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Abstract Stone Duality

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Type theory and category theory

Given any (possibly dependent) type theory
with all the structural rules (in particular weakening),
there is an **elementary sketch**:

its **objects** or **vertices** are the **contexts**

its **generating morphisms** or **edges** are of two kinds:
display maps $\widehat{x} : [\Gamma,\ x : X] \longrightarrow\!\!\triangleright\ \Gamma$ for each **type** $\Gamma\ \vdash\ X$ type
cuts $[a/x] : \Gamma \to [\Gamma,\ x : X]$ for each **term** $\Gamma\ \vdash\ a : X$

Extended substitution lemma

$$
\begin{array}{lcl}
\widehat{x} \,;\, \widehat{y} & = & \widehat{y} \,;\, \widehat{x} \\
\widehat{x} \,;\, [b/y] & = & [b/y] \,;\, \widehat{x} \\
[a/x] \,;\, \widehat{x} & = & \mathsf{id} \\
[a/x] \,;\, [b/y] & = & [[a/x]^* b/y] \,;\, [a/x] \\
[y/x] \,;\, \widehat{x} \,;\, [x/y] \,;\, \widehat{y} & = & \mathsf{id}
\end{array}
$$

where y is not free in a.

Normalisation theorem: every $[x_1 : X_1, \ldots, x_n : X_n] \to [y_1 : Y_1, \ldots, y_m : Y_m]$ is
$[a_m/z_m] \,;\, \cdots \,;\, [a_1/z_1] \,;\, \widehat{x}_n \,;\, \cdots \,;\, \widehat{x}_1 \,;\, [y_m/z_m] \,;\, \cdots \,;\, [y_1/z_1] \,;\, \widehat{y}_m \,;\, \cdots \,;\, \widehat{y}_1$

Universal properties and introduction/elimination rules can then be compared directly.

Where to find the details?

Practical Foundations of Mathematics

Cambridge Strudies in Advanced Mathematics 59

Cambridge University Press, 1999

ISBN 0521337798

Today: the Monadic λ-calculus

Why it's needed — in the Russian School of recursive analysis,
Cantor space $\mathbf{2}^{\mathbb{N}}$ and the closed real interval $\mathbb{I} \equiv [0, 1] \subset \mathbb{R}$ **are not compact**.

Subspace topology for locally compact spaces — $\mathbf{\Sigma}$-split subspaces.

How to construct models of the monadic λ-calculus.

Cantor space $\mathbf{2}^{\mathbb{N}} \rightarrowtail \mathbf{2}^{\mathbb{N}}_{\perp}$ and Dedekind reals $\mathbb{R} \rightarrowtail \mathbf{\Sigma}^{\mathbb{Q}} \times \mathbf{\Sigma}^{\mathbb{Q}}$.

Textbook recursive analysis — Enumeration

A type theory is written in a certain **alphabet**: x, $\wedge$, $\Rightarrow$, (, $\vdash$, :, ...
in which we can enumerate all strings of letters.

There is a **primitive recursive** function

$$v(n) \equiv \begin{cases} 1 & \text{if } n \text{ encodes a valid deduction} \\ 0 & \text{otherwise} \end{cases}$$

If the type theory has a "universal type"
then for any definable type X (*e.g.* $\mathbf{2}^{\mathbb{N}}$) with a constant x_0
there is a function $\mathbb{N} \to X$ defined by

$$u_n \equiv \begin{cases} a & \text{if } n \text{ encodes a valid deduction of } (\vdash a : X) \\ x_0 & \text{otherwise} \end{cases}$$

A pathological open set

Cantor space and the closed unit interval are both metric and measure spaces.
Let u_n be an enumeration of the definable elements of either of them.

Define $\phi x \;\equiv\; \exists n.\, |x - u_n| < 2^{-n-1}$.

By construction, if a is definable, so $a = u_m$ for some m, so $\phi a \Leftrightarrow \top$.

Does this make ϕ an open cover?

If ϕ is a cover then the space is not compact.

Observe that ϕ is the directed union of ϕ_m
where each $\phi_m x \;\equiv\; \exists n < m.\, |x - u_n| < 2^{-n-1}$
is a union of m intervals of total length $1 - 2^{-m-1} < 1$, so certainly doesn't cover.

Another way to see this: **König's Lemma fails**.
ϕ defines an infinite binary (**Kleene**) tree with no infinite **computable** path.

Is ϕ a cover?

Is the following (ω-) rule valid?

$$\frac{\text{for all definable } \vdash a : X, \quad \vdash \phi a \Leftrightarrow \top}{x : X \ \vdash \ \phi x \Leftrightarrow \top}$$

Maybe the system in which we're working does allow this — maybe it doesn't.

The problem is not an absolute one

The Dedekind reals and Cantor space are defined by univeral properties.
Cantor space is the exponential $\mathbf{2}^{\mathbb{N}}$.
The Dedekind reals form an equaliser (which we'll see later).

These universal properties need not be preserved by functors.

Let $\mathcal{S}$ be a category
that has objects C and R which have these universal properties ("by definition")
but other "undesirable" features (failure of compactness) "by Proof".

Let $i : \mathcal{S} \to \mathcal{S}'$ be some construction, together with its comparison functor.

iC and iR are objects of $\mathcal{S}'$,
where they may still have some of the properties that they had before,
but not others.

$\mathcal{S}'$ have have "its own" C' and R' satisfying the same universal properties *verbatim*,
but maybe now they have "desirable" properties (compactness).

But $C' \neq iC$, $R' \neq iR$ — probably iC and iR are not interesting.

How can we construct a new category with “good” R and C?

We’ll come back to this later.

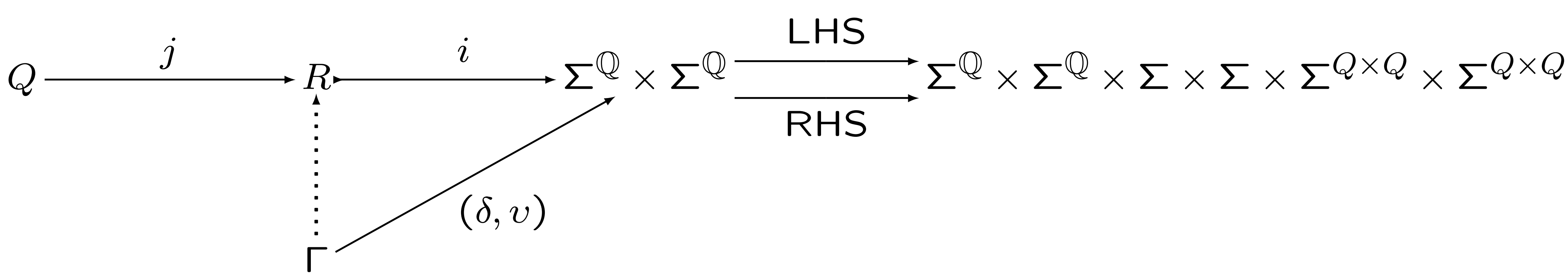
First we’ll find out what a “good” Dedekind real line looks like.

The Dedekind reals

A **Dedekind cut** (δ, υ) is a pair of predicates $\Gamma,\ q : Q \vdash \delta q,\ \upsilon q : \Sigma$, such that

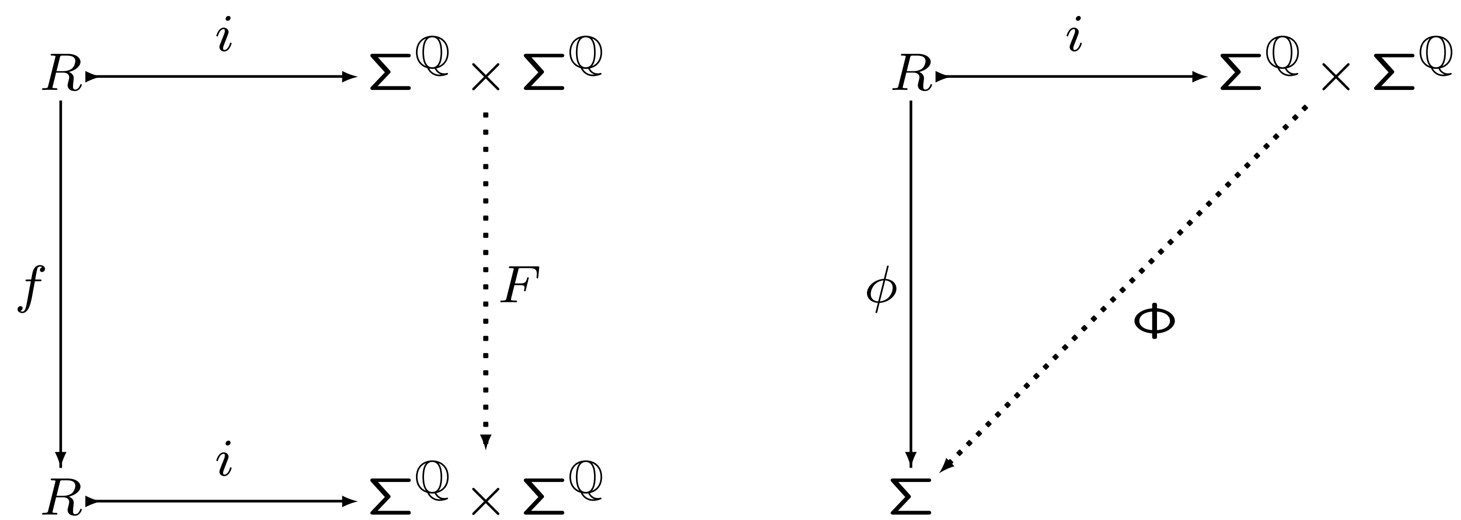
$$\begin{array}{lll}
\Gamma,\ u : Q & \vdash\ \upsilon u \Leftrightarrow \exists t{:}Q.\ \upsilon t \wedge (t < u) & \upsilon \text{ rounded upper} \\
\Gamma,\ d : Q & \vdash\ \delta d \Leftrightarrow \exists e{:}Q.\ (d < e) \wedge \delta e & \delta \text{ rounded lower} \\
\Gamma & \vdash\ \exists u{:}Q.\ \upsilon u \Leftrightarrow \top & \text{bounded above} \\
\Gamma & \vdash\ \exists d{:}Q.\ \delta d \Leftrightarrow \top & \text{bounded below} \\
\Gamma,\ d, u : Q & \vdash\ \delta d \wedge \upsilon u \Rightarrow (d < u) & \text{disjoint} \\
\Gamma,\ d, u : Q & \vdash\ (d < u) \Rightarrow (\delta d \vee \upsilon u) & \text{located}
\end{array}$$

We therefore expect R to be the **equaliser**



The Dedekind reals

Since real numbers $a : R$ are represented by cuts, $(\delta, \upsilon) : \Sigma^{\mathbb{Q}} \times \Sigma^{\mathbb{Q}}$,
we also need to represent all functions $f : R \to R$ (and, more fundamentally, $\phi : R \to \Sigma$)
in the λ-calculus by terms whose variables and values also have type $\Sigma^{\mathbb{Q}} \times \Sigma^{\mathbb{Q}}$



So R should have the **subspace topology**
(another way of saying this is that Σ is **injective**).

The Dedekind reals

Instead of a merely existential property of the extension of functions or open subspaces *one at a time*,
we postulate a (Scott) *continuous function* $I : \Sigma^R \rightarrowtail \Sigma^{\Sigma^{\mathbb{Q}} \times \Sigma^{\mathbb{Q}}}$
that does the extension in a "canonical" way.
This is to satisfy

$$x : R,\ \phi : \Sigma^R \ \vdash \ I\phi(ix) \ \Leftrightarrow \ \phi x : \Sigma,$$

making Σ^R a retract of $\Sigma^{\Sigma^{\mathbb{Q}} \times \Sigma^{\mathbb{Q}}}$.

Therefore Σ^R, and so R itself, are determined abstractly by an idempotent

$$\mathcal{E} \equiv I \cdot \Sigma^i : \Sigma^{\Sigma^{\mathbb{Q}} \times \Sigma^{\mathbb{Q}}} \to \Sigma^{\Sigma^{\mathbb{Q}} \times \Sigma^{\mathbb{Q}}}.$$

We shall need to define $\mathcal{E}$ as a λ-term involving Q, its order ($<$) and its powers ($\Sigma^{\mathbb{Q}}$ *etc.*)
— but not R, as that's what we're trying to define.

The Dedekind reals

Dedekind cuts embed $i : \mathbb{R} \rightarrowtail \Sigma^{\mathbb{Q}} \times \Sigma^{\mathbb{Q}}$ by

$$a \mapsto (\lambda d.\, d < a, \lambda u.\, a < u)$$

and the map $I : \Sigma^{\mathbb{R}} \rightarrowtail \Sigma^{\Sigma^{\mathbb{Q}} \times \Sigma^{\mathbb{Q}}}$ is defined by

$$(U \subset \mathbb{R}) \text{ open } \mapsto \lambda \delta \upsilon.\, \exists du.\, \delta d \wedge \upsilon u \wedge ([d, u] \subset U).$$

The map I is *Scott continuous*
(*i.e.* it takes directed unions of open subsets of $\mathbb{R}$ to directed joins in the lattice $\Sigma^{\Sigma^{\mathbb{Q}} \times \Sigma^{\mathbb{Q}}}$)
by the Heine–Borel theorem (*compactness* of the closed interval $[d, u]$).

Then

$$\phi a \;\equiv\; (a \in U) \;\mapsto\; I\phi(ia) \;\equiv\; (\exists du.\, a \in (d, u) \subset [d, u] \subset U) \iff (a \in U) \;\equiv\; \phi a.$$

which expresses *local compactness* of $\mathbb{R}$.

Hence $\Sigma^{\mathbb{R}} \lhd \Sigma^{\Sigma^{\mathbb{Q}} \times \Sigma^{\mathbb{Q}}}$, where the lattice $\Sigma^{\mathbb{R}}$ of open subsets of $\mathbb{R}$
is itself equipped with the Scott topology.

The Dedekind reals

The next task is to formulate the idempotent $\mathcal{E} \equiv I \cdot \Sigma^i$
entirely in terms of higher order predicates on $\mathbb{Q}$. We have

$$\Phi : \Sigma^{\Sigma^{\mathbb{Q}} \times \Sigma^{\mathbb{Q}}} \;\vdash\; \mathcal{E}\Phi \;=\; I(\lambda x.\, \Phi(ix)) \;=\; \lambda\delta\upsilon.\, \exists du.\, \delta d \wedge \upsilon u \wedge [d,u]\Phi : \Sigma^{\Sigma^{\mathbb{Q}} \times \Sigma^{\mathbb{Q}}}.$$

where $[d,u]\Phi \;\equiv\; \forall x \in [d,u].\, \Phi(\delta_x, \upsilon_x)$ is a pun:
it is the common notation for both the closed interval and the necessity modal operator.

$$[d,u]\Phi \;\Leftrightarrow\; \forall x \in [d,u].\, \exists et.\, (e < x < t) \wedge \Phi(\delta_e, \upsilon_t),$$

which means that the closed interval $[d,u]$
is covered by open intervals (e,t) for each of which $\Phi(\delta_e, \upsilon_t)$ holds.

The Dedekind reals

Finitely many (n) overlapping open intervals suffice to define $[d,u]\Phi$, so we may adopt a more explicit notation for them:

$$[d,u] \;\equiv\; [q_1, q_{2n}] \;\subset\; (q_0,q_3)\cup(q_2,q_5)\cup(q_4,q_7)\cup\cdots\cup(q_{2n-2},q_{2n+1}),$$

Using this we have, classically,

$$[d,u]\Phi \;\Leftrightarrow\; \exists q_0<\cdots<q_{2n+1}.\,(q_1=d)\wedge(q_{2n}=u)\wedge\bigwedge_{k=0}^{n-1}\Phi(\lambda x.\,x<q_{2k},\ \lambda x.\,q_{2k+3}<x)$$

$$\mathcal{E}\Phi(\delta,\upsilon) \;\Leftrightarrow\; \exists q_0<\cdots<q_{2n+1}.\;\delta q_1\wedge\upsilon q_{2n}\;\wedge\;\bigwedge_{k=0}^{n-1}\Phi(\lambda x.\,x<q_{2k},\ \lambda x.\,q_{2k+3}<x).$$

Admissibility

A point $\Gamma \vdash (\delta, \upsilon) : \Sigma^{\mathbb{Q}} \times \Sigma^{\mathbb{Q}}$ of the larger space
is called **admissible** with respect to $\mathcal{E}$ if

$$\Gamma,\ \Phi : \Sigma^{\Sigma^{\mathbb{Q}} \times \Sigma^{\mathbb{Q}}} \ \vdash\ \Phi(\delta, \upsilon)\ \Leftrightarrow\ \mathcal{E}\Phi(\delta, \upsilon) : \Sigma.$$

The operator $\mathcal{E}$ "normalises" open subspaces of the larger space
by restricting them to the smaller one and then re-expanding them.
A point of the larger space therefore belongs to the smaller one
iff its membership of *any* open subspace Φ of the larger space
is unaffected by normalisation.

If (δ, υ) is a cut then it is admissible

$$\mathcal{E}\Phi(\delta,\upsilon) \;\Leftrightarrow\; \exists q_0 \ldots q_{2n+1}.\, \delta q_1 \wedge (\upsilon q_1 \vee \bigvee_{j=1}^{2n-2} (\delta q_j \wedge \upsilon q_{j+2}) \vee \delta q_{2n}) \wedge \upsilon q_{2n} \wedge \bigwedge_{k=0}^{n-1} \Phi(\delta_{q_{2k}}, \upsilon_{q_{2k+3}}),$$

$$\begin{aligned}
\mathcal{E}\Phi(\delta,\upsilon) \;&\Rightarrow\; \exists q_0 \ldots q_{2n+1}.\, \bigvee_{k=0}^{n-1} (\delta q_{2k} \wedge \upsilon q_{2k+3} \wedge \Phi(\delta_{q_{2k}}, \upsilon_{q_{2k+3}})) \\
&\Rightarrow\; \exists d < u.\, \delta d \wedge \upsilon u \wedge \Phi(\delta_d, \upsilon_u) \;\Leftrightarrow\; \Phi(\delta,\upsilon) \qquad\qquad d \equiv q_{2k},\ u \equiv q_{2k+3} \\
&\Rightarrow\; \exists q_0 < \cdots < q_3.\, \delta q_1 \wedge \upsilon q_2 \wedge \Phi(\delta_{q_0}, \upsilon_{q_3}) \;\Rightarrow\; \mathcal{E}\Phi(\delta,\upsilon),
\end{aligned}$$

so a single interval $[q_1, q_2] \subset (q_0, q_3)$ would have been enough for the expansion.

If (δ, υ) is admissible then it is a cut

We have to deduce each of the parts of the definition of a cut from instances of admissibility,

$$\mathcal{E}\Phi(\delta, \upsilon) \Leftrightarrow \Phi(\delta, \upsilon),$$

with respect to carefully chosen Φ.

If (δ, υ) is admissible then it is a cut

For *boundedness*, consider $\Phi \equiv \lambda\alpha\beta.\top$. Then

$$\top \equiv \Phi(\delta, \upsilon) \Leftrightarrow \mathcal{E}\Phi(\delta, \upsilon) \equiv \exists q_0 < \cdots < q_{2n+1}.\, \delta q_1 \wedge \upsilon q_{2n}.$$

If (δ, υ) is admissible then it is a cut

For *roundedness*, consider $\Phi \equiv \lambda\alpha\beta.\, \alpha d \wedge \beta u$.

Then $\delta d \wedge \upsilon u \;\equiv\; \Phi(\delta, \upsilon) \;\Leftrightarrow\; \mathcal{E}\Phi(\delta, \upsilon)$, whose expansion is

$$\exists q_0 < \cdots < q_3.\, \delta q_1 \wedge \upsilon q_2 \wedge (d < q_0) \wedge (q_3 < u) \;\Leftrightarrow\; \exists q_1 q_2.\, (d < q_1) \wedge \delta q_1 \wedge (q_2 < u) \wedge \upsilon q_2.$$

If (δ, υ) is admissible then it is a cut

For *disjointness*, consider $\Phi \equiv \lambda\alpha\beta.\, \bot$.

Then the big conjunction is either empty ($n = 0$) or false ($n \geq 1$), so

$$\bot \;\equiv\; \Phi(\delta, \upsilon) \;\Leftrightarrow\; \mathcal{E}\Phi(\delta, \upsilon) \;\equiv\; \exists q_0 < \cdots < q_{2n+1}.\, \delta q_1 \wedge \upsilon q_{2n} \wedge (n = 0) \;\Leftrightarrow\; \exists q_0 < q_1.\, \delta q_1 \wedge \upsilon q_0.$$

If (δ, υ) is admissible then it is a cut

For *locatedness*, let $d < u$.

Since (δ, υ) are rounded and bounded, there is a sequence of rationals

$$q_0 < q_1 < d < q_2 < q_3 < u < q_4 < q_5 \quad \text{with} \quad \delta q_1 \quad \text{and} \quad \upsilon q_4.$$

Then $\Phi(\alpha, \beta) \equiv \alpha d \vee \beta u$ gives

$$\delta d \vee \upsilon u \ \equiv \ \Phi(\delta, \upsilon) \ \Leftrightarrow \ \mathcal{E}\Phi(\delta, \upsilon),$$

whose expansion is implied by

$$(q_0 < \cdots < q_5) \wedge \delta q_1 \wedge \upsilon q_4 \wedge (d < q_0 \vee q_3 < u) \wedge (d < q_2 \vee q_5 < u).$$

The closed interval is compact

$$\frac{\Gamma,\ x : [d,u]\ \vdash\ \sigma\ \Rightarrow\ \phi x}{\Gamma\ \vdash\ \sigma\ \Rightarrow\ \forall x{:}[d,u].\ \phi x\ \equiv\ I\phi(\delta_d, \upsilon_u)}$$

Let $(\delta, \upsilon) \equiv (\delta_x, \upsilon_x) \geq (\delta_d, \upsilon_u)$ be the cut corresponding to $x : R$. Then

$$I\phi(\delta_d, \upsilon_u)\ \Rightarrow\ I\phi(\delta, \upsilon)\ \equiv\ I\phi(ix)\ \Leftrightarrow\ \phi x,$$

which establishes the upward direction.

The closed interval is compact

The hypothesis says that

$$\Gamma,\ x : R,\ \sigma \Leftrightarrow \top\ \vdash\ \phi x \vee \psi x\ \Leftrightarrow\ \top,$$

which captures universal quantification as a *judgement*.

This is $\Gamma,\ \sigma \Leftrightarrow \top\ \vdash\ \Sigma^i(\Phi \vee \Psi) \Leftrightarrow \top : \Sigma^R$, where $\Phi \equiv I\phi$.

$$\Gamma,\ \sigma \Leftrightarrow \top\ \vdash\ \mathcal{E}(\Phi \vee \Psi)(\delta_d, \upsilon_u)\ \Leftrightarrow\ \mathcal{E}\top(\delta_d, \upsilon_u) : \Sigma.$$

Now, $\mathcal{E}\top(\alpha, \beta) \Leftrightarrow \top$ iff (α, β) is bounded, which (δ_d, υ_u) is.
In the expansion of $\mathcal{E}(\Phi \vee \Psi)(\delta_d, \upsilon_u)$, on the other hand,

$$(\Phi \vee \Psi)(\delta_{q_{2k}}, \upsilon_{q_{2k+3}})\ \Leftrightarrow\ \Phi(\delta_{q_{2k}}, \upsilon_{q_{2k+3}})\ \vee\ (u < q_{2k} \vee q_{2k+3} < d)\ \Leftrightarrow\ \Phi(\delta_{q_{2k}}, \upsilon_{q_{2k+3}})\ \vee\ \bot$$

since the conjuncts $\upsilon_u q_{2k}$ and $\delta_d q_1$ in $\mathcal{E}$ make $q_{2k} < q_{2n} < u$ and $d < q_1 < q_{2k+3}$.

$$\top\ \Leftrightarrow\ \mathcal{E}\top(\delta_d, \upsilon_u)\ \Leftrightarrow\ \mathcal{E}(\Phi \vee \Psi)(\delta_d, \upsilon_u)\ \Leftrightarrow\ \mathcal{E}\Phi(\delta_d, \upsilon_u)\ \Leftrightarrow\ I\phi(\delta_d, \upsilon_u).$$

What have we done?

We were working in a λ-calculus with types Σ, $\mathbb{N}$, $X \times Y$ and Σ^X.

But the object R was not one of these types.

It was a "subspace" of $\Sigma^{\mathbb{Q}} \times \Sigma^{\mathbb{Q}}$ that was defined by an idempotent $\mathcal{E}$ on $\Sigma^{\Sigma^{\mathbb{Q}} \times \Sigma^{\mathbb{Q}}}$.

Restricted λ-calculus

Just the type-formation rules

$$\mathbf{1}\ \mathsf{type} \qquad \frac{X_1\ \mathsf{type} \quad \ldots X_k\ \mathsf{type}}{\Sigma^{X_1\times\cdots\times X_k}\ \mathsf{type}}\ \Sigma^{(-)}F$$

but with the normal rules for λ-abstraction and application,

$$\frac{\Gamma, x : X \ \vdash\ \sigma : \Sigma^{Y}}{\Gamma\ \vdash\ \lambda x{:}X.\ \sigma : \Sigma^{X\times Y}}\ \Sigma^{(-)}I \qquad \frac{\Gamma\ \vdash\ \phi : \Sigma^{X\times Y} \quad \Gamma\ \vdash\ a : X}{\Gamma\ \vdash\ \phi[a] : \Sigma^{Y}}\ \Sigma^{(-)}E$$

together with the usual α, β and η rules.

Sober λ-calculus

$\Gamma \vdash P : \Sigma^{\Sigma^X}$ is **prime** if

$$\Gamma,\ \mathcal{F} : \Sigma^3 X \ \vdash\ \mathcal{F}P \ =\ P(\lambda x.\, \mathcal{F}(\lambda \phi.\, \phi x)).$$

In the context of the lattice structure and Scott continuity, P is prime iff it preserves $\top$, $\bot$, $\wedge$ and $\vee$.

The **sober λ-calculus** has the additional rules

$$\frac{\Gamma \ \vdash\ P : \Sigma^{\Sigma^X} \qquad P \text{ is prime}}{\Gamma \ \vdash\ \mathsf{focus}\, P : X} \qquad \mathsf{focus}\, I$$

$$\frac{\Gamma \ \vdash\ P : \Sigma^{\Sigma^X} \qquad P \text{ is prime}}{\Gamma,\ \phi : \Sigma^X \ \vdash\ \phi(\mathsf{focus}\, P) = P\phi : \Sigma} \qquad \mathsf{focus}\, \beta$$

$$\frac{\Gamma \ \vdash\ a, b : X \qquad \Gamma,\ \phi : \Sigma^X \ \vdash\ \phi a = \phi b}{\Gamma \ \vdash\ a = b} \qquad \mathsf{T}_0$$

Monadic λ-calculus

$\mathcal{E} : \Sigma^Y \to \Sigma^Y$ is called a **nucleus** if

$$\mathcal{F} : \Sigma^3 Y \quad \vdash \quad E(\lambda y.\, \mathcal{F}(\lambda\phi.\, E\phi y)) \;=\; E(\lambda y.\, \mathcal{F}(\lambda\phi.\, \phi y)).$$

In the context of the lattice structure and Scott continuity, E is a nucleus iff

$$\phi, \psi : \Sigma^X \;\vdash\; E(\phi \wedge \psi) \;=\; E(E\phi \wedge E\psi) \quad \text{and} \quad E(\phi \vee \psi) = E(E\phi \vee E\psi).$$

Monadic λ-calculus

The $\{\}$-rules of the **monadic λ-calculus** define the subspace itself.

$$\frac{X \text{ type} \qquad x : X,\ \phi : \Sigma^X \ \vdash\ E\phi x : \Sigma \qquad E \text{ is a nucleus}}{\{X \mid \mathcal{E}\} \text{ type}} \qquad \{\}F$$

$$\frac{\Gamma \ \vdash\ a : X \qquad \Gamma,\ \phi : \Sigma^X \ \vdash\ \phi a = E\phi a}{\Gamma \ \vdash\ \mathsf{admit}_{X,E}\, a : \{X \mid \mathcal{E}\}} \qquad \{\}I$$

$$x : \{X \mid \mathcal{E}\} \ \vdash\ i_{X,E} x : X \qquad \{\}E_0$$

$$x : \{X \mid \mathcal{E}\},\ \phi : \Sigma^X \ \vdash\ \phi(i_{X,E} x) = E\phi(i_{X,E} x) \qquad \{\}E_1$$

$$\frac{\Gamma \ \vdash\ a : X \qquad \Gamma,\ \phi : \Sigma^X \ \vdash\ \phi a = E\phi a}{\Gamma \ \vdash\ a = i_{X,E}(\mathsf{admit}_{X,E}\, a) : X} \qquad \{\}\beta$$

$$x : \{X \mid E\} \ \vdash\ x = \mathsf{admit}_{X,E}(i_{X,E} x) \qquad \{\}\eta$$

Monadic λ-calculus

The $\Sigma^{\{\}}$-rules say that it has the subspace topology, where $I_{X,E}$ expands open subsets of the subspace to the whole space.

$$\theta : \Sigma^{\{X|\mathcal{E}\}} \;\vdash\; I_{X,E}\theta : \Sigma^X \qquad \Sigma^{\{\}}E$$

The β-rule says that the composite $\Sigma^X \longrightarrow\!\!\!\!\!\!\blacktriangleright\; \Sigma^{\{X|\mathcal{E}\}} \rightarrowtail \Sigma^X$ is E:

$$\phi : \Sigma^X \;\vdash\; I_{X,E}(\lambda x{:}\{X \mid \mathcal{E}\}.\ \phi(i_{X,E}x)) = E\phi \qquad \Sigma^{\{\}}\beta$$

Notice that this is the only rule that introduces E into the λ-expressions. The η-rule says that the other composite $\Sigma^{\{X|\mathcal{E}\}} \rightarrowtail \Sigma^X \longrightarrow\!\!\!\!\!\!\blacktriangleright\; \Sigma^{\{X|\mathcal{E}\}}$ is the identity:

$$\theta : \Sigma^{\{X|\mathcal{E}\}},\ x : \{X \mid \mathcal{E}\} \;\vdash\; \theta x = I_{X,E}\theta(i_{X,E}x) \qquad \Sigma^{\{\}}\eta.$$

Some constructions

$$
\begin{array}{lcl}
X & \cong & \{\Sigma^2 X \mid \lambda \mathcal{F} F.\, F(\lambda x.\, \mathcal{F}(\lambda \phi.\, \phi x))\} \\
\{X \mid E_0\} \times \{Y \mid E_1\} & = & \{X \times Y \mid E_1^X \cdot E_0^Y\} \\
\{X \mid E_0\} + \{Y \mid E_1\} & = & \{\Sigma^{\Sigma^X \times \Sigma^Y} \mid E\} \\
\text{where} \quad E \mathcal{H} H & = & H\langle \lambda x.\, \mathcal{H}(\lambda \phi \psi.\, E_0 \phi x), \lambda y.\, \mathcal{H}(\lambda \phi \psi.\, E_1 \phi y)\rangle \\
\Sigma^{\{X \mid E\}} & = & \{\Sigma^X \mid \Sigma^E\} \\
\{\{X \mid E_1\} \mid E_2\} & = & \{X \mid E_2\} \\
\mathsf{pts}(A, \alpha) & = & \{\Sigma^A \mid \lambda F \phi.\, \phi(\alpha F)\} \\
X/R & = & \{\Sigma^2 X \mid \lambda \mathcal{F} F.\, F(\lambda x.\, \mathcal{F}(\lambda \phi.\, R \phi x))\} \\
U \cap (X \setminus V) & = & \{X \mid \lambda \phi.\, U \wedge \phi \vee V\} \\
X/\delta & = & \{\Sigma^2 X \mid \lambda \mathcal{F} F.\, F(\lambda x.\, \mathcal{F}(\lambda \phi.\, \exists y.\, \delta(x, y) \wedge \phi y))\} \\
 & \cong & \{\Sigma^X \mid \lambda F \phi.\, \exists x.\, F(\lambda y.\, \delta(x, y)) \wedge \phi x\}
\end{array}
$$

The "basic" category.

Category $\mathcal{S}$ with an object Σ for which all powers Σ^X exist.

$\mathcal{S}$ doesn't have to be cartesian closed.

It does have to have finite products, because
$\Gamma \to \Sigma^X$ corresond to $\Gamma \times X \to \Sigma$.

$\Sigma^{\mathbb{N}}$ must look like $\mathcal{P}(\mathbb{N})$ with the Scott topology.

So Σ is an internal distributive lattice with the Phoa principle
and Scott continuity (actually, this implies Phoa).

Some sort of topological structure can be defined with this.

There are lots of examples —
domains, **Dcpo**, Scott domains, algebraic lattices
topological spaces, locales
PERs, realisability toposes
programming languages modulo observable equivalence
recursive analysis.

Improving the “basic” category $\mathcal{S}$ using Stone duality

Some ideas about the “topology” SX on X.

From classical topology, we expect SX to be a distributive lattice, with $\bot$, $\top$, $\wedge$, $\vee$.

The inverse image operator for $f : X \to Y$ is a lattice homomorphism $Sf : SY \to SX$.

(Classical topology asks for more than this — “arbitrary” or at least infinitary unions.)

So the category of “frames” is some category of algebras.
As we only the the “basic category” $\mathcal{S}$ and not sets,
the carriers of algebras have to belong to $\mathcal{S}$.

Then we define the category of “locales” (the new category of “spaces”)
to be the opposite of the category of “frames”.

We hope that this will enjoy some of the advantages
that locales have over traditional spaces,
such as a Cantor space and a closed real interval that are compact.

The adjunction $\Sigma^{(-)} \dashv \Sigma^{(-)}$

An idea from domain theory: SX is the exponential Σ^X
where Σ is the Sierpiński space.

In **any** category with finite products and powers of Σ,
we have a contravariant adjunction

$$\begin{array}{c} X \longrightarrow \Sigma^Y \\ \hline\hline X \times Y \longrightarrow \Sigma \\ \hline\hline Y \longrightarrow \Sigma^X \end{array} \qquad\qquad \begin{array}{c} \mathcal{S}^{\mathsf{op}} \\ \Sigma^{(-)} \Big\uparrow \dashv \Big\downarrow \Sigma^{(-)} \\ \mathcal{S} \end{array}$$

whose units are both

$$\eta : X \longrightarrow \Sigma^{\Sigma^X} \quad \text{by} \quad \eta x \equiv \lambda\phi.\, \phi x.$$

The adjunction defines a monad

Hence there is a **strong monad** (T, η, μ, σ) with

$$TX \;\equiv\; \Sigma^{\Sigma^X}$$

$$Tf : \Sigma^{\Sigma^X} \to \Sigma^{\Sigma^Y} \quad \text{by} \quad TfF \;\equiv\; \lambda\psi.\, F(\lambda x.\, \psi(fx))$$

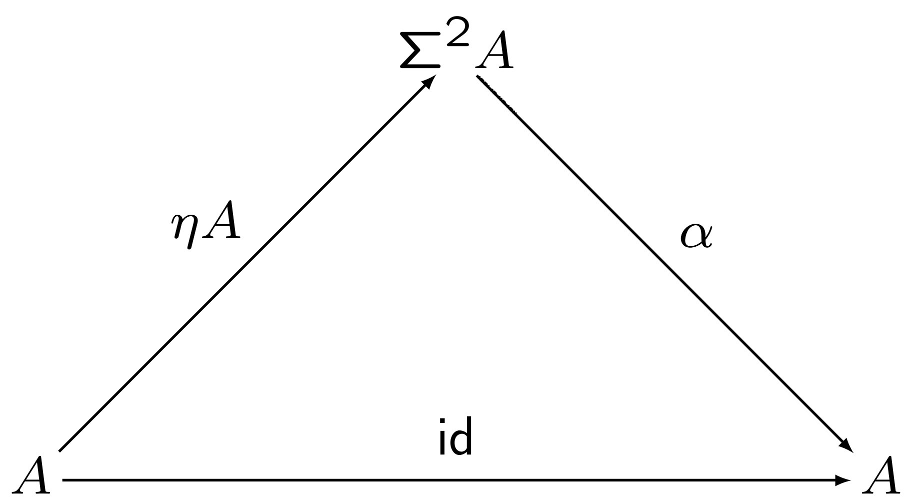
$$\eta : X \longrightarrow \Sigma^{\Sigma^X} \quad \text{by} \quad \eta x \;\equiv\; \lambda\phi.\, \phi x.$$

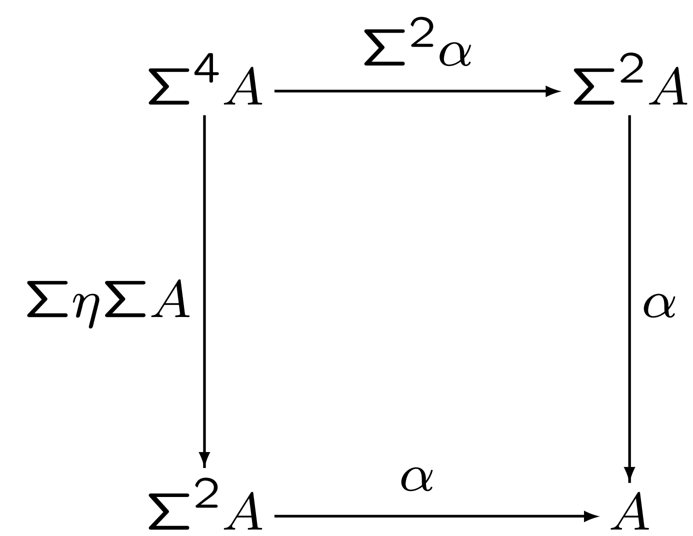
$$\mu : \Sigma^{\Sigma^{\Sigma^{\Sigma^X}}} \to \Sigma^{\Sigma^X} \quad \text{by} \quad \mu\mathsf{F} \;\equiv\; \lambda\phi.\, \mathsf{F}(\lambda F.\, F(\lambda x.\, \phi x))$$

$$\sigma : X \times \Sigma^{\Sigma^Y} \to \Sigma^{\Sigma^{X\times Y}} \quad \text{by} \quad \sigma(x, G) \;\equiv\; \lambda\theta.\, G(\lambda y.\, \theta x y).$$

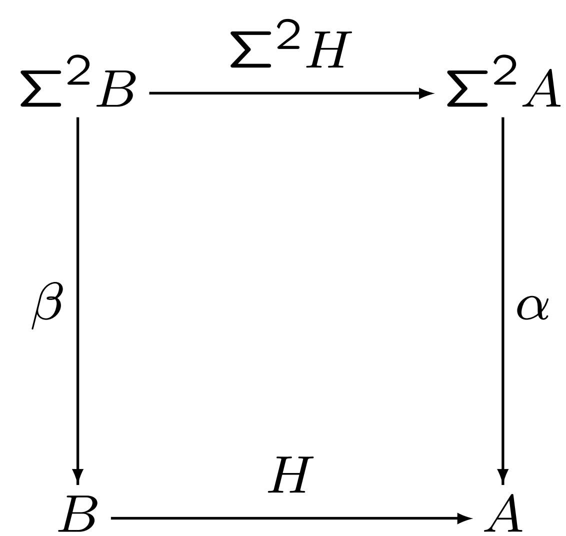
Algebras for the monad

An **algebra** is a pair (A, α) such that these diagrams commute:





A **homomorphism** is a map $H : B \to A$ such that this diagram commutes:



Embedding the original category

The two categories are linked by the functor $i : \mathcal{S} \to \mathcal{S}'$ defined by

$$
X \xrightarrow{\;f\;} Y \quad \mapsto \quad
\begin{array}{ccc}
\Sigma^3 X & \xleftarrow{\;\Sigma^3 f\;} & \Sigma^3 Y \\
{\scriptstyle \Sigma\eta\Sigma X}\downarrow & & \downarrow{\scriptstyle \Sigma\eta\Sigma Y} \\
\Sigma X & \xleftarrow{\;\Sigma f\;} & \Sigma Y
\end{array}
$$

This is the topological inverse image map.

It is also the **continuation passing interpretation**:
given a continuation $\psi : \Sigma^Y$ or $k : Y \to \Sigma$,
the function $f : X \to Y$ is represented by $\lambda x\psi.\, \psi(fx)$ or $\lambda xk.\, (fx)$,
but it is **central** — it has no **computational** (side) **effects**.

What does the new category give us?

Is it another "basic category" (with $\times$, $\Sigma^{(-)}$, $\mathbb{N}$ and $\Sigma^{\mathbb{N}}$)?
Yes — but $\times$ is difficult.

What happens if we do the construction again?
We get the same thing.

Can we characterise the resulting category?
Yes — Beck's theorem.

Can we characterise the resulting category symbolically?
Yes — the monadic λ-calculus.

Are Cantor space and the unit interval compact?
Yes, but we've got a long way to go.

Beck's theorem

The forgetful functor $U : (A, \alpha) \mapsto A$ from the category of algebras
is characterised by two properties:

(1) It **reflects invertibility**: if $H : (B, \beta) \to (A, \alpha)$ is a homomorphism and $H : B \cong A$ is an isomorphism of carriers
then $H : (B, \beta) \cong (A, \alpha)$ is an isomorphism of algebras.

(2) It **creates** U-**split coequalisers**.

(1) is about **sobriety** and (2) is about **subspaces**.