On the Reaxiomatisation of General Topology

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Topological spaces

A topological space is a set X (of points)
equipped with a set of ("open") subsets of X
closed under finite intersection and arbitrary union.

Wood and chipboard

A topological space is a set X (of points)
equipped with a set of ("open") subsets of X
closed under finite intersection and arbitrary union.

Chipboard is a set X of particles of sawdust
equipped with a quantity of glue
that causes the sawdust to form a cuboid.

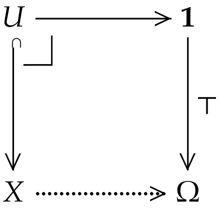
Classifying subobjects

In a topos there is a bijective correspondence

- between subobjects $U \rightarrowtail X$
- and morphisms $X \longrightarrow \Omega$.

The exponential Ω^X is the powerset.

Similarly upper subsets of a poset or CCD-lattice.



Classifying open subspaces

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In topology there is a three-way correspondence

- amongst open subspaces $U \hookrightarrow X$,
- morphisms $X \longrightarrow \Sigma \equiv \binom{\odot}{\bullet}$,
- and closed subspaces $C \hookrightarrow X$.

This is not set-theoretic complementation.

The exponential Σ^X is the topology.

Topology as λ-calculus — Basic Structure

The category $\mathcal{S}$ (of "spaces") has

- an internal distributive lattice $(\Sigma, \top, \bot, \wedge, \vee)$
- and all exponentials of the form Σ^X

We do not ask for all exponentials (cartesian closure).
At least, not as an axiom.

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The category $\mathcal{S}$ (of "spaces") has

- finite products
- an internal distributive lattice $(\Sigma, \top, \bot, \wedge, \vee)$
- and all exponentials of the form Σ^X
- satisfying
  - for sets, the Euclidean principle

$$\sigma \wedge F\sigma \iff \sigma \wedge F\top$$

  - for posets and CCD-lattices, the Euclidean principle and monotonicity
  - for spaces, the Phoa principle

$$F\sigma \iff F\bot \vee \sigma \wedge F\top$$

The Euclidean and Phoa principles capture uniqueness of the correspondence amongst open and closed subspaces of X and maps $X \to \Sigma$ (extensionality).

Advantages of this approach

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This duality is obscured in

- traditional topology and locale theory by $\bigvee / \wedge$
- constructive and intuitionistic analysis by $\neg\neg$.

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General topology is unified with recursion theory.
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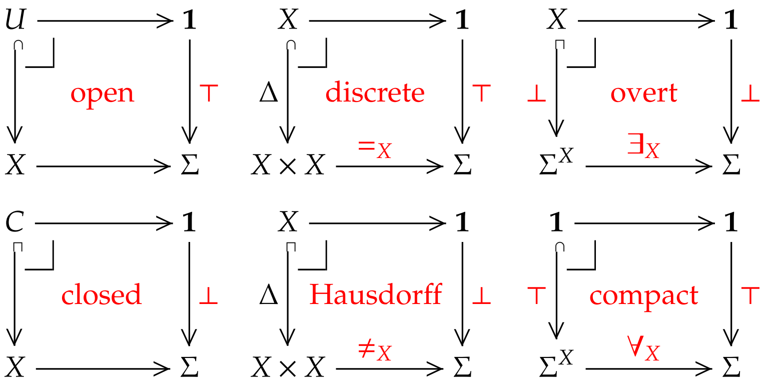
However, extracting executable programs is not obvious.

Some familiar definitions



The Frobenius laws for $\exists_X \dashv \Sigma^{!_X} \dashv \forall_X$,

$$\exists_X(\sigma \wedge \phi) \iff \sigma \wedge \exists_X(\phi) \quad \text{and} \quad \forall_X(\sigma \vee \phi) \iff \sigma \vee \forall_X(\phi),$$

are special cases of the Phoa principle.

Some familiar theorems

Any closed subspace of a compact space is compact.

Any compact subspace of a Hausdorff space is closed.

The inverse image of any closed subspace is closed.

The direct image of any compact subspace is compact.

Some less familiar theorems

Any open subspace of a overt space is overt.

Any overt subspace of a discrete space is open.

The inverse image of any open subspace is open.

The direct image of any overt subspace is overt.

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Dcpo has the basic structure, plus equalisers and all exponentials.

$\mathbf{2}^{\mathbb{N}}$ exists, and carries the discrete order.

The Dedekind and Cauchy reals may be defined.
They also carry the discrete order.

In this category, the order determines the topology.
The topology is discrete.

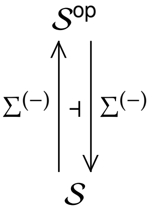
$\mathbf{2}^{\mathbb{N}}$ and $\mathbb{I}$ are not compact.

Abstract Stone Duality

The category of topologies is $\mathcal{S}^{\mathsf{op}}$,
the dual of the category $\mathcal{S}$ of "spaces".

Monadic axiom: It's also the category of
algebras for a monad on $\mathcal{S}$.

Inspired by Robert Paré, *Colimits in topoi*, 1974.



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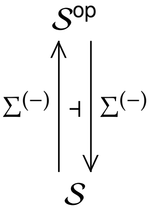

Jon Beck (1966) characterised monadic adjunctions:

- $\Sigma^{(-)} : \mathcal{S}^{\mathsf{op}} \to \mathcal{S}$ reflects invertibility,
  i.e. if $\Sigma^f : \Sigma^Y \cong \Sigma^X$ then $f : X \cong Y$, and
- $\Sigma^{(-)} : \mathcal{S}^{\mathsf{op}} \to \mathcal{S}$ creates $\Sigma^{(-)}$-split coequalisers.

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Category theory is a strong drug —
it must be taken in small doses.

As in homeopathy (?),
it gets more effective the more we dilute it!

Diluting Beck's theorem (first part)

If $\Sigma^f : \Sigma^Y \cong \Sigma^X$ then $f : X \cong Y$.

X is the equaliser of

$$X \rightarrowtail^{\eta_X} \Sigma^2 X \equiv \Sigma^{\Sigma^X} \underset{\Sigma^2 \eta_X}{\overset{\eta_{\Sigma^2 X}}{\rightrightarrows}} \Sigma^4 X$$

where $\eta_X : x \mapsto \lambda\phi.\,\phi x$.
(Without the axiom, an object X that has this property is called abstractly sober.)

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For $X \equiv \mathbb{N}$ this is definition by description
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For $X \equiv \mathbb{R}$ it is Dedekind completeness.

Diluting Beck's theorem (second part)

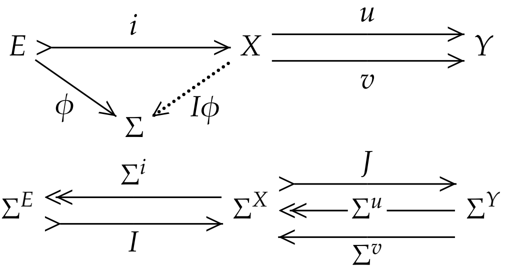
$\Sigma^{(-)} : \mathcal{S}^{\mathsf{op}} \to \mathcal{S}$ creates $\Sigma^{(-)}$-split coequalisers.

Recall that a Σ-split pair (u, v) has some J such that

$$\Sigma^u ; J ; \Sigma^v = \Sigma^v ; J ; \Sigma^v \quad \text{and} \quad \mathsf{id}_{\Sigma^X} = J ; \Sigma^u$$

Then their equaliser i has a splitting I such that

$$i ; u = i ; v, \qquad \mathsf{id}_{\Sigma^E} = I ; \Sigma^i \quad \text{and} \quad \Sigma^i ; I = J ; \Sigma^v.$$



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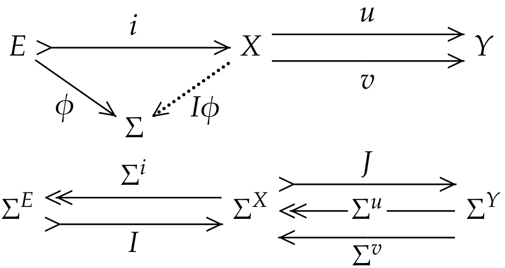

This means that (*certain*) subspaces exist, and they have the subspace topology — every open subspace of E is the restriction of one of X, in a canonical way.

Applications of Σ-split subspaces

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It can, however, be used to prove that Σ is a dominance or classifier for open inclusions (closed ones too).

We may also construct

- the lift or partial map classifier $X_\perp$,
- Cantor space $\mathbf{2}^{\mathbb{N}}$, and
- the Dedekind reals $\mathbb{R}$.

Moreover, $\mathbf{2}^{\mathbb{N}}$ and $\mathbb{I}$ are compact.

More generally, it can be used to develop an abstract, finitary axiomatisation of the $\ll$ relation for continuous lattices.

The free model is equivalent to the category of computably based locally compact locales and computable continuous functions.

Overt discrete objects

Recall: discrete spaces have equality ($=$),
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These play the role of sets.
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This is a miracle.
None of the usual structure of categorical logic
was assumed in order to make it happen.

Lists and finite subsets

On any overt discrete object X, there exist

- the free semilattice $\mathsf{K}X$ or "set of Kuratowski-finite subsets" and
- the free monoid $\mathsf{List}X$ or "set of lists".

So $\mathcal{E}$ (the full subcategory of overt discrete objects) is an Arithmetic Universe.

Kuratowski-finite = overt, discrete and compact.

Finite = overt, discrete, compact and Hausdorff.

Models of the monadic axiom

It is easy to find models of the monadic axiom.

If $\mathcal{S}_0$ has $\mathbf{1}$, $\times$ and $\Sigma^{(-)}$, then $\mathcal{S} \equiv \mathcal{A}^{\mathsf{op}}$ also has them, *and* the monadic property, where $\mathcal{A}$ is the category of Eilenberg–Moore algebras for the monad on $\mathcal{S}$.

It also inherits

- the other basic structure ($\top$, $\bot$, $\wedge$, $\vee$ and the Euclidean or Phoa axioms),
- $\mathbb{N}$ (with recursion and description),
- the Scott principle.

However, it need not inherit other structure such as being cartesian closed or (a reflective subcategory of) a topos.

We call $\mathcal{S}$ the monadic completion of $\mathcal{S}_0$ and write $\overline{\mathcal{S}_0}$ for it.

Escaping from local compactness

Most of the ideas that you try take you back in again!

Escaping from local compactness

The extended calculus should include

- all finite limits (in particular equalisers),
- something to control the relationship between equalisers and exponentials ($\Sigma^{(-)}$).

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Less ambitiously, we look for axioms that ensure that $\mathcal{S}$ includes the category $\mathbf{Loc}(\mathcal{E})$ of locales, or at least the category $\mathbf{Sob}(\mathcal{E})$ of sober spaces or spatial locales.

An interim model

Dana Scott's category **Equ** of equilogical spaces

- has the basic structure, $\mathbb{N}$ and the Scott principle,
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- satisfies the underlying set axiom (to follow).

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This is not the definitive model.
We just use it to guarantee consistency of the proposed axioms.

The Underlying Set Axiom

Recall that the underlying set functor U from the classical category **Sp** of (not necessarily T_0) spaces has adjoints

$$\begin{array}{c} \mathbf{Sp} \\ \text{discrete} \equiv \Delta \uparrow \dashv \mathsf{U} \downarrow \dashv \uparrow \text{indiscriminate} \\ \mathbf{Set} \end{array}$$

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Again, there's a corresponding type theory:

$$\frac{a : X}{\overline{\tau.\, a : \mathsf{U}X}} \qquad a = \varepsilon(\tau.\, a)$$

so long as the free variables of a are all of overt discrete type.

Overt discrete objects form a topos

Lemma: Any mono $X \to D$ from an overt object to a discrete one is an open inclusion, and therefore classified by Σ.

Theorem:

- The underlying set axiom $\Delta \dashv \mathsf{U}$ holds
- iff $\mathcal{S}$ is enriched over $\mathcal{E}$, where

$$\mathcal{S}(X, Y) \rightarrowtail \mathsf{U}\Sigma^{\Sigma^Y \times X} \rightrightarrows \mathsf{U}\Sigma^{\Sigma^3 Y \times X}$$

  is an equaliser in $\mathcal{E}$,
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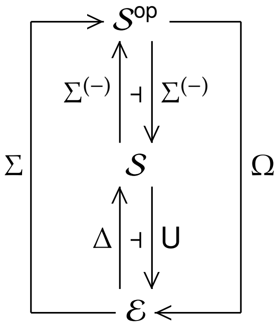
Now we can compare our category $\mathcal{S}$ with $\mathbf{Loc}(\mathcal{E})$ and $\mathbf{Sob}(\mathcal{E})$.

Comparing the monads

We have a composite of adjunctions over the topos $\mathcal{E}$:



The monad $\Omega \cdot \Sigma$ on $\mathcal{E}$
is (isomorphic to) that for frames
iff the general Scott principle holds,

$$\Phi\xi \iff \exists \ell : \mathsf{K}(N).\ \Phi(\lambda n.\, n \in \ell) \wedge \forall n \in \ell.\ \xi n,$$

where N is any object of the topos $\mathcal{E}$, not necessarily countable,
$\xi : \Sigma^N$ and $\Phi : \Sigma^{\Sigma^N}$.

Comparing $\mathcal{S}$ with $\mathbf{Loc}(\mathcal{E})$

Assuming the general Scott principle as an axiom,

$\mathbf{Loc}(\mathcal{E})$ is the opposite of the category of Eilenberg–Moore algebras for the monad $\Omega \cdot \Sigma$ on $\mathcal{E}$.

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$\mathcal{S}$ is too big — the functor is not full or faithful.

Comparing $\mathcal{S}$ with $\mathbf{Loc}(\mathcal{E})$

Consider the full subcategory $\mathcal{L} \subset \mathcal{S}$
of objects X that are expressible as equalisers

$$X \rightarrowtail \Sigma^N \rightrightarrows \Sigma^M$$

where $N, M \in \mathcal{E}$.

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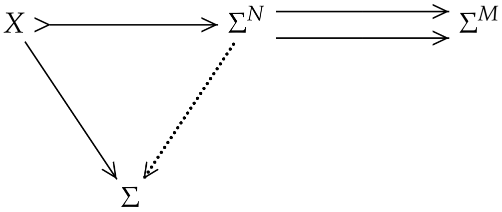

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Warning: It cannot be injective with respect to all regular monos in whole of $\mathcal{S}$.

Example: $\Sigma^{\mathbb{N}^\mathbb{N}} \times \mathbb{N}^\mathbb{N} \rightarrowtail \Sigma^{\mathbb{N}^\mathbb{N}} \times \mathbb{N}_\perp^\mathbb{N}$.

Characterising sober spaces and locales

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Indeed $\mathcal{L} \cap \mathcal{P} \simeq \mathbf{Sob}(\mathcal{E})$,
where $\mathcal{P} \subset \mathcal{S}$ is the full subcategory of spaces X
with enough points, *i.e.* $\varepsilon : \mathsf{U}X \twoheadrightarrow X$.

Recall that $\mathcal{S} \equiv \overline{\mathbf{Equ}}$ provides a model of these assumptions over any elementary topos $\mathcal{E}$.

Corollary: We have a complete axiomatisation of $\mathbf{Sob}(\mathcal{E})$ over an elementary topos $\mathcal{E}$.

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Using a stronger injectivity axiom we would be able to force $\mathcal{L} \equiv \mathbf{Loc}(\mathcal{E})$ and so completely axiomatise locales

if we had a model or other proof of consistency.

The extended computable theory

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What is the extended form of the monadic axiom
that axiomatised computably based locally compact locales?

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Nevertheless, there is plenty to do to develop the interim theory.