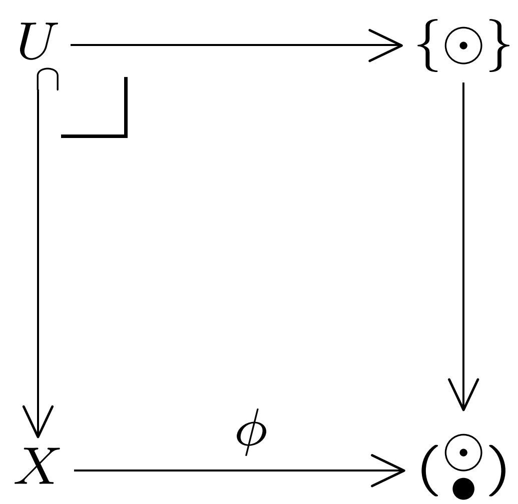
But U and ϕ also uniquely determine each other.
(Extensionality — here we part company with Per Martin-Löf.)



We say that $\phi : X \to \{\top, \bot\}$ **classifies** $U \subset X$.

In the **intuitionistic logic** of an **elementary topos**,
$\{\top, \bot\}$ is replaced by another object, called Ω, with the same property.
(It is often **much** more complicated.)

The same thing in topology



Now let $U \subset X$ be an open subspace.
Then $\phi : X \to \binom{\odot}{\bullet}$ is a continuous function,

$\{\odot\} \subset \binom{\odot}{\bullet}$ is open.
Again, classically,
the correspondence $U \leftrightarrow \phi$ is unique.

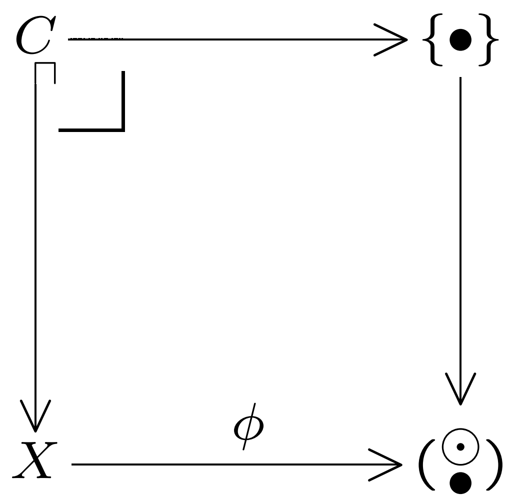
The **Sierpiński space** $\binom{\odot}{\bullet}$ appeared in topology textbooks for decades
as a pathetic (counter)example.

It is **key** to domain theory and Abstract Stone Duality.

Again, constructively, we replace $\binom{\odot}{\bullet}$ by something more complicated,
called Σ, with the same property.

But it's **nowhere near** as complicated as the set Ω.

The same thing for closed subspaces



Now let $C \subset X$ be a **closed** subspace.

Then $\phi : X \to \binom{\odot}{\bullet}$ is a continuous function,

so long as $\{\bullet\} \subset \binom{\odot}{\bullet}$ is closed.

So $\odot \in \binom{\odot}{\bullet}$ classifies open subspaces

and $\bullet \in \binom{\odot}{\bullet}$ **co-classifies** classifies subspaces.

Using the same map $\phi : X \to \binom{\odot}{\bullet}$.

So (by uniqueness) **open and closed subspaces are in bijection**.

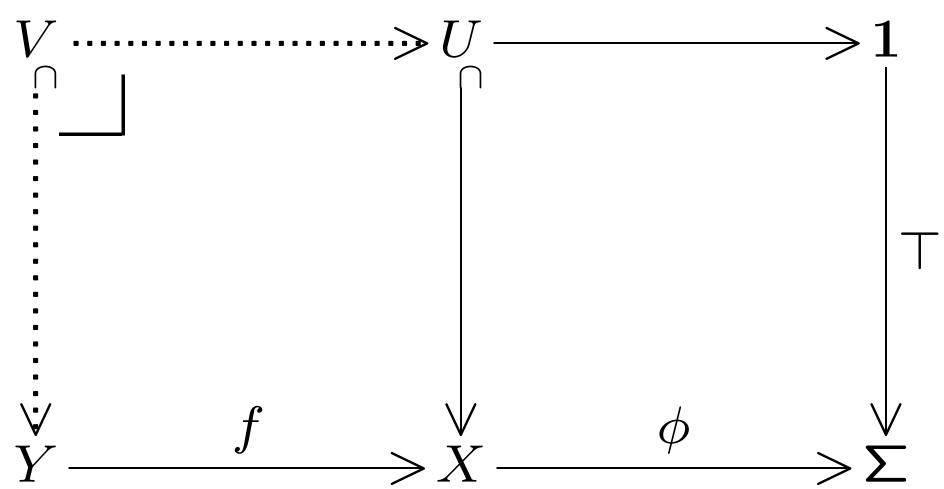
Inverse images

As exponentials are defined by a universal property,
the assignment $X \mapsto \Sigma^X$ extends to a contravariant endofunctor,
$\Sigma^{(-)} : \mathcal{C} \to \mathcal{C}^{\mathsf{op}}$.

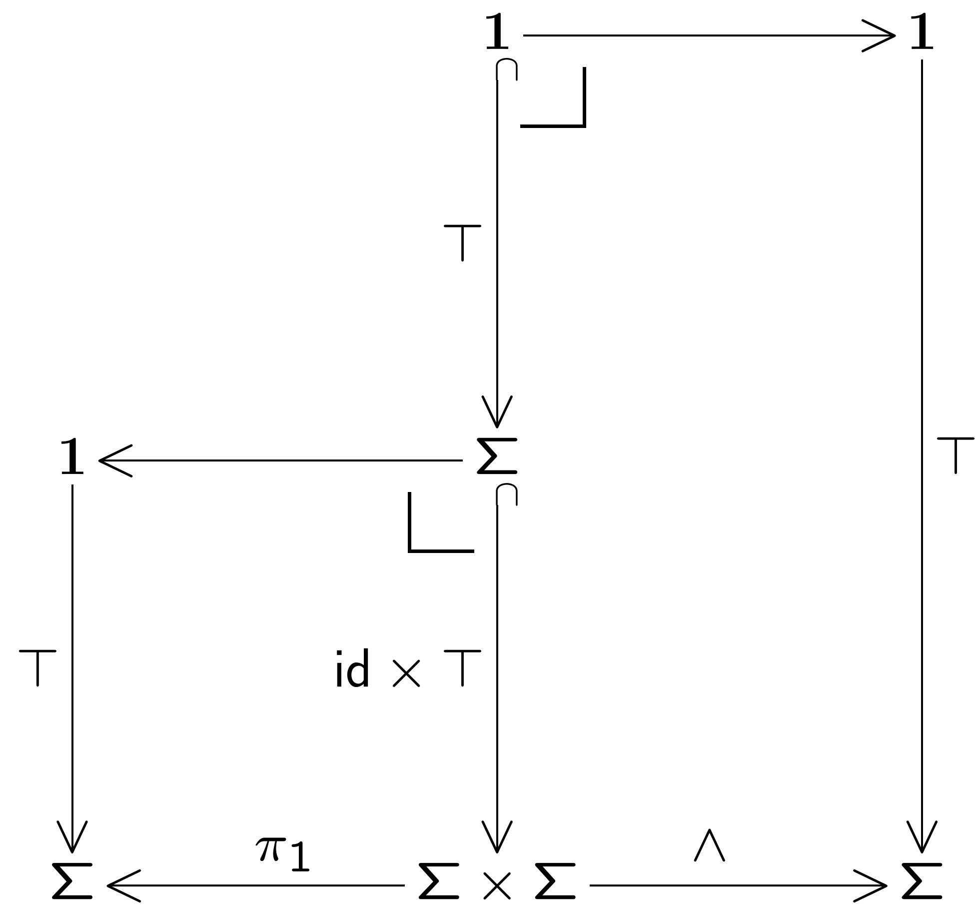
It takes $f : Y \to X$ to $\Sigma^f : \Sigma^X \to \Sigma^Y$ by

$$\Sigma^f(\phi) = \phi \circ f = f\,;\phi = \lambda y.\, \phi(fy).$$

The effect of Σ^f is to form the pullback or inverse image along f:

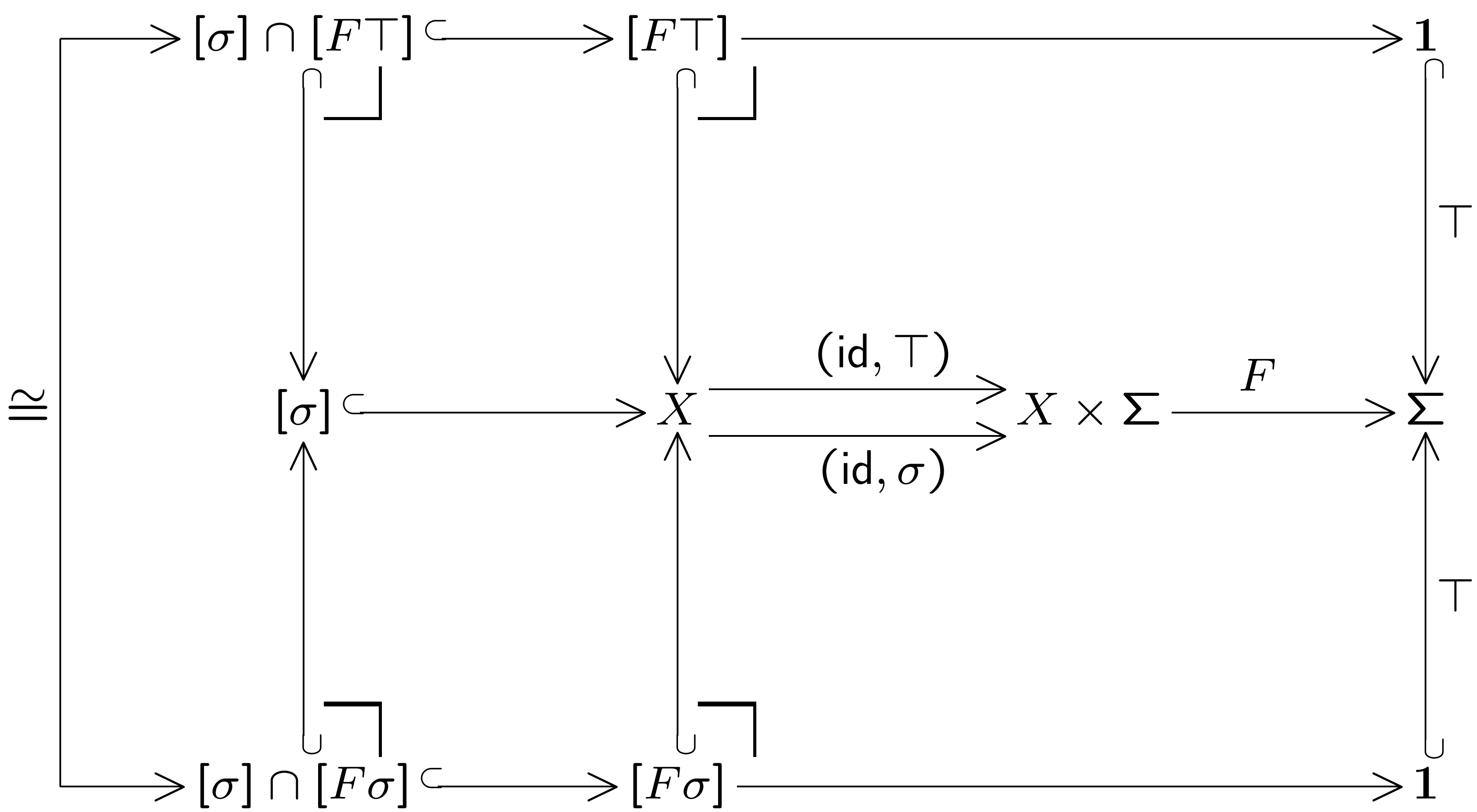


Intersections and conjunctions



The Euclidean Principle

Since classifiers for isomorphic subobjects are equal,



$$\sigma(x) \wedge F(x, \sigma(x)) \iff \sigma(x) \wedge F(x, \top)$$

The Phoa Principle — topologically

Since $\top : \Sigma$ **uniquely** classifies open subspaces,	$\sigma \wedge F\sigma \;\Leftrightarrow\; \sigma \wedge F\top$
Since $\bot : \Sigma$ **uniquely** (co)classifies closed subspaces,	$\sigma \vee F\sigma \;\Leftrightarrow\; \sigma \vee F\bot$
Since all $F : \Sigma \to \Sigma$ preserve the order,	$F\bot \;\Rightarrow\; F\sigma \;\Rightarrow\; F\top$
Hence	$F\sigma \;\Leftrightarrow\; F\bot \vee \sigma \wedge F\top$

Proof, assuming distributivity:

$$F\sigma = (F\sigma \vee \sigma) \wedge F\sigma = (F\bot \vee \sigma) \wedge F\sigma = (F\bot \wedge F\sigma) \vee (\sigma \wedge F\sigma) = F\bot \vee (\sigma \wedge F\top).$$

Conversely, this equation entails the laws of a distributive lattice, both Euclidean principles and monotonicity.

The Phoa Principle — computationally

Σ is called `unit` in ML and `void` in C and Java.

A program of type Σ may or may not terminate. Just that.

It is known as an **observation**.

A program of type $\Sigma \to \Sigma$ turns observations into observations.

It may	terminate even though its input doesn't	$F\sigma \Leftrightarrow F\bot \Leftrightarrow \top$
	terminate iff its input does	$F\sigma \Leftrightarrow \sigma$
	not terminate even though its input does	$F\sigma \Leftrightarrow F\top \Leftrightarrow \bot$

But that's all.

So $F\sigma \Leftrightarrow F\bot \vee \sigma \wedge F\top$.

The Phoa Principle — logically

The Phoa principle justifies rules for negation like those of Gentzen's classical sequent calculus:

$$\frac{\Gamma,\ \sigma \Leftrightarrow \top\ \vdash\ \alpha\ \Rightarrow\ \beta}{\Gamma\ \vdash\ \sigma \wedge \alpha\ \Rightarrow\ \beta} \qquad \frac{\Gamma,\ \sigma \Leftrightarrow \bot\ \vdash\ \beta\ \Rightarrow\ \alpha}{\Gamma\ \vdash\ \beta\ \Rightarrow\ \sigma \vee \alpha}$$

Proof: The intersection of the open/closed subspaces co/classified by σ and α is contained in that co/classified by β.

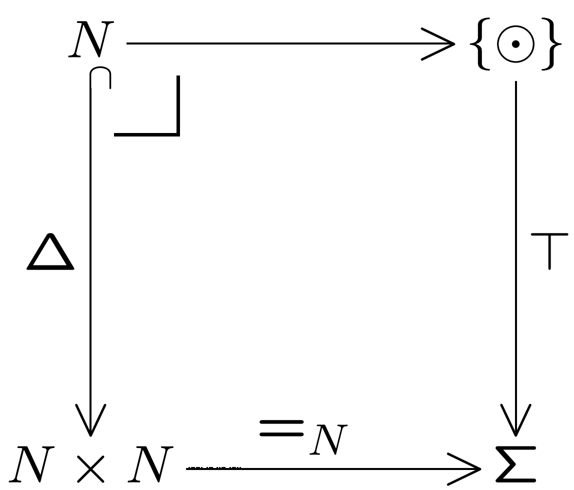
Remarkably, we can even prove statements by cases,

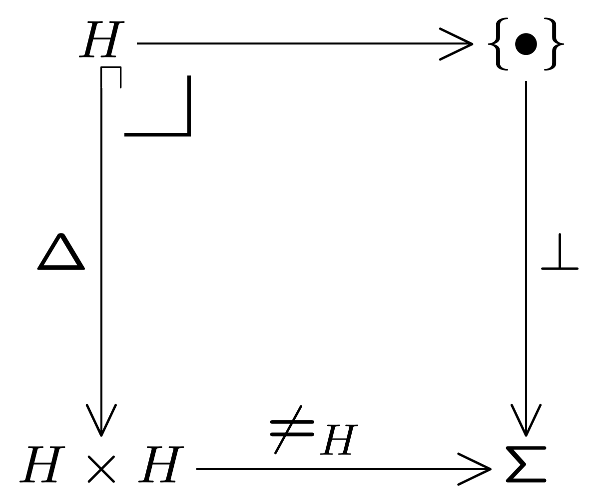
$$\frac{\Gamma,\ \sigma \Leftrightarrow \top\ \vdash\ \alpha \Rightarrow \beta \qquad \Gamma,\ \sigma \Leftrightarrow \bot\ \vdash\ \alpha \Rightarrow \beta}{\Gamma\ \vdash\ \alpha \Rightarrow \beta}$$

but the proof isn't obvious.

Discrete and Hausdorff Spaces

What if $X \subset X \times X$ is open or closed?





$$\frac{\Gamma \vdash n = m : N}{\Gamma \vdash (n =_N m) \Leftrightarrow \top : \Sigma} \qquad \frac{\Gamma \vdash h = k : H}{\Gamma \vdash (h \neq_H k) \Leftrightarrow \bot : \Sigma}$$

Properties of Equality and Apartness

Substitution, reflexivity, symmetry and transitivity, and their duals:

$$
\begin{array}{rclcrcl}
\phi m \wedge (n = m) & \Rightarrow & \phi n & \quad & \phi h \vee (h \neq k) & \Leftarrow & \phi k \\
(n = n) & \Leftrightarrow & \top & & (h \neq h) & \Leftrightarrow & \bot \\
(n = m) & \Leftrightarrow & (m = n) & & (h \neq k) & \Leftrightarrow & (k \neq h) \\
(n = m) \wedge (m = k) & \Rightarrow & (n = k) & & (h \neq k) \vee (k \neq \ell) & \Leftarrow & (h \neq \ell)
\end{array}
$$

Proof: $\lambda mn.\, \phi m \wedge (n = m)$ and $\lambda mn.\, \phi n \wedge (n = m)$ classify the same open subspace of $N \times N$.

$\lambda hk.\, \phi \vee (h \neq k)$ and $\lambda hk.\, \phi k \wedge (h \neq k)$ co-classify the same closed subspace of $H \times H$.

Overt and compact spaces

A space X is called **overt** or **compact**
if there is a term $\exists_X : \Sigma^X \to \Sigma$ or $\forall_X : \Sigma^X \to \Sigma$
that satisfies the type-theoretic rules

$$\frac{\Gamma,\ x : X \ \vdash\ \phi x \ \Rightarrow\ \sigma}{\Gamma\ \vdash\ \exists x.\, \phi x \ \Rightarrow\ \sigma} \qquad \frac{\Gamma,\ x : X \ \vdash\ \sigma \ \Rightarrow\ \phi x}{\Gamma\ \vdash\ \sigma \ \Rightarrow\ \forall x.\, \phi x}$$

which correspond to the adjunctions

$$\begin{array}{c} X \\ {\scriptstyle !}\big\downarrow \\ \mathbf{1} \end{array} \qquad\qquad \begin{array}{ccccc} & & \Sigma^X & & \\ \exists_X \big\downarrow & \dashv & \big\uparrow \Sigma^! & \dashv & \big\downarrow \forall_X \\ & & \Sigma & & \end{array}$$

They do not mean "there exists" and "for every" on points!

Frobenius and modal laws

$$\exists x.\, \sigma \wedge \phi x \iff \sigma \wedge \exists x.\, \phi x \qquad (\forall x.\, \phi x) \wedge (\exists x.\, \psi x) \Rightarrow \exists x.\, (\phi x \wedge \psi x)$$

$$\forall x.\, \sigma \vee \phi x \iff \sigma \vee \forall x.\, \phi x \qquad (\forall x.\, \phi x) \vee (\exists x.\, \psi x) \Leftarrow \forall x.\, (\phi x \vee \psi x)$$

Proof: First, $(\exists x.\, \bot) \iff \bot$,
by putting $\sigma \equiv \bot$ in the definition.

Now let $F\sigma \equiv \exists x.\, \sigma \wedge \phi x$
in the **Phoa principle** $F\sigma \iff F\bot \vee \sigma \wedge F\top$,
so $\exists x.\, \sigma \wedge \phi x \equiv F\sigma \iff F\bot \vee \sigma \wedge F\top \iff (\exists x.\, \bot) \vee \sigma \wedge (\exists x.\, \phi x) \iff \sigma \wedge (\exists x.\, \phi x)$.

Exercise to prove the others.

Reasoning with the quantifiers

So long as the types of the variables really are overt or compact,
we may reason with the quantifiers in the usual ways:

If we find a particular $\Gamma \vdash a : X$ that satisfies $\Gamma \vdash \phi a \Leftrightarrow \top$,
then we may of course assert $\Gamma \vdash \exists x.\, \phi x \Leftrightarrow \top$.
This simple step tends to pass unnoticed in the middle of an argument,
often in the form $\phi a \Rightarrow \exists x.\, \phi x$.

Similarly, if the judgement $\Gamma \vdash \forall x.\, \phi x \Leftrightarrow \top$ has been proved,
and we have a particular value $\Gamma \vdash a : X$,
then we may deduce $\Gamma \vdash \phi a \Leftrightarrow \top$.
Again, we often write $\phi a \Leftarrow \forall x.\, \phi x$.

The familiar mathematical idiom "there exists" is valid:
$\exists x.\, \phi x$ is asserted and then x is temporarily used in the subsequent argument.

The λ-calculus formulation automatically allows substitution under the quantifiers,
whereas in categorical logic this property must be stated separately,
and is known as the Beck–Chevalley condition.

Lattice duality in topology

The theory so far was motivated partly by the category $\mathbf{Set}$.
Most of it still applies to $\mathbf{Set}$ and $\mathbf{Pos}$.

Recall the three parts of the Phoa principle:

		sets	posets	spaces
Euclid	$\sigma \wedge F\sigma \Leftrightarrow \sigma \wedge F\top$	✓	✓	✓
dual Euclid	$\sigma \vee F\sigma \Leftrightarrow \sigma \vee F\bot$	$\sigma \vee \neg\sigma$	$\neg\neg\sigma \Rightarrow \sigma$	✓
monotonicity	$F\bot \Rightarrow F\sigma \Rightarrow F\top$	$\times$	✓	✓

Lattice ("de Morgan") holds in **classical** set and order theory.
The Phoa (and dual Euclidean) principles hold in **intuitionistic** locale theory.

Quite a lot of topology can be developed with the Phoa principle.
Lattice duality is very illuminating.

Breaking the duality in topology

Traditional topology and locale theory treat $\bigvee$ and $\wedge$ asymmetrically:
infinitary $\bigvee$ but finitary $\wedge$.

Consequently, it's difficult to see the analogies between
sets and compact Hausdorff spaces
open maps and proper maps

Of course, the duality **must** be broken.

But making everything preserve **directed** joins automatically,
we put them in the background
and otherwise treat $\vee$ and $\wedge$ symmetrically.

There are several ways of stating the **Scott continuity** axiom:
as continuity, fixed points, limit–colimit coincidence, bases, *etc.*
We shall do so in whatever way is clearest in each circumstance.

The full subcategory of overt discrete objects

	$=$	$\exists$
$\mathbf{1}$	$(x_1 = x_2) \equiv \top$	$\exists_{\mathbf{1}} \equiv \mathsf{id} : \mathbf{\Sigma}^{\mathbf{1}} \to \mathbf{\Sigma}$
$X \times Y$	$((x,y) = (x',y')) \equiv ((x = x') \wedge (y = y'))$	$\exists (x,y).\, \phi(x,y) \;\equiv\; \exists x.\, \exists y.\, \phi(x,y)$
$i : U \subset X$	$(u_1 =_U u_2) \;\equiv\; (iu_1 =_X iu_2)$	$\exists u.\, \phi u \;\equiv\; \exists x.\, \theta x \wedge \phi x$
$X + Y$	$(\nu_0 x = \nu_0 x') \equiv (x = x')$	$\exists z{:}X+Y.\, \phi z \;\equiv\; \exists x.\, \phi(\nu_0 x) \vee \exists y.\, \phi(\nu_1 y)$
$Q \equiv X/{\sim}$	$[x] =_Q [x'] \;\equiv\; (x \sim x')$	$\cdots$
$\mathsf{List} X$	$\cdots$	$\cdots$

where U is classified by θ and Scott continuity is needed for $\mathsf{List}(X)$.

This structure is called an **Arithmetic Universe**.

Making ASD agree with traditional topology

Traditional topology and locale theory
are built on top of a theory of “sets” (objects with no topology)
either Set Theory or (Martin-Löf or other) Type Theory or a Topos.

ASD axiomatises topology directly — there are no (pre-existing) sets.

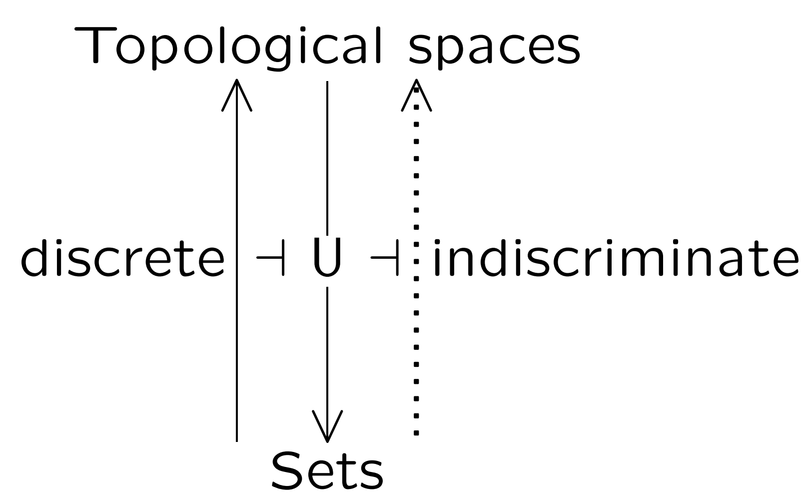
If they are to “agree” we must first identify the “sets” in ASD
i.e. which of the objects in ASD have a “trivial” topology.

We choose **overt discrete** objects to play to role of “sets”.

Does this subcategory look at all like set theory?
Can we reconstruct traditional topology on top of it?
How does the result compare to the ASD category?

Discrete and Indiscriminate Topologies

In classical point-set topology we have the triple adjunction



where the middle functor yields the **underlying set of points** of a space
its left adjoint equips a set with the **discrete topology**, in which all subsets are open
its right adjoint equips a set X with the **indiscriminate topology**,
in which only $\emptyset$ and X are open (but this isn't sober, so we shall not use it).

(NB: $\mathsf{U} : \mathbf{Loc} \to \mathbf{Set}$ also exists in locale theory — it just isn't faithful.)

Discrete topology and underlying sets in ASD

Consider the same adjunction in ASD

$\mathcal{S}$ (the whole ASD category)

$$\Delta \uparrow \dashv \downarrow \mathsf{U}$$

$\mathcal{E}$ (overt discrete objects)

Δ is the inclusion of a full subcategory — usually anonymous

What happens if we assume that it has a right adjoint $\Delta \dashv \mathsf{U}$?

The structure induced by $\Delta \dashv \mathsf{U}$

For any type (space, object of $\mathcal{S}$) X,
$\mathsf{U}X$ is another type, the **underlying set** of X.

But $\mathsf{U}X$ is overt discrete, whatever X was.

So it has
an **existential quantifier**, $(\exists_{\mathsf{U}X}) : \Sigma^{\mathsf{U}X} \to \Sigma$
and an **equality**, $(=_{\mathsf{U}X}) : \mathsf{U}X \times \mathsf{U}X \to \Sigma$.

$(=_{\mathsf{U}\Sigma^{\mathbb{N}}})$ solves the Halting Problem,
so this has no computational interpretation.
(That's why we listed it as an Axiom in brackets.)

A Type Theory for Underlying Sets

The adjunction $\Delta \dashv \mathsf{U}$ also has an effect on terms.

$$\frac{\Delta\Gamma \xrightarrow{a} X \text{ in } \mathcal{S}}{\Gamma \xrightarrow{\tau.\,a} \mathsf{U}X \text{ in } \mathcal{E}} \qquad \text{or} \qquad \frac{\Gamma \ \vdash\ a : X}{\Gamma \ \vdash\ \tau.\,a : \mathsf{U}X}$$

The context Γ must belong to $\mathcal{E}$ — *i.e.* its types must be overt discrete.

So, we have a transformation (U-**introduction**)

$$\frac{\Gamma \ \vdash\ a : X}{\Gamma \ \vdash\ \tau.\,a : \mathsf{U}X}$$

the counit (U-**elimination**), which is a function-symbol

$$x : \mathsf{U}X \ \vdash\ \varepsilon x : X$$

satisfying the β- and η-rules

$$\Gamma \ \vdash\ \varepsilon(\tau.\,a) = a : X \qquad \text{and} \qquad x : \mathsf{U}X \ \vdash\ x = (\tau.\,\varepsilon x) : \mathsf{U}X.$$

In short, $\tau.$ may be applied to any term
so long as all of its free variables are of overt discrete type.

Returning to topology

Any mono $i : X \to D$ from an overt object to a discrete one
is an open inclusion.

So maybe:

The classifier Ω for **all** monos in $\mathcal{E}$
is the same as
The classifier Σ for **open** monos in $\mathcal{S}$.

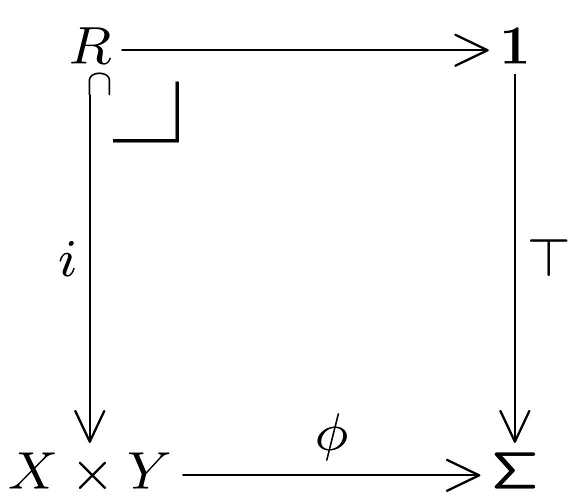
Not quite — Ω is a set, whilst Σ is a space.

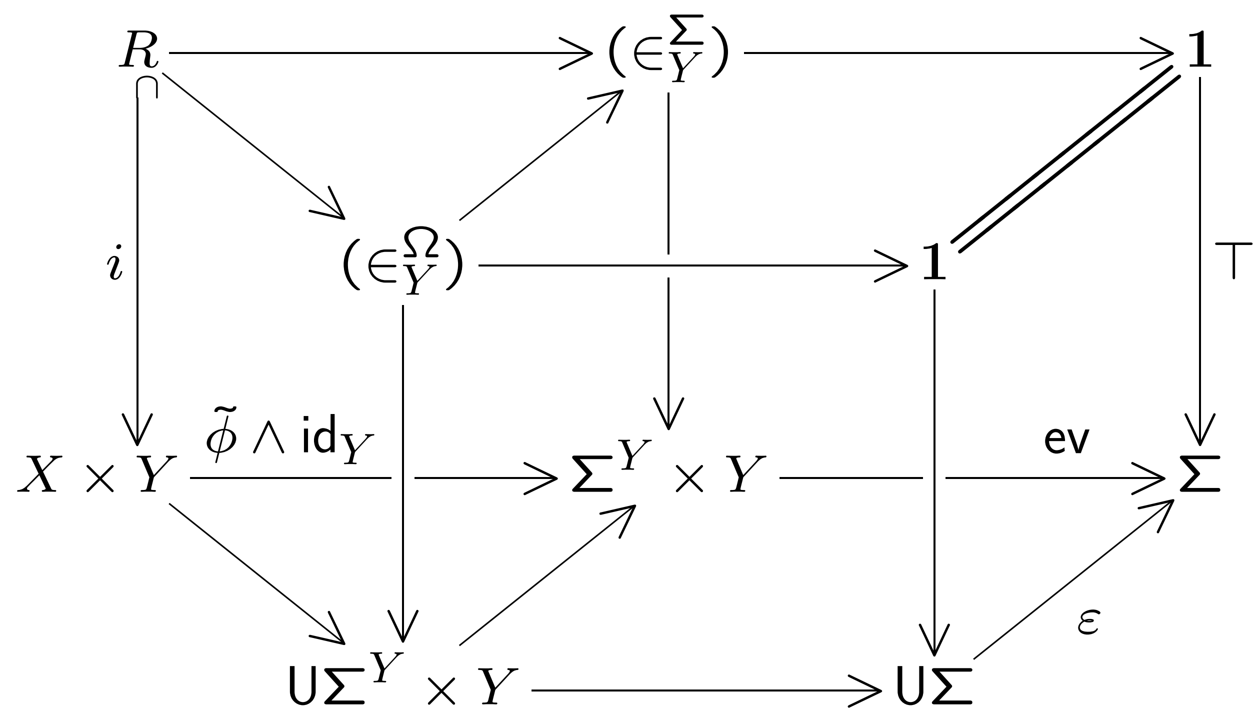
In fact, $\Omega = \mathsf{U}\Sigma$.

Overt discrete objects form a topos

Theorem If $\Delta \dashv \mathsf{U}$ exists then $\mathcal{E}$ is an elementary topos with subobject classifier $\Omega \equiv \mathsf{U}\Sigma$.

Proof Since $\mathcal{E}$ has finite limits, we show that $\mathsf{U}\Sigma^Y$ is the powerset of $Y \in \mathsf{ob}\mathcal{E}$.





A bigger subcategory than $\mathcal{E}$

Let $\mathcal{L} \subset \mathcal{S}$ be the full subcategory of objects L of the form

$$L \rightarrowtail \Sigma^N \rightrightarrows \Sigma^M$$

(an equaliser) with N and M overt discrete.

($\mathcal{L}$ will be the category of **locales** in the extended calculus.)

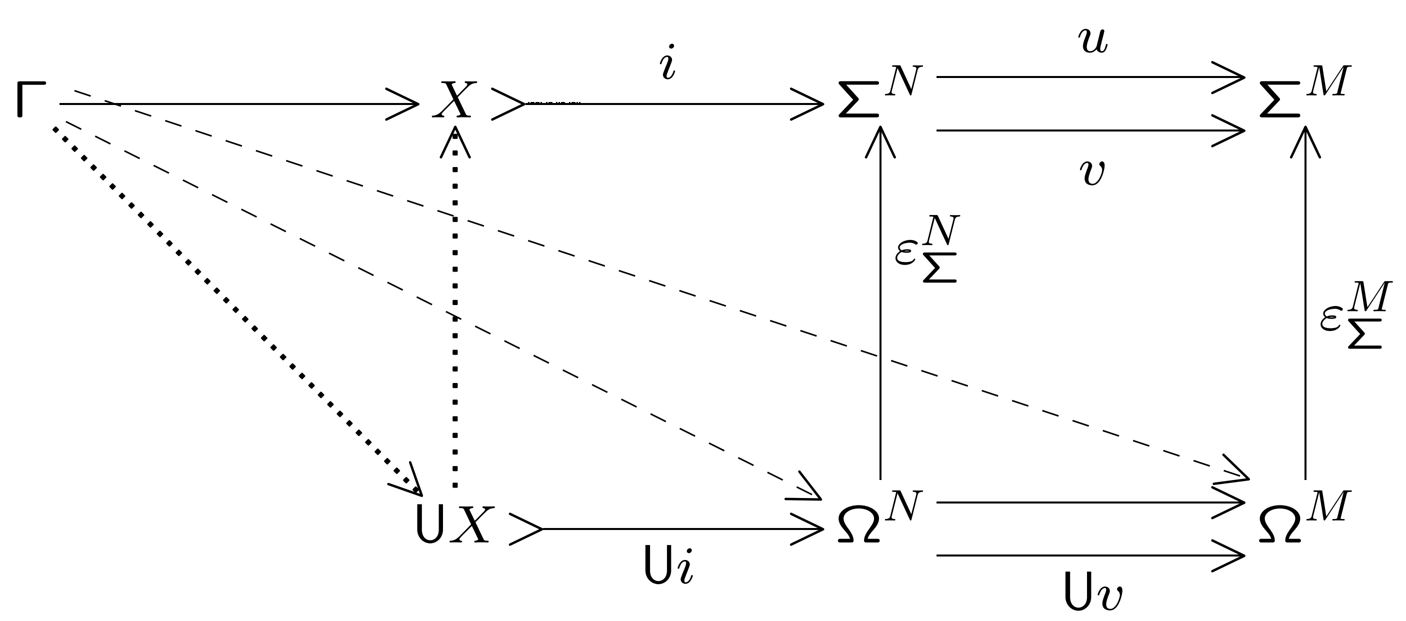
Converse

If $\mathcal{E}$ is a topos then the inclusion $\mathcal{E} \subset \mathcal{L}$ has a right adjoint.

The mono $\top : \mathbf{1} \hookrightarrow \Omega$ in $\mathcal{E}$ — is open in $\mathcal{S}$.
It's classified by some $\varepsilon_\Sigma : \Omega\Sigma \to \Sigma$.

Put $\mathsf{U}\Sigma \equiv \Omega$.

Since U, $\Sigma^{(-)}$ and equalisers are all right adjoints,
the construction lifts to Σ^N and its equalisers.



A Curious Corollary

To set this up, we only needed $\top$ and $\wedge$ in $\mathbf{\Sigma}$.

$\bot$ and $\vee$ in $\mathbf{\Sigma}$ are definable.

$\mathcal{S}$ has finite coproducts (from the monadic assumption).

Any coreflective subcategory $\mathcal{E} \subset \mathcal{S}$ is closed under (colimits, but in particular) finite coproducts.

So $\mathbf{0}$ and $\mathbf{2}$ are overt.
Their existential quantifiers are
$\exists_{\mathbf{0}} \equiv \bot : \mathbf{\Sigma}^{\mathbf{0}} \equiv \mathbf{1} \to \mathbf{\Sigma}$ and $\exists_{\mathbf{2}} \equiv \vee : \mathbf{\Sigma}^{\mathbf{2}} \equiv \mathbf{\Sigma} \times \mathbf{\Sigma} \to \mathbf{\Sigma}$.

Distributivity follows from the Euclidean principle.

Hence there is no "sequential" version (*i.e.* without $\vee$) of ASD with the underlying set axiom.
(There might well still be one without it.)

Intrinsic and Imposed Structure

$\mathcal{S}$ is a category that we have (partly, so far) axiomatised as a type theory.
We claim that it is an **intrinsic** notion of "topological space".

For example, Σ has an **intrinsic** lattice structure $(\top, \wedge, \bot, \vee)$.

All maps $\Sigma^Y \to \Sigma^X$ in $\mathcal{S}$ preserve the order arising from this.

$\mathcal{E}$ is a topos, in which we can do mathematics
in the traditional (at least, late 20th century) way.

For example, $\Omega \equiv \mathsf{U}\Sigma$ has an **imposed** lattice structure $(\top, \curlywedge, \bot, \curlyvee)$,
where $\curlywedge : \mathsf{U}\Sigma \times \mathsf{U}\Sigma \to \mathsf{U}\Sigma$ is $\mathsf{U}(\wedge : \Sigma \times \Sigma \to \Sigma)$.

Maps in $\mathcal{E}$ don't have to preserve this
unless we **impose** an extra condition to say so.

$\Omega X \equiv \mathsf{U}\Sigma^X$ as a complete Heyting algebra

Using the type theory for underlying sets, we can define **Heyting implication**, $(\Rightarrow) : \Omega X \times \Omega X \to \Omega X$, by

$$\phi, \psi : \mathsf{U}\Sigma^X \;\vdash\; (\phi \Rightarrow \psi) \;\equiv\; \tau.\, \lambda x.\, \exists \theta{:}\mathsf{U}\Sigma^X.\, \varepsilon\theta x \wedge (\phi \curlywedge \theta \preccurlyeq \psi) : \mathsf{U}\Sigma^X,$$

so $$\varepsilon(\phi \Rightarrow \psi)x \;\Leftrightarrow\; \exists\theta.\, \varepsilon\theta x \wedge (\phi \curlywedge \theta \preccurlyeq \psi),$$

Heyting negation, $(\neg) : \Omega X \to \Omega X$, by $\neg\phi \equiv (\phi \Rightarrow \bot)$, the **lower sets**, $\downarrow : \Omega X \to \Omega\Omega X$, by

$$\phi : \mathsf{U}\Sigma^X \;\vdash\; {\downarrow}\phi \;\equiv\; \tau.\, \lambda\psi{:}\mathsf{U}\Sigma^X.\, (\psi \preccurlyeq \phi) : \mathsf{U}\Sigma^{\mathsf{U}\Sigma^X},$$

and the **join**, $\bigvee : \Sigma^{\Omega X} \to \Sigma^X$, by

$$F : \Sigma^{\mathsf{U}\Sigma^X} \;\vdash\; \bigvee F \;\equiv\; \lambda x{:}X.\, \exists\theta{:}\mathsf{U}\Sigma^X.\, F\theta \wedge (\varepsilon\theta)x : \Sigma^X.$$

Direct and inverse image maps

Let $f : X \to Y$ in $\mathcal{S}$. Then $f^* \dashv f_*$, where
the **inverse image**, $f^* \equiv \Omega f \equiv \mathsf{U}\Sigma^f : \Omega Y \to \Omega X$, is defined by

$$\psi : \mathsf{U}\Sigma^Y \;\vdash\; f^*\psi \;\equiv\; \tau.\, \lambda x{:}X.\, (\varepsilon\psi)(fx) : \mathsf{U}\Sigma^X$$

and the **direct image**, $f_* : \Omega X \to \Omega Y$, by

$$\phi : \mathsf{U}\Sigma^X \;\vdash\; f_*\phi \;\equiv\; \tau.\, \lambda y{:}Y.\, \exists\theta{:}\mathsf{U}\Sigma^Y.\, (\varepsilon\theta)y \wedge (f^*\theta \preccurlyeq \phi) : \mathsf{U}\Sigma^Y.$$

What you're supposed to remember from today

Euclidean principle: $\sigma \wedge F\sigma \;\Leftrightarrow\; \sigma \wedge F\top$ (equivalent to subobject classifier)

Phoa principle: $F\sigma \;\Leftrightarrow\; F\bot \vee \sigma \wedge F\top$ (for $\left(\begin{smallmatrix}\odot \\ \bullet\end{smallmatrix}\right)$ as a double classifier)

The "underlying set" functor $\Delta \dashv \mathsf{U}$ gives the topos-based theory
which will match the standard theory (locales over an elementary topos)
but has no computational interpretation.

Something we haven't axiomatised yet

Subspaces.

You don't get much from $\mathbf{1}$, $\mathbb{N}$, $\times$, $\mathbf{\Sigma}^{(-)}$ and U.

(They're needed to make sense of what we've done today.)

A job for tomorrow.