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Abstract Stone Duality

Paul Taylor

University of Manchester

Funded by UK EPSRC GR/S58522

www.cs.man.ac.uk/~pt/ASD

pt@cs.man.ac.uk

077 604 625 87

The Classical Intermediate Value Theorem

Any continuous $f : [0, 1] \to \mathbb{R}$ with $f(0) \leq 0 \leq f(1)$ has a zero.

Indeed, $f(x_0) = 0$ where $x_0 \equiv \mathsf{sup}\,\{x \mid f(x) \leq 0\}$.

A so-called "closed formula".

A program: interval halving

Let $a_0 \equiv 0$ and $e_0 \equiv 1$.

By recursion, consider $c_n \equiv \frac{1}{2}(a_n + e_n)$ and

$$a_{n+1},\ e_{n+1} \ \equiv \ \begin{cases} a_n,\ c_n & \text{if } f(c_n) > 0 \\ \\ c_n,\ e_n & \text{if } f(c_n) \leq 0, \end{cases}$$

so by induction $f(a_n) \leq 0 \leq f(e_n)$.

But a_n and e_n are respectively (non-strictly) increasing and decreasing sequences, whose differences tend to 0.

So they converge to a common value c.

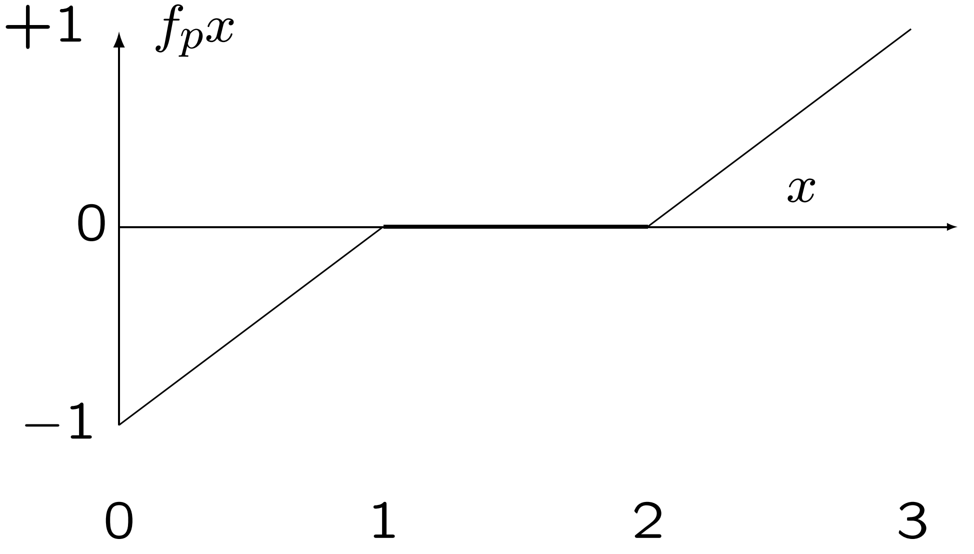
By continuity, $f(c) = 0$.

Where is the zero?

For $-1 \leq p \leq +1$ and $0 \leq x \leq 3$ consider

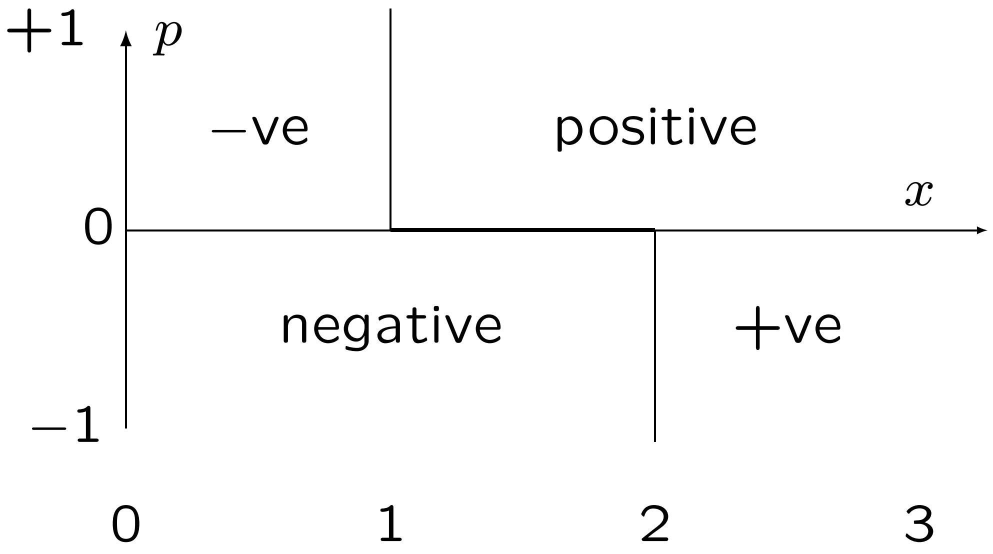
$$f_p x \ \equiv \ \mathsf{min}\,(x-1,\ \mathsf{max}\,(p, x-2))$$

Here is the graph of $f_p(x)$ against x for $p \approx 0$.



Where is the zero?

The behaviour of $f_p(x)$ depends qualitatively on p and x like this:



$$f(1) = 0 \iff p \geq 0$$

$$f(2) = 0 \iff p \leq 0$$

$$f(\tfrac{3}{2}) = 0 \iff p = 0$$

If there is some way of finding a zero of f_p,
as a side-effect it will decide how p stands in relation to 0.

Let's bar the monster

Definition $f : \mathbb{R} \to \mathbb{R}$ **doesn't hover** if,

for any $e < t$, $\quad \exists x.\, (e < x < t) \;\wedge\; (fx \neq 0)$.

Exercise Any nonzero polynomial doesn't hover.

Interval halving again

Suppose that f doesn't hover.

Let $a_0 \equiv 0$ and $e_0 \equiv 1$.

By recursion, consider

$$b_n \equiv \tfrac{1}{3}(2a_n + e_n) \quad \text{and} \quad d_n \equiv \tfrac{1}{3}(a_n + 2e_n).$$

Then $f(c_n) \neq 0$ for some $b_n < c_n < d_n$, so put

$$a_{n+1},\ e_{n+1} \equiv \begin{cases} a_n,\ c_n & \text{if } f(c_n) > 0 \\ c_n,\ e_n & \text{if } f(c_n) < 0, \end{cases}$$

so by induction $f(a_n) < 0 < f(e_n)$.

But a_n and e_n are respectively (non-strictly) increasing and decreasing sequences, whose differences tend to 0.

So they converge to a common value c.

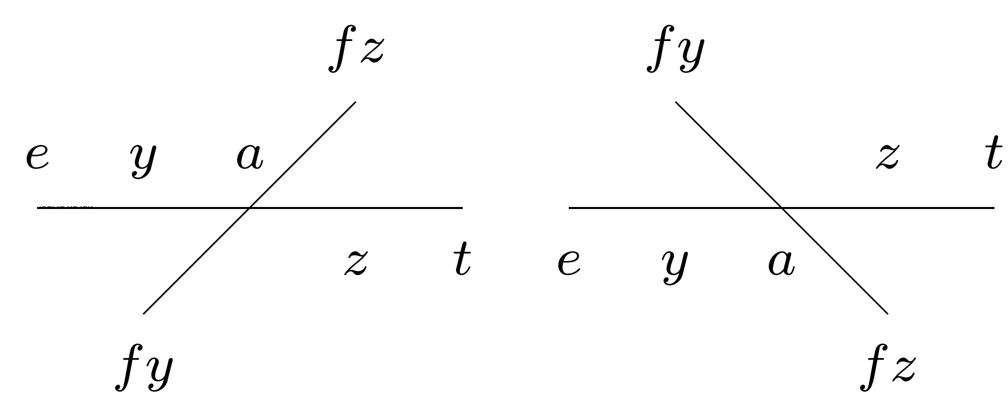
By continuity, $f(c) = 0$.

Stable zeroes

The revised interval halving algorithm finds zeroes with this property:

Definition $a \in \mathbb{R}$ is a **stable zero** of f
if, for all $e < a < t$,

$$\exists yz.\,(e < y < a < z < t) \ \wedge\ (fy < 0 < fz \ \vee fy > 0 > fz).$$



Exercise Check that a stable zero of a continuous function really is a zero.

Classically, a zero is stable iff
every nearby function (in the sup or ℓ_∞ norm) has a nearby zero.

Straddling intervals

Proposition An open subspace $U \subset \mathbb{R}$ **touches** S, *i.e.* contains a stable zero, $a \in U \cap S$, iff U contains a **straddling interval**,

$$[e,t] \subset U \quad \text{with} \quad fe < 0 < ft \quad \text{or} \quad fe > 0 > ft.$$

Proof [$\Leftarrow$] The straddling interval is an intermediate value problem in miniature.

If an interval $[e,t]$ straddles with respect to f
then it also does so with respect to any nearby function.

The possibility operator

Notation Write $\lozenge U$ if U contains a straddling interval.

By hypothesis, $\lozenge I \Leftrightarrow \top$ (where I is some open interval containing $\mathbb{I}$).

Trivially, $\lozenge \emptyset \Leftrightarrow \bot$.

Theorem $\lozenge \bigcup_{i \in I} U_i \iff \exists i.\ \lozenge U_i$.

Consider

$$V^{\pm} \;\equiv\; \{x \mid \exists y\!:\!\mathbb{R}.\ \exists i\!:\!I.\ (fy \gtrless 0) \wedge [x,y] \subset U_i\}$$

so $\mathbb{I} \subset V^{+} \cup V^{-}$.

Then $x \in (a,c) \subset V^{+} \cap V^{-}$ by connectedness, with $fx \neq 0$ and $[x,y] \subset U_i$.

The Possibility Operator as a Program

Let $\Diamond$ be a property of open subspaces of $\mathbb{R}$
that preserves unions and satisfies $\Diamond\, U_0$ for some open interval U_0.

Then $\Diamond$ has an "accumulation point" $c \in U_0$,
i.e. one of which every open neighbourhood $c \in U \subset \mathbb{R}$ satisfies $\Diamond\, U$.
In the example of the intermediate value theorem, any such c is a stable zero.

Interval halving again: let $a_0 \equiv 0$, $e_0 \equiv 1$
and, by recursion, $b_n \equiv \frac{1}{3}(2a_n + e_n)$ and $d_n \equiv \frac{1}{3}(a_n + 2e_n)$, so

$$\Diamond(a_n, e_n) \;\equiv\; \Diamond\,((a_n, d_n) \cup (b_n, e_n)) \;\Leftrightarrow\; \Diamond(a_n, d_n) \vee \Diamond(b_n, e_n).$$

Then at least one of the disjuncts is true,
so let (a_{n+1}, e_{n+1}) be either (a_n, d_n) or (b_n, e_n).

Hence a_n and e_n converge from above and below respectively to c.

If $c \in U$ then $c \in (a_n, e_n) \subset (c \pm \epsilon) \subset U$ for some $\epsilon > 0$ and n,
but $\Diamond(a_n, e_n)$ is true by construction,
so $\Diamond\, U$ also holds, since $\Diamond$ takes $\subset$ to $\Rightarrow$.

Enclosing cells of higher dimensions

Straddling intervals can be generalised.

Let $f : \mathbb{R}^{\mathbf{n}} \to \mathbb{R}^{\mathbf{m}}$ with $n \geq m$.

Let $C \subset \mathbb{R}^{\mathbf{n}}$ be a sphere, cube, *etc*.

Definition C is an **enclosing cell** if
$0 \in \mathbb{R}^{\mathbf{m}}$ lies in the interior of the image $f(C) \subset \mathbb{R}^{\mathbf{m}}$.

(There is a definition for locally compact spaces too.)

Notation Write $\Diamond U$ if $U \subset \mathbb{R}^{\mathbf{n}}$ contains an enclosing cell.

Theorem If $\Diamond \left(\bigcup_{i \in I} U_i \right) \Leftrightarrow \exists i.\ \Diamond U_i$ then
cell halving finds stable zeroes of f.

Modal operators, separately

$Z \equiv \{x \in \mathbb{I} \mid fx = 0\}$ is closed and compact.
$W \equiv \{x \mid fx \neq 0\}$ is open.
S is the subspace of stable zeroes.

Notation For $U \subset \mathbb{R}$ open, write $\Box U$ if $Z \subset U$ (or $U \cup W = \mathbb{R}$).

$$\Box X \text{ is true} \quad \text{and} \quad \Box U \wedge \Box V \;\Rightarrow\; \Box(U \cap V)$$

$$\Diamond \emptyset \text{ is false} \quad \text{and} \quad \Diamond(U \cup V) \;\Rightarrow\; \Diamond U \vee \Diamond V.$$

$$(Z \neq \emptyset) \quad \text{iff} \quad \Box \emptyset \text{ is false}$$

$$(S \neq \emptyset) \quad \text{iff} \quad \Diamond \mathbb{R} \text{ is true}$$

Both operators are Scott continuous.

Modal operators, together

The modal operators $\lozenge$ and $\square$ for the subspaces $S \subset Z$ are related in general by:

$$\square U \wedge \lozenge V \;\Rightarrow\; \lozenge(U \cap V)$$

$$\square U \iff (U \cup W = X)$$

$$\lozenge V \;\Rightarrow\; (V \not\subset W)$$

S is dense in Z iff

$$\square(U \cup V) \;\Rightarrow\; \square U \vee \lozenge V$$

$$\lozenge V \;\Leftarrow\; (V \not\subset W)$$

In the intermediate value theorem for functions that don't hover (*e.g.* polynomials):
$S = Z$ in the **non-singular** case
$S \subset Z$ in the **singular** case (*e.g.* double zeroes).

Open maps

For continuous $f : X \to Y$,
if $V \subset Y$ is open, so is $f^{-1}(V) \subset X$
if $V \subset Y$ is closed, so is $f^{-1}(V) \subset X$
if $U \subset X$ is compact, so is $f(U) \subset Y$
(if $U \subset X$ is overt, so is $f(U) \subset Y$)

Definition $f : X \to Y$ is **open** if,
whenever $U \subset X$ is open, so is $f(U) \subset Y$.

Proposition If $f : X \to Y$ is open then
if $V \subset Y$ is overt, so is $f^{-1}(V) \subset X$.

Corollary If $f : X \to Y$ is open then all zeroes are stable.

Examples of open maps

If $f : \mathbb{R}^{\mathbf{n}} \to \mathbb{R}^{\mathbf{n}}$ is continuously differentiable, and det $\left(\frac{\partial f_j}{\partial x_i}\right) \neq 0$.

If $f : \mathbb{C} \to \mathbb{C}$ is analytic and not constant — even if it has coincident zeroes.

Cauchy's integral formula:
a disc $C \subset \mathbb{C}$ is enclosing iff $\oint_{\partial C} \frac{dz}{f(z)} \neq 0$.

Stokes's theorem!

Possibility operators classically

Define $\Diamond U$ as $U \cap S \neq \emptyset$,
for *any subset* $S \subset \mathbb{R}$ *whatever*.

Then $\Diamond (\bigcup_{i \in I} U_i)$ iff $\exists i.\ \Diamond U_i$.

Conversely, if $\Diamond$ has this property, let

$$S \;\equiv\; \{a \in \mathbb{R} \mid \text{for all open } U \subset \mathbb{R}, \quad a \in U \Rightarrow \Diamond U\}.$$

$$W \;\equiv\; \mathbb{R} \setminus S \;=\; \bigcup \{U \text{ open} \mid \neg \Diamond U\}$$

Then W is open and S is closed.
$\neg \Diamond W$ by preservation of unions.
Hence $\Diamond U$ holds iff $U \not\subset W$, *i.e.* $U \cap S \neq \emptyset$.

If $\Diamond$ had been derived from some S'
then $S = \overline{S'}$, its closure.

Possibility operators: summary

$\Diamond$ is defined, like compactness, in terms of unions of open subspaces, so it is a concept of **general topology**

The proof that $\Diamond$ preserves joins uses ideas from **geometric topology**, like connectedness and sub-division of cells.

$\Diamond$ is like a bounded existential quantifier, so it's **logic**.

A very general **algorithm** uses $\Diamond$ to find solutions of problems.

But classical point-set topology is too clumsy to take advantage of this.

Overt and compact subspaces

Overt subspace

$\Diamond U$ means U **touches** $\Diamond$

for any U, $(a \in U) \Rightarrow \Diamond U$

a is an **accumulation point** of $\Diamond$

a is in the **closure** of $\Diamond$

Overt subspace of discrete space is open

Open subspace of overt space is overt

Compact subspace

$\Box U$ means U **covers** $\Box$

for any U, $\Box U \Rightarrow (a \in U)$

a is in the **saturation** of $\Box$

Compact subspace of Hausdorff space is closed

Closed subspace of compact space is Hausdorff

Overt and compact subspaces

Overt subspace of discrete space

$\Diamond\,\phi$ means ϕ **touches** $\Diamond$

$$\phi_x y \;\equiv\; (y \in \{x\}) \;\equiv\; (x = y)$$

$$\alpha x \;\equiv\; \Diamond(\lambda y.\, x = y)$$

Compact subspace of Hausdorff space

$\Box\,\phi$ means ϕ **covers** $\Box$

$$\phi_x y \;\equiv\; (y \in \overline{\{x\}}) \;\equiv\; (x \neq y)$$

$$\omega x \;\equiv\; \Box(\lambda y.\, x \neq y)$$

Open subspace of overt space

$$\Diamond\,\phi \;\equiv\; \exists_N(\alpha \wedge \phi)$$

Closed subspace of compact space

$$\Box\,\phi \;\equiv\; \forall_K(\omega \vee \phi)$$

Overt and compact subspaces

Overt subspace		Compact subspace
$\Diamond U$ means U **touches** $\Diamond$		$\Box U$ means U **covers** $\Box$
$\dfrac{U \subset W}{\neg \Diamond U}$	Closed subspace $X \setminus W$	$\dfrac{U \cup W = X}{\Box U}$
$\dfrac{A \cap U = \emptyset}{\neg \Diamond U}$	Open subspace A	$\dfrac{A \subset U}{\Box U}$

Overt and compact subspaces

Overt subspace		Compact subspace
defined by $\lozenge : \Sigma^{\Sigma^X}$		defined by $\Box : \Sigma^{\Sigma^X}$
$a \in \lozenge$ if $\phi a \;\Rightarrow\; \lozenge\,\phi$		$a \in \Box$ if $\Box\,\phi \;\Rightarrow\; \phi a$
$\dfrac{\phi \;\le\; \omega}{\lozenge\,\phi \;\Leftrightarrow\; \bot}$	Closed subspace co-classified by $\omega : \Sigma^X$ $a \in \omega$ if $\omega a \Leftrightarrow \bot$	$\dfrac{\phi \vee \omega \;\Leftrightarrow\; \top}{\Box\,\phi \;\Leftrightarrow\; \top}$
$\dfrac{\alpha \wedge \phi \;\Leftrightarrow\; \bot}{\lozenge\,\phi \;\Leftrightarrow\; \bot}$	Open subspace classified by $\alpha : \Sigma^X$. $a \in \alpha$ if $\alpha a \Leftrightarrow \top$	$\dfrac{\alpha \;\le\; \phi}{\Box\,\phi \;\Leftrightarrow\; \top}$
$\lozenge\,(\lambda x.\, \theta(x, \lozenge)) \;\Leftarrow$ $\lozenge\,(\lambda x.\, \theta(x, \lambda\phi.\ \lozenge\,\phi \vee \phi x))$,	general case	$\Box\,(\lambda x.\, \theta(x, \Box)) \;\Rightarrow$ $\Box\,(\lambda x.\, \theta(x, \lambda\phi.\ \Box\,\phi \wedge \phi x))$,

Modal laws

Overt subspace	Compact subspace
$\Diamond\bot \;\Leftrightarrow\; \bot$	$\Box\top \;\Leftrightarrow\; \top$
$\Diamond(\phi\vee\psi) \;\Leftrightarrow\; \Diamond\phi\vee\Diamond\psi$	$\Box(\phi\wedge\psi) \;\Leftrightarrow\; \Box\phi\wedge\Box\psi$
$\sigma\wedge\Diamond\phi \;\Leftrightarrow\; \Diamond(\sigma\wedge\phi)$	$\sigma\vee\Box\phi \;\Leftrightarrow\; \Box(\lambda x.\,\sigma\vee\phi x)$
$\Diamond\,(\lambda x.\,\blacklozenge(\lambda y.\,\phi xy)) \;\Leftrightarrow\; \blacklozenge(\lambda y.\;\Diamond(\lambda x.\,\phi xy))$	$\Box\,(\lambda x.\,\blacksquare(\lambda y.\,\phi xy)) \;\Leftrightarrow\; \blacksquare(\lambda y.\;\Box(\lambda x.\,\phi xy))$

Mixed modal laws

$\Box\phi\vee\Diamond\psi \;\Leftarrow\; \Box(\phi\vee\psi)$ and $\Box\phi\wedge\Diamond\psi \;\Rightarrow\; \Diamond(\phi\wedge\psi)$

Empty/inhabited is decidable

Overt subspace		Compact subspace
$\Diamond\top \Leftrightarrow \bot$	empty	$\Box\bot \Leftrightarrow \top$
$\Diamond\top \Leftrightarrow \top$	inhabited	$\Box\bot \Leftrightarrow \bot$
$\Box\bot \vee \Diamond\top \Leftarrow$	complementary	$\Box\bot \wedge \Diamond\top \Rightarrow$
$\Box(\bot \vee \top) \Leftrightarrow \Box\top \Leftrightarrow \top$	(mixed modal laws)	$\Diamond(\bot \wedge \bot) \Leftrightarrow \Diamond\bot \Leftrightarrow \bot$

The dichotomy means that the parameter space Γ is a disjoint union.

So, if it is connected, like $\mathbb{R}^{\mathbf{n}}$, **something** must break at singularities.
It is the modal law $\Box(\phi \vee \psi) \Rightarrow \Box\phi \vee \Diamond\psi$.

Compact overt subspace of $\mathbb{R}$ defines a Dedekind cut

Overt subspace $\Diamond$		Compact subspace $\Box$
$\bot$, $\vee$, $\bigvee\!\!\!/$ and so $\exists_{\mathbb{R}}$	commutes with	$\top$, $\wedge$ and $\bigvee\!\!\!/$
$\delta d \ \equiv\ \Diamond(\lambda k.\, d < k)$	Dedekind cut	$\upsilon u \ \equiv\ \Box(\lambda k.\, k < u)$
$(d < e) \wedge \delta e \ \equiv$	lower/upper	$\upsilon t \wedge (t < u) \ \equiv$
$(d < e) \wedge \Diamond(\lambda k.\, e < k)$		$\Box(\lambda k.\, k < t) \wedge (t < u)$
$\Leftrightarrow\ \Diamond(\lambda k.\, d < e < k)$	(Frobenius/$\Box\top$)	$\Leftrightarrow\ \Box(\lambda k.\, k < t < u)$
$\Rightarrow\ \Diamond(\lambda k.\, d < k)\ \equiv\ \delta d$	(transitivity)	$\Rightarrow\ \Box(\lambda k.\, k < u)\ \equiv\ \upsilon u$
$\Leftarrow$	rounded (interpolation)	$\Leftarrow$
$\exists d.\, \delta d\ \equiv\ \exists d.\ \Diamond(\lambda k.\, d < k)$	inhabited	$\exists u.\, \upsilon u\ \equiv\ \exists u.\ \Box(\lambda k.\, k < u)$
$\Leftrightarrow\ \Diamond(\lambda k.\, \exists d.\, d < k)$	(directed joins)	$\Leftrightarrow\ \Box(\lambda k.\, \exists u.\, k < u)$
$\Leftrightarrow\ \Diamond\top\ \Leftrightarrow\ \top$ (inhabited)	(extrapolation)	$\Leftrightarrow\ \Box\top\ \Leftrightarrow\ \top$

Compact overt subspace of $\mathbb{R}$ defines a Dedekind cut

δ and υ are disjoint, by transitivity of $<$

$$(\delta d \wedge \upsilon u) \ \equiv\ \Diamond(\lambda k.\, d < k) \wedge \Box(\lambda k.\, k < u) \ \Rightarrow\ \Diamond(\lambda k.\, d < k \wedge k < u) \ \Rightarrow\ (d < u)$$

δ and υ are located (touch), by locatedness of $<$

$$(\delta d \vee \upsilon u) \ \equiv\ \Diamond(\lambda k.\, d < k) \vee \Box(\lambda k.\, k < u) \ \Leftarrow\ \Box(\lambda k.\, d < k \vee k < u) \ \Leftarrow\ (d < u). \qquad \Box$$

The proofs are dual, each using one of the mixed modal laws, and $\Diamond\,\sigma \Rightarrow \sigma \Rightarrow \Box\,\sigma$.

Compact overt subspace of $\mathbb{R}$ defines a Dedekind cut

Hence there is some $a : \mathbb{R}$ with

$$\delta d \;\Leftrightarrow\; (d < a) \;\Leftrightarrow\; \Diamond(\lambda k.\, d < k) \quad \text{and} \quad \upsilon u \;\Leftrightarrow\; (a < u) \;\Leftrightarrow\; \Box(\lambda k.\, k < u)$$

Moreover, $a \in K$.

Recall that K is the closed subspace co-classified by $\omega x \equiv \Box(\lambda k.\, x \neq k)$,
so we must show that $\omega a \Leftrightarrow \bot$.

$$\begin{aligned}
\omega a \;\equiv\; \Box(\lambda k.\, a \neq k) \;&\Leftrightarrow\; \Box(\lambda k.\, a < k) \;\vee\; (k < a) \\
&\Rightarrow\; \Diamond(\lambda k.\, a < k) \;\vee\; \Box(\lambda k.\, k < a) \\
&\equiv\; \delta a \vee \upsilon a \\
&\Leftrightarrow\; (a < a) \vee (a < a) \;\Leftrightarrow\; \bot.
\end{aligned}$$

Compact overt subspace of $\mathbb{R}$ has a maximum

Any overt compact subspace $K \subset \mathbb{R}$ is
either empty
or has a greatest element $\mathsf{max}\, K \equiv a \in K$.

This satisfies, for $\Gamma \vdash x : \mathbb{R}$,

$$
\begin{array}{rcl}
(x < \mathsf{max}\, K) & \Leftrightarrow & (\exists k\!:\!K.\ x < k) \\
(\mathsf{max}\, K < x) & \Leftrightarrow & (\forall k\!:\!K.\ k < x) \\
k : K & \vdash & k \le \mathsf{max}\, K
\end{array}
$$

$$
\frac{\Gamma,\ k : K \ \vdash\ k \le x}{\Gamma \ \vdash\ \mathsf{max}\, K \le x}
$$

The Bishop-style proof

K is **totally bounded** if,
for each $\epsilon > 0$, there's a finite subset $S_\epsilon \subset K$
such that $\forall x\!:\!K.\ \exists y \in S_\epsilon.\ |x - y| < \epsilon$.

If K is closed and totally bounded,
either the set S_1 is empty, in which case K is empty too,
or $x_n \equiv \max S_{2^{-n}}$ defines a Cauchy sequence that converges to $\max K$.

But K is also overt, with
$\lozenge\,\phi \;\equiv\; \exists \epsilon > 0.\ \exists y \in S_\epsilon.\ \phi y$.

K is **located** if, for each $x \in X$,
$\inf\{|x - k| \mid k \in K\}$ is defined.
(A different usage of the word "located".)

closed and totally bounded $\Rightarrow$ compact and overt $\Rightarrow$ located

Total boundedness and locatedness are **metrical concepts**.
Compactness and overtness are **topological**.

The real interval is connected

Any closed subspace of a compact space is compact.
Any open subspace of an overt space is overt.

Any clopen subspace of an overt compact space is overt compact,
so it's either empty or has a maximum.

Since the clopen subspace is open, its elements are interior,
so maximum can only be the right endpoint of the interval.

Any clopen subspace has a clopen complement.
They can't both be empty, but
in the interval they can't both have maxima (the right endpoint).

Hence one is empty and the other is the whole interval.

Connectedness in modal notation

Using $\Box\,\theta \;\equiv\; \forall x\!:\![0,1].\;\theta x$ and $\Diamond\,\theta \;\equiv\; \exists x\!:\![0,1].\;\theta x$,

$$\Diamond(\phi\wedge\psi)=\bot \;\vdash\; \Box(\phi\vee\psi) \;\Rightarrow\; \Box\,\phi\vee\Box\,\psi \qquad\qquad \Box(\phi\vee\psi) \;\Rightarrow\; \Box\,\phi\vee\Box\,\psi\vee\Diamond(\phi\vee\psi)$$

$$\Diamond(\phi\wedge\psi)=\bot \;\vdash\; \Box(\phi\vee\psi)\wedge\Diamond\,\phi\wedge\Diamond\,\psi \;\Rightarrow\; \bot \qquad\qquad \Box(\phi\vee\psi)\wedge\Diamond\,\phi\wedge\Diamond\,\psi \;\Rightarrow\; \Diamond(\phi\wedge\psi)$$

$(\Diamond\,\phi\wedge\Box\,\psi \;\Rightarrow\; \Diamond(\phi\wedge\psi))$ (Gentzen-style rule for $\Diamond(\phi\wedge\psi)$)

Weak intermediate value theorems

Let $f : [0,1] \to \mathbb{R}$, and use two of these forms of connectedness.

$$\Diamond(\phi \wedge \psi) = \bot \ \vdash \ \Box(\phi \vee \psi) \wedge \Diamond\,\phi \wedge \Diamond\,\psi \ \Rightarrow \ \bot$$

$\phi x \equiv (0 < fx)$ and $\psi x \equiv (fx < 0)$
$\Diamond(\phi \wedge \psi) \ \Leftrightarrow \ \bot$ by disjointness.

$$(f0 < 0 < f1) \ \wedge \ (\forall x{:}[0,1].\ fx \neq 0) \ \Leftrightarrow \ \bot$$

so the closed, compact subspace
$Z \equiv \{x : \mathbb{I} \mid fx = 0\}$
is not empty.

$$\Box(\phi \vee \psi) \wedge \Diamond\,\phi \wedge \Diamond\,\psi \ \Rightarrow \ \Diamond(\phi \wedge \psi)$$

$\phi x \equiv (e < fx)$ and $\psi x \equiv (fx < t)$
$\Box(\phi \vee \psi)$ by locatedness.

$$(f0 < e < t < f1) \ \Rightarrow \ (\exists x{:}[0,1].\ e < fx < t)$$
or $\epsilon > 0 \ \vdash \ \exists x.\,|fx| < \epsilon$

so the open, overt subspace
$\{x \mid e < fx < t\}$
is inhabited.

Straddling intervals in ASD

Recall that

$f : \mathbb{R} \to \mathbb{R}$ **doesn't hover** if $(e < t) \;\Rightarrow\; \exists x.\,(e < x < t) \;\wedge\; (fx \neq 0)$.

$a : \mathbb{R}$ is a **stable zero** if $(e < a < t) \;\Rightarrow\; \exists yz.\,(e < y < a < z < t) \;\wedge\; (fy < 0 < fz \;\vee fy > 0 > fy)$.

$\Diamond\,\phi \;\equiv\; \exists et\colon[d,u].\;(e < t) \;\wedge\; (\forall x\colon[e,t].\;\phi x) \;\wedge\; (fe < 0 < ft \;\vee fe > 0 > ft)$.

Then a is a stable zero iff it is an accumulation point of $\Diamond$ $(\phi a \;\Rightarrow\; \Diamond\,\phi)$.

If f doesn't hover then $\Diamond$ preserves joins, $\Diamond(\exists n.\,\theta_n) \;\Leftrightarrow\; \exists n.\,\Diamond\,\theta_n$.

Consider $\phi^{\pm}x \;\equiv\; \exists n.\,\exists y.\,(x < y < u) \wedge (fy \underset{<}{>} 0) \wedge \forall z\colon[x,y].\;\theta_n z$.

Then $\exists x.\,\phi^{+}x \wedge \phi^{-}x$ by connectness and continue as before.