# Computable Real Analysis without Set Theory or Turing Machines


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# Russian Recursive Analysis

The recursive real number $a : \mathbb{R}$ is one for which
there is a program that, given $k : \mathbb{N}$ as input,
yields $p, q : \mathbb{Q}$ with $p < a < q$ and $q - p < 2^{-k}$.

The recursive real line is the set of all such $a : \mathbb{R}$.

Using some standard recursion theory...
there exists a singular cover of $\mathbb{R}$, *i.e.* a recursively enumerable sequence of intervals $(p_n, q_n) \subset \mathbb{R}$ with $p_n < q_n : \mathbb{Q}$ such that

- each recursive real number $a$ lies in some interval $(p_n, q_n)$,
- but $\sum_n q_n - p_n < 1$.

There is no finite subcover of $\mathbb{I} \equiv [0, 1]$.

Measure theory also goes badly wrong.

# One solution: Weihrauch's Type Two Effectivity

Consider all real numbers.
Represent them (for example) by signed binary expansions

$$a = \sum_{k=-\infty}^{+\infty} d_k \cdot 2^{-k} \qquad \text{with} \quad d_k \in \{+1, 0, -1\}.$$

Think of $\{\ldots, 0, 0, 0, \ldots, d_{-2}, d_{-1}, , d_0, d_1, d_2, \ldots\}$ as a Turing tape with finitely many nonzero digits to the left, but possibly *infinitely* many to the right.

Do real analysis in the usual way.

Do computation with the sequences of digits.

Klaus Weihrauch, *Computable Analysis*, Springer, 2000.
Vasco Brattka, Peter Hertling, Martin Ziegler, ...

# Another solution: Bishop's Constructive Analysis

Live without the Heine–Borel theorem.

Compact = closed and totally bounded.

($X$ is *totally bounded* if, for any $\epsilon > 0$, there's a finite set $S_\epsilon \subset X$ such that for any $x \in X$ there's $s \in S_\epsilon$ with $d(x, s) < \epsilon$.)

Errett Bishop, *Foundations of Constructive Analysis*, 1967

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He developed remarkably much of analysis in a "can do" way, without dwelling on counterexamples that arise from wrong classical definitions.

Consistent with both Russian Recursive Analysis and Classical Analysis. Uses Intuitionistic Logic (Brouwer, Heyting).

Douglas Bridges, Hajime Ishihara, Mark Mandelkern, Ray Mines, Fred Richman, Peter Schuster, ...

No explicit computation, but
the issues that Constructive Analysis raises
are often the same ones that Numerical Analysts experience.

# Disadvantages of these methods

Point-set topology and recursion theory separately are complicated subjects that lack conceptual structure.

Together, they give pathological results.

Intuitionism makes things even worse —
the natural relationship between open and closed subspaces is replaced by double negation.

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Category theory can do better than this!

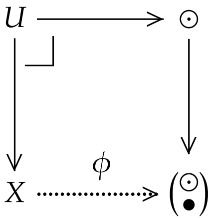
# Some topology — the Sierpiński space

Classically, it's just $\binom{\odot}{\bullet}$.

For every open subspace $U \subset X$

there's a unique continuous function $\phi : X \to \binom{\odot}{\bullet}$

for which $U = \phi^{-1}(\odot)$.

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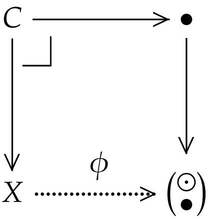


This is a bijective correspondence too.

# The Sierpiński space

For every open subspace $C \subset X$
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for which $U = \phi^{-1}(\top)$

For every closed subspace $C \subset X$
there's a unique continuous function $\phi : X \to \Sigma$
for which $C = \phi^{-1}(\bot)$.

There is a three-way correspondence.

It's not set-theoretic complementation.

It doesn't involve double negation or excluded middle.

It's topology, not set theory.

# Relative containment of open subspaces

Let $\sigma, \alpha, \beta$ be propositions (terms of type $\Sigma$) with parameters $x_1 : X_1, ..., x_k : X_k$.

They define open subspaces of $\Gamma$.

The correspondence is supposed to be bijective.

So they should satisfy a Gentzen-style rule of inference:

$$\frac{\Gamma,\ \sigma \Leftrightarrow \top \ \vdash\ \alpha \ \Rightarrow\ \beta}{\Gamma\ \vdash \sigma \wedge \alpha\ \Rightarrow\ \beta}$$

in which the top line means

*within the open subspace of $\Gamma$ defined by $\sigma$,*
*the open subspace defined by $\alpha$*
*is contained in the open subspace defined by $\beta$.*

and the bottom line means

*the intersection of the open subspaces defined by $\sigma$ and $\alpha$*
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# Relative containment of closed subspaces

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$$\frac{\Gamma,\ \sigma \Leftrightarrow \bot \ \vdash\ \alpha \ \Rightarrow\ \beta}{\Gamma\ \vdash \alpha\ \Rightarrow\ \sigma \vee \beta}$$

in which the top line means

> *within the closed subspace of $\Gamma$ defined by $\sigma$,*
> *the closed subspace defined by $\alpha$*
> *contains the closed subspace defined by $\beta$.*

and the bottom line means

> *the intersection of the closed subspaces defined by $\sigma$ and $\beta$*
> *is contained in that defined by $\alpha$.*

# The Euclidean & Phoa Principles

The Gentzen-style rule for open subspaces,

$$\frac{\Gamma,\ \sigma \Leftrightarrow \top\ \vdash\ \alpha\ \Rightarrow\ \beta}{\Gamma\ \vdash \sigma \wedge \alpha\ \Rightarrow\ \beta}$$

with $\alpha \equiv F\top$ and $\beta \equiv \sigma \wedge F\sigma$ gives the Euclidean principle

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$$\sigma \wedge F\top \iff \sigma \wedge F\sigma.$$

Combining this with monotonicity, $\alpha \Rightarrow \beta\ \vdash\ F\alpha \Rightarrow F\beta$,
and the Gentzen-style rule for closed subspaces,
we obtain the Phoa principle,

$$F\sigma \iff F\bot \vee \sigma \wedge F\top.$$

Paul Taylor, *Geometric and Higher Order Logic*, 2000.

# The topology as a function space

The topology on $X$ is the set of functions $X \to \Sigma \equiv \binom{\odot}{\bullet}$.

Function spaces $X \to Y$ have a (compact–open) topology too.

But it's only well behaved when $X$ is locally compact.

Ralph Fox, *Topologies on function spaces*, 1945.

To prove this, the critical case is $Y \equiv \Sigma$.

Then $X \to \Sigma$ carries the Scott topology.

Dana Scott, *Continuous Lattices*, 1972.

# Compactness and Scott continuity

A function $F : L_1 \to L_2$ between complete lattices
is Scott continuous iff it preserves directed joins.

For example, let $K \subset X$ be any subspace
and $F : (X \to \Sigma) \to \Sigma$ the function for which
$F(U) = \top$ if $K \subset U$ and $\bot$ otherwise.

Then $F$ is Scott continuous iff $K$ is compact.

(This is just the "finite open subcover" definition in another form.)

Popularised by Martín Escardó, *Synthetic Topology*, 2004.

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In set theory $\exists$ satisfies the Frobenius law,

$$\exists x.\, \sigma \wedge \phi(x) \iff \sigma \wedge \exists x.\, \phi x$$

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In set theory and topology $\exists$ satisfies the Frobenius law,

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In topology $\forall$ also satisfies the dual Frobenius law

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The Frobenius law for $\exists$ is a special case of the Euclidean principle, with

$$F\sigma \equiv \exists x.\, \sigma \wedge \phi x.$$

# This is still not enough to axiomatise topology

Consider the category **Dcpo** of posets with directed joins.
It has all limits, colimits and function-spaces.

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It has all limits, colimits and function-spaces.

The Dedekind and Cauchy reals can be defined.

They carry the discrete order, and the discrete topology.

$\mathbb{I} \equiv [0,1]$ and Cantor space are not compact.

This is just as bad as Russian Recursive Analysis.

# Stone Duality and Locales

Marshall Stone, 1934: topology is dual to algebra.

The topology on $X$ is an algebraic structure (finite meets and infinitary joins).

Continuous functions $X \to Y$ correspond bijectively to homomorphisms from topology on $Y$ to topology on $X$.

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Locale theory redefines topology as algebra.

Peter Johnstone, *Stone Spaces*, CUP, 1983.

Eliminates many of the uses of the Axiom of Choice that plague point-set topology.

Can be defined for sheaves,
and satisfies the Heine–Borel theorem.

# Abstract Stone Duality

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"What if" the algebras (topologies) are spaces too?

In category theory we may define algebras over any category we please, using a monad.

The category of topologies is $\mathcal{S}^{\mathsf{op}}$, the dual of the category $\mathcal{S}$ of "spaces".

It's also a category of algebras for a monad on $\mathcal{S}$.

$$\begin{array}{c} \mathcal{S}^{\mathsf{op}} \\ \Sigma^{(-)} \uparrow \dashv \downarrow \Sigma^{(-)} \\ \mathcal{S} \end{array}$$

Paul Taylor, 1993.

Inspired by Robert Paré, *Colimits in topoi*, 1974.

# Some generalised abstract nonsense

Jon Beck (1966) characterised monadic adjunctions:

- $\Sigma^{(-)} : \mathcal{S}^{\mathsf{op}} \to \mathcal{S}$ reflects invertibility,
  *i.e.* if $\Sigma^f : \Sigma^Y \cong \Sigma^X$ then $f : X \cong Y$, and
- $\Sigma^{(-)} : \mathcal{S}^{\mathsf{op}} \to \mathcal{S}$ creates $\Sigma^{(-)}$-split coequalisers.

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Category theory is a strong drug —
it must be taken in small doses.

As in homeopathy (?),
it gets more effective the more we dilute it!

# Diluting Beck's theorem (first part)

If $\Sigma^f : \Sigma^Y \cong \Sigma^X$ then $f : X \cong Y$.

$X$ is the equaliser of

$$X \rightarrowtail^{\eta_X} \Sigma^2 X \equiv \Sigma^{\Sigma^X} \underset{\Sigma^2 \eta_X}{\overset{\eta_{\Sigma^2 X}}{\rightrightarrows}} \Sigma^4 X$$

where $\eta_X : x \mapsto \lambda\phi.\, \phi x$.

# Diluting Beck's theorem (first part)

There's an equivalent type theory for general spaces $X$.
$P : \Sigma^{\Sigma^X}$ is prime if

$$\Gamma, \mathcal{F} : \Sigma^3 X \vdash \mathcal{F} P = P\Big(\lambda x. \mathcal{F}(\lambda \phi. \phi x)\Big).$$

This says that the composites

$$\Gamma \xrightarrow{P} \Sigma^{\Sigma^X} \rightrightarrows \Sigma^4 X$$

are equal.

So we should have a map $\Gamma \to X$.

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$$\frac{\Gamma \vdash P : \Sigma^{\Sigma^X} \qquad P \text{ is prime}}{\Gamma \vdash \mathsf{focus}\, P : X} \quad \mathsf{focus}\, I$$

$$\frac{\Gamma \vdash P : \Sigma^{\Sigma^X} \qquad P \text{ is prime}}{\Gamma,\ \phi : \Sigma^X \vdash \phi(\mathsf{focus}\, P) = P\phi : \Sigma} \quad \mathsf{focus}\, \beta$$

$$\frac{\Gamma \vdash a, b : X \qquad \Gamma,\ \phi : \Sigma^X \vdash \phi a = \phi b}{\Gamma \vdash a = b} \quad \mathrm{T}_0$$

The definition $\mathsf{thunk}\, a = \eta_X(a) = \lambda\phi.\, \phi a$ serves as the elimination rule for $\mathsf{focus}$. Using this, equivalent ways of writing the $\mathsf{focus}\, \beta$ and $\eta$ ($\mathrm{T}_0$) rules are

$$\mathsf{thunk}\,(\mathsf{focus}\, P) = P \qquad \text{and} \qquad \mathsf{focus}\,(\mathsf{thunk}\, x) = x,$$

where $P$ is prime.

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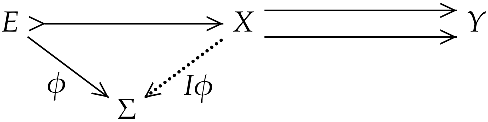
# Diluting Beck's theorem (second part)

$\Sigma^{(-)} : \mathcal{S}^{\mathsf{op}} \to \mathcal{S}$ creates $\Sigma^{(-)}$-split coequalisers.

This means that (*certain*) subspaces exist,
and they have the subspace topology.




Every open subspace of $E$ is the restriction of one of $X$,
in a canonical way.

There's a corresponding type theory.

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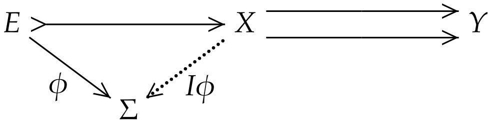


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It can be used to develop an abstract, finitary axiomatisation of the "way below" relation for continuous lattices.

Paul Taylor, *Computably based locally compact spaces*, 2006.

# Diluting Beck's theorem (second part) even further

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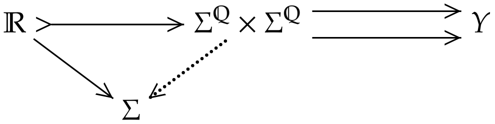
In particular, the Dedekind reals can be expressed as an equaliser




where, classically, the map $I$ takes
an open subspace $O \subset \mathbb{R}$ to the open subspace

$$\{(D,U) \in \Sigma^{\mathbb{Q}} \times \Sigma^{\mathbb{Q}} \mid \exists d,u : \mathbb{Q}.\ d \in D \wedge u \in U \wedge [d,u] \subset O\}.$$

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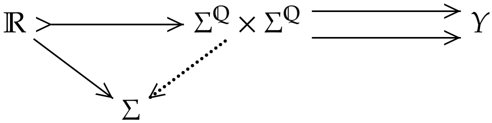


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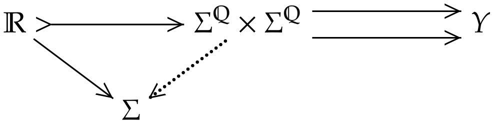


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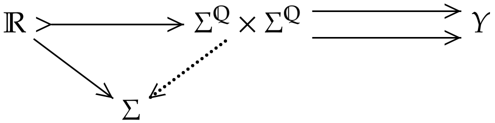


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$\mathbb{R}$ can be defined abstractly from this.

It satisfies the Heine–Borel theorem!

# Natural axioms for the reals

The arithmetic operations: $0, 1, +, -, \times, \div$.
The arithmetic relations: $<, >, \neq$.
(Not $\leq, \geq$ and $=$ since they're not
topologically open or computationally observable)

Geometric logic: $\top, \bot, \wedge, \vee, \exists_{\mathbb{N}}, \exists_{\mathbb{R}}$.
(Not $\neg$ or $\Rightarrow$ since they would solve the halting problem.)

Primitive recursion over $\mathbb{N}$.

Dedekind completeness and the Heine–Borel theorem.

Universal quantification ($\forall$) is defined over compact subspaces.

Andrej Bauer and Paul Taylor, *The Dedekind Reals in ASD*, 2005.