In defence of Dedekind and Heine–Borel

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www.cs.man.ac.uk/~pt/ASD

Abstract

As one who has been doing analysis for only two years, I hesitate to offer an axiomatisation of something so venerable as the real line.
But at a time when a number of disciplines that are constructive, computable, both or neither are at last talking to one another, we badly need such a definition so that we can agree on what we're talking about.
Let me say in my own defence that my axioms are at least headline properties in traditional analysis:
the only unfamiliar statement is that the line is overt, but there the constroversial thing would be to say otherwise.
The problem is that some of my constructive allies disagree with some of the traditional properties.

Abstract

Formalists support Cauchy and Cantor against Dedekind because they like numbers and sequences but not sets. Yet familiar examples such as Riemann integration give cuts naturally but sequences artificially. I shall show that Dedekind completeness and definition by description can naturally be expressed as λ-calculi.

Bishop abandoned the Heine–Borel theorem because it fails in recursive analysis, but mathematics seems to be very strange without it.

In Abstract Stone Duality this theorem is more or less an axiom. However, this axiom has a rich background, combining categorical algebra with the fundamental theorem of interval analysis.

Abstract

So what is the *real* real line? Can we devise a experiment to justify the axioms?
Such a test is whether open subsets look like we expect them to look. Traditionally, any open subset of the real line is a countable union of disjoint open intervals. Can I prove this in ASD? Can Bishop prove it?
I shall show how a polished version of the modal notation that I introduced at CCA in Kyoto in 2005 can be used to give *two* definitions of connectedness, each of them linked to an approximate intermediate value theorem. This will be applied to the classification of open subspaces and of connected ones, and I shall conclude with some examples and counterexamples.

Disclaimer

The presentation of this lecture as a "court case" with "witnesses" must be understood light-heartedly. In particular, you must not assume that the "evidence" attributed to the "witnesses" actually represents their views.

Generally speaking, the attibutions are to be understood in the usual academic way, albeit highly abbreviated since this was a lecture and not a paper.

In some cases the connection between the person and the ideas is quite tenuous. In particular, the syntax of a language for the fragment of ASD for $\mathbb{R}$ is linked to John Cleary's *Logical Arithmetic via* my work on *Interval Analysis without Intervals*, which is still in progress.

Axioms for the real line

The axioms that I propose are all headline properties in traditional analysis, apart from overtness, but there the controversial thing would be to say otherwise.

$\mathbb{R}$ is

- overt, with $\exists$;
- Hausdorff, with $\neq$;
- totally ordered, *i.e.* $(x \neq y) \Leftrightarrow (x < y) \vee (y < x)$;
- a field, where x^{-1} is defined iff $x \neq 0$;
- Dedekind complete; and
- Archimedean;
- and the closed interval is compact, with $\forall$.

However, some of my constructive allies disagree with some of the traditional properties.

La legge è uguale per tutti

The case for Cauchy against Dedekind.

A practical one:
you want to see his *figures*!

A formalist one:

- he uses general subsets or predicates;
- he's impredicative.

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A formalist one:

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The practical defence:
Familiar examples such as Riemann integration give cuts naturally but sequences artificially.

The formalist defence:
Dedekind completeness can naturally be expressed as λ-calculi.

The counter-claim:
Cauchy sequences are much more complicated to define.

First witness for the defence: Archimedes

Theorem: The area of a circle (K for $\kappa\upsilon\kappa\lambda o\varsigma$) is equal to that of the right triangle Δ formed from the radius and circumference.

Proof: Compare K and Δ with the areas of the inscribed (I_n) and circumscribed (E_n) regular n-gons.

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Aha! Two Cauchy sequences!

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Proof: Compare K and Δ with the areas of the inscribed (I_n) and circumscribed (E_n) regular n-gons.

No. First we bound the ratios I_n/K and E_n/K.

Suppose that $K > \Delta$.
Then $I_n > \Delta$ for some n, which we show to be impossible.

Similarly if $K < \Delta$ then $E_n < \Delta$ for some n, which is also impossible.

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Similarly if $K < \Delta$ then $E_n < \Delta$ for some n, which is also impossible.

Actually, we show that any upper or lower bound for Δ is also one for K.

Witness: Giuseppe Peano

Il primo Formalista!

We argue the formalist defence of the axioms for $\mathbb{R}$
(in particular Dedekind completeness)
by analogy with those for $\mathbb{N}$:

- $0 : \mathbb{N}$
- $n : \mathbb{N} \vdash n + 1 : \mathbb{N}$
- $0 \neq n + 1$
- $n = m \iff n + 1 = m + 1$
- induction

This defines primitive recursion.

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- $0 \neq n + 1$
- $n = m \iff n + 1 = m + 1$
- induction
- definition by description or unique choice.

This defines general recursion (more or less).

Le Discrizioni secondo Peano

Studii di Logica Matematica, 1897, §22.

..., sia α una classe contenente un solo individuo, cioè:

- esistano degli α, e
- comunque si prendano due individui x ed y di α, essi siano sempre eguali.

Questo individuo lo indicheremo con $\bar{\iota}\alpha$. Sicchè

$$\exists a \quad : \; x, y \in \alpha . \supset_{x,y} . x = y \quad :\supset: \quad x = \bar{\iota}\alpha \; . = . \; \alpha = \iota x \quad \text{Def.}$$

Veramente questa definizione dà il significato
di tutta la formula $x = \bar{\iota}\alpha$, e non del solo gruppo $\bar{\iota}\alpha$.
Ma ogni proposizione contenente $\bar{\iota}\alpha$
è riduttibile alla forma $\bar{\iota}\alpha \in \phi$, ove ϕ è una classe;
e questa ad $\alpha \supset \phi$, ove è scomparso il segno $\bar{\iota}$;
quantunque non ci riesca formare un'eguaglianza il cui primo membro sia $\bar{\iota}\alpha$, ed il secondo un gruppo di segni noti.

Descriptions according to Peano

Studies in Mathematical Logic, 1897, §22.

..., let α be a class containing a single member, that is:

- there is an α, and
- whenever we take two things x and y from α, these must always be equal.

We call this member $\bar{\iota}\alpha$. That is

$$(\exists x.\ x \in \alpha),\ (\forall xy.\ x, y \in \alpha \Rightarrow x = y) \quad \vdash \quad (x = \bar{\iota}\alpha) \iff (\alpha = \{x\}).$$

This definition really gives a meaning to the whole formula $x = \bar{\iota}\alpha$, and not just to the combination $\bar{\iota}\alpha$.
Any *proposition* containing $\bar{\iota}\alpha$ is reducible to the form $\bar{\iota}\alpha \in \phi$, where ϕ is a class, and hence to $\alpha \Rightarrow \phi$, from which the sign $\bar{\iota}$ has disappeared,
even though we can't form an equality whose first member is $\bar{\iota}\alpha$ and the second is a group of known symbols
[*i.e. define* $\bar{\iota}\alpha$ in terms of known symbols].

A lambda-calculus for Descriptions

Given any predicate α
for which the axioms of a description are provable,
we may introduce its witness:

$$\dfrac{\begin{matrix}[n:\mathbb{N}] \\ \vdots \\ \alpha n : \Sigma\end{matrix} \qquad (\exists n.\ \alpha n) \qquad \begin{matrix}[\alpha n, \quad \alpha m] \\ \vdots \\ n = m : \mathbb{N}\end{matrix}}{(\bar{\imath} n.\ \alpha n) : \mathbb{N}}$$

A lambda-calculus for Descriptions

The elimination rules recover the axioms.

The β-rule says that $(\bar{\iota}n.\, \alpha n)$ has the property that α specifies:

$$(\bar{\iota}n.\, \alpha n) = m \qquad \Longleftrightarrow \qquad \alpha m.$$

As in the λ-calculus, this simply substitutes part of the context for the bound variables.

The η-rule says that any number m defines a Dedekind cut in the obvious way:

$$\alpha n \;\equiv\; (n = m).$$

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There is a normalisation theorem by which, as Peano says, ogni proposizione ... è riduttibile alla forma ... $\alpha \supset \phi$, ove è scomparso il segno $\bar{\iota}$, although I prefer $\exists x.\, \alpha x \wedge \phi x$.

A lambda-calculus for Dedekind cuts

Our formulation of Dedekind cuts does not use set theory, or type-theoretic predicates of arbitrary logical strength.

Given any pair $[\delta, \upsilon]$ of predicates for which the axioms of a Dedekind cut are provable,
we may introduce a real number:

$$\dfrac{\begin{array}{c}[d : \mathbb{R}] \\ \vdots \\ \delta d : \Sigma\end{array} \qquad \begin{array}{c}[u : \mathbb{R}] \\ \vdots \\ \upsilon u : \Sigma\end{array} \qquad \text{axioms for Dedekind cut}}{(\mathsf{cut}\, du.\, \delta d \wedge \upsilon u) : \mathbb{R}}$$

A λ-calculus for Dedekind cuts

The elimination rules recover the axioms.

The β-rule says that $(\mathsf{cut}\, du.\, \delta d \wedge \upsilon u)$ obeys the order relations that δ and υ specify:

$$e < (\mathsf{cut}\, du.\, \delta d \wedge \upsilon u) < t \qquad \Longleftrightarrow \qquad \delta e \wedge \upsilon t.$$

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$$\delta d \equiv (d < a), \qquad \text{and} \qquad \upsilon u \equiv (a < u).$$

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$$\delta d \;\equiv\; (d < a), \qquad \text{and} \qquad \upsilon u \;\equiv\; (a < u).$$

There is a normalisation theorem whereby
this syntax for individual real numbers
can be translated into interval computation.

Witness: John Cleary

		$\mathbb{N}$	$\mathbb{R}$	$\mathbb{N}\&\Sigma$	$\mathbb{R}\&\Sigma$	$\mathbb{N}\&?$	Σ
$\mathbb{N}$	0	succ				rec	the
$\mathbb{R}$	$0, 1$	n	$+, -, \times, \div$			rec	cut
Σ	$\top, \bot$	$=, \leq, \geq$ $<, >, \neq$	$<, >, \neq$	$\exists n$	$\exists x : \mathbb{R}$ $\forall x : [a, b]$	rec	$\wedge, \vee$

This syntax can be manipulated using constraint logic programming.

Summary of the formalist defence: precedent

set theory:	$\{- \mid -\}$	membership
λ-calculus:	λ	application
descriptions:	$\bar{\iota}$	equality
Dedekind cuts:	cut	order

Witness: Marshall Stone

A term $P : \Sigma^{\Sigma^X}$ or $P : (X \to \Sigma) \to \Sigma$ is prime if

$$P\top \Leftrightarrow \top \qquad P(\phi \wedge \psi) \Leftrightarrow P\phi \wedge P\psi$$

$$P\bot \Leftrightarrow \bot \qquad P(\phi \vee \psi) \Leftrightarrow P\phi \vee P\psi$$

(This idea was in Aleš Pultr's first lecture on Monday.)

The space X is sober if it has introduction and β-rules

$$\frac{P : \Sigma^{\Sigma^X} \quad \text{prime}}{(\mathsf{focus}\, P) : X} \qquad \frac{P : \Sigma^{\Sigma^X} \quad \text{prime} \quad \phi : \Sigma^X}{\phi(\mathsf{focus}\, P) \Leftrightarrow P\phi}$$

where elimination is application and the η-rule is

$$P \;\equiv\; \mathsf{thunk}\, a \;\equiv\; \eta_X a \;\equiv\; \lambda\phi.\, \phi a.$$

(thunk and force are used in extensions of functional programming languages that allow *computational effects* such as **goto**.)

Descriptions as primes

If $\alpha : \Sigma^{\mathbb{N}}$ is a description then

$$P \;\equiv\; \lambda\phi.\, \exists x.\, \alpha x \wedge \phi x$$

is prime.

If $P : \Sigma^{\Sigma^{\mathbb{N}}}$ is prime then

$$\alpha \;\equiv\; \lambda x.\, P(\lambda y.\, x = y)$$

is a description.

If one satisfies the relevant rules then so does the other.

Dedekind cuts as primes

If (δ, υ) is a Dedekind cut then

$$P \;\equiv\; \lambda\phi.\, \exists du.\, \delta d \wedge (\forall x : [d, u].\, \phi x) \wedge \upsilon u$$

is prime (relying on the co-defendants, Heine–Borel).

If $P : \Sigma^{\Sigma^{\mathbb{R}}}$ is prime then

$$\delta \;\equiv\; \lambda d.\, P(\lambda x.\, d < x) \qquad \upsilon \;\equiv\; \lambda u.\, P(\lambda x.\, x < u)$$

is a Dedekind cut.

If one satisfies the relevant rules then so does the other.

Witness: Peter Schuster

Let $f : [0,1] \to [0,1]$ be continuous. Suppose that

- $\inf \{fx \mid x : [0,1]\} = 0$, and
- $x \neq y \Rightarrow (fx > 0) \vee (fy > 0)$.

Then $fx = 0$ for some (unique) x.

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Then $fx = 0$ for some (unique) x.

Then $\omega \equiv \lambda x.\,(fx \neq 0)$ is a codescription:

$$\begin{array}{lll} \big(\forall x : [0,1].\ \omega x\big) \Leftrightarrow \bot & \textit{cf.} & \big(\exists n : \mathbb{N}.\ \alpha n\big) \Leftrightarrow \top \\ x \neq y \Rightarrow \omega x \vee \omega y & \textit{cf.} & n = m \Leftarrow \alpha n \wedge \alpha m \end{array}$$

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Also

- $P \equiv \lambda \phi.\, \forall x : [0,1].\ \omega x \vee \phi x$ is prime;
- *cf.* $P \equiv \lambda \phi.\, \exists n : \mathbb{N}.\ \alpha n \vee \phi n$;
- $\delta \equiv \lambda d.\, \forall x : [0,d].\ \omega x$ and $\upsilon \equiv \lambda u.\, \forall x : [u,1].\ \omega x$ define a Dedekind cut.

Witnesses: Jon Beck and Joachim Lambek

A space X is sober if
every homomorphism $\Sigma^X \to \Sigma^\Gamma$
is Σ^f for some unique function $f : \Gamma \to X$.

A space X is sober iff the diagram

$$X \xrightarrow{x \mapsto \lambda\phi.\, \phi x} \Sigma^{\Sigma^X} \underset{F \mapsto \lambda\Phi.\, F(\lambda x.\, \Phi(\lambda\phi.\, \phi x))}{\overset{F \mapsto \lambda\Phi.\, \Phi(\lambda\phi.\, F\phi)}{\rightrightarrows}} \Sigma^{\Sigma^{\Sigma^{\Sigma^X}}}$$

is an equaliser.

Witnesses: Jon Beck and Robert Paré

Every homomorphism $\Sigma^X \to \Sigma^\Gamma$ is Σ^f for some unique function $f : \Gamma \to X$.

Every algebra is Σ^X for some unique space X.

Lindenbaum–Tarksi–Paré: the category of sets or any elementary topos has this property.

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The court will adjourn for eight years 1997-2005,

while I prepare the formalist defence of Heine–Borel:

There is an algebra that

- has Dedekind cuts as its points; and
- obeys Heine–Borel: $[0, 1] \subset \mathbb{R}$ is compact.

The topology on $\mathbb{R}$ as an algebra

The topology, $\Sigma^{\mathbb{R}}$, on $\mathbb{R}$ is a retract
of the topology on the space $\Sigma^{\mathbb{Q}} \times \Sigma^{\mathbb{Q}}$ of Dedekind cuts:

$$\Sigma^{\mathbb{R}} \underset{\Sigma^i}{\overset{I}{\rightleftarrows}} \Sigma^{\Sigma^{\mathbb{Q}} \times \Sigma^{\mathbb{Q}}}$$

This says that

$$\mathbb{R} \overset{i}{\rightarrowtail} \Sigma^{\mathbb{Q}} \times \Sigma^{\mathbb{Q}}$$

has the subspace topology in a canonical way.

We shall look at this classically first.

Then we show how to define the retract just using rationals.

Witness: Ramon Moore

In order to use Dedekind cuts for real computation,
we must extend the definitions of the arithmetic operations.

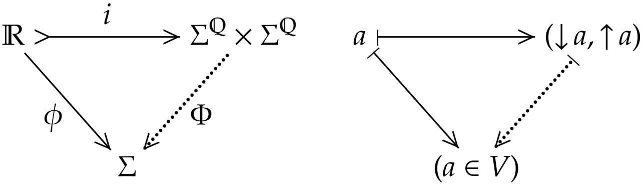
$$
\begin{array}{ccc}
\mathbb{R} \times \mathbb{R} & \xrightarrow{\;i \times i\;} & \Sigma^{\mathbb{Q}} \times \Sigma^{\mathbb{Q}} \times \Sigma^{\mathbb{Q}} \times \Sigma^{\mathbb{Q}} \\
{\scriptstyle +}\downarrow & & \vdots\, {\scriptstyle \oplus} \\
\mathbb{R} & \xrightarrow{\;\;\;\;i\;\;\;\;} & \Sigma^{\mathbb{Q}} \times \Sigma^{\mathbb{Q}}
\end{array}
$$

For the arithmetic operations, this was done classically by
Ramon Moore, Interval Analysis, 1966.

How does this work for open sets?

Extending open subspaces classically

Recall that $\phi : \Sigma^{\mathbb{R}}$ defines an open subspace $V \subset \mathbb{R}$.



We require $(a \in V) \equiv \phi a \iff \Phi(ia) \equiv \Phi(\downarrow a, \uparrow a)$.

So $\mathbb{R}$ has the subspace topology inherited from $\Sigma^{\mathbb{Q}} \times \Sigma^{\mathbb{Q}}$.

$$V \mapsto \{(D, U) \mid \exists d \in D.\, \exists u \in U.\, d < u \wedge ([d, u] \subset V)\}$$

$$\phi \mapsto \lambda \delta \upsilon.\, \exists du.\, \delta d \wedge \upsilon u \wedge d < u \wedge \forall x : [d, u].\, \phi x$$

We can settle this argument rationally

We have defined the idempotent $\mathcal{E} \equiv I \cdot \Sigma^i$ on $\Sigma^{\Sigma^{\mathbb{Q}} \times \Sigma^{\mathbb{Q}}}$ by

$$\begin{aligned}\mathcal{E}\Phi(\delta, \upsilon) \;&\equiv\; I\left(\lambda x.\, \Phi(ix)\right)(\delta, \upsilon) \\ &\Leftrightarrow\; \exists du : \mathbb{R}.\; \delta d \wedge \upsilon u \wedge \forall x : [d, u].\; \Phi(\delta_x, \upsilon_x) : \Sigma^{\Sigma^{\mathbb{Q}} \times \Sigma^{\mathbb{Q}}}.\end{aligned}$$

Since Φ is Scott continuous and $[d, u]$ is compact, this is

$$\exists q_0 < \cdots < q_{2n+1} : \mathbb{Q}. \quad \delta q_1 \wedge \upsilon q_{2n} \;\wedge \bigwedge_{k=0}^{n-1} \Phi(\lambda e.\, e < q_{2k},\; \lambda t.\, q_{2k+3} < t)$$

(See *Dedekind Reals in ASD*.)

This only depends on rational numbers and predicates.

The case for and against Heine–Borel

Let $\mathcal{E}$ be the rationally defined idempotent on $\Sigma^{\Sigma^{\mathbb{Q}}\times\Sigma^{\mathbb{Q}}}$.
This is the same in all foundational situations.

In each situation, let $i : R \rightarrowtail \Sigma^{\mathbb{Q}} \times \Sigma^{\mathbb{Q}}$ be the subspace of Dedekind cuts.

Classically, there is a Scott continuous function
$I : \Sigma^{\mathbb{R}} \rightarrowtail \Sigma^{\Sigma^{\mathbb{Q}}\times\Sigma^{\mathbb{Q}}}$ such that $\Sigma^i \cdot I = \mathsf{id}$ and $I \cdot \Sigma^i = \mathcal{E}$.

In other situations, *e.g.* Russian Recursive Analysis,
I need not exist.

Indeed, it exists iff R is locally compact iff $[0, 1]$ is compact.

The "subspace" is an equaliser that depends on what objects exist in the category.

Witness: Paul Taylor

This argument is useless if it only applies to $\mathbb{R}$ in isolation.

We must construct a new category whose objects are formal Σ-split subspaces $\{X \mid E\} \rightarrowtail X$.

(*cf.* constructing a new field containing a formal root of a polynomial).

The good news:

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The bad news:

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The bad news: all of this takes over 200 journal pages [A,B,G].

Further differences of opinion

There are many human objectives
that are best achieved by co-operation with your alies,
even if they only agree on a few things.

Designing a system of mathematical axioms is not one of them.

We borrow ideas and try to talk comparable languages.

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Formal topology is founded on Martin-Löf type theory.
This has, in particular, $\Longrightarrow$ and Π.

Locale theory is founded on the theory of elementary toposes.
This has, in particular, powersets, $\mathcal{P}(X) = \Omega^X$.

These are both (different) logics of discrete sets,
on top of which topology is defined.

Yet more differences of opinion

Abstract Stone Duality is a logic of pure topology, and of computation.

$\Rightarrow$ is neither continuous nor computable.

In ASD, Σ just has $\wedge$, $\vee$, $\exists_{\mathbb{N}}$ and $\forall_{[0,1]}$.

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Abstract Stone Duality, locale theory and formal topology
all define spaces *via* their algebras of open sets.

In ASD, this algebra is another space,
in locale theory it's a set or object of a topos,
in formal topology it is generated by a Martin-Löf type.

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Abstract Stone Duality, locale theory and formal topology all define spaces *via* their algebras of open sets.

They all prove the Heine–Borel theorem.

Martín Escardó has developed some ideas about topology and computation using a similar logic on Σ.

However, he does not define spaces *via* algebras.

He has different opinions about the Heine–Borel theorem.

Witnesses: André Joyal and Milly Maietti

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The overt discrete objects (those with $\exists$ and $=$) admit

- products $\mathbf{1}$ and $\times$;
- equalisers (sets of solutions of equations);
- stable disjoint unions $\emptyset$ and $+$;
- stable effective quotients of equivalence relations;
- free monoids (sets of lists), with (general) recursion.

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This too depends on the definition of spaces *via* algebras.

Since the logic of ASD is very weak, the proofs are very long.

Witness: Karl Weierstraß

What can we do with this logic for $\mathbb{R}$?

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What can we do with this logic for $\mathbb{R}$?

Theorem: $\mathbb{R}$ is locally compact:

$$\phi x \iff \exists \delta > 0.\ \forall y : [x \pm \delta].\ \phi y$$

Witness: Karl Weierstraß

What can we do with this logic for $\mathbb{R}$?

Theorem: $\mathbb{R}$ is locally compact:

$$\phi x \iff \exists \delta > 0.\ \forall y : [x \pm \delta].\ \phi y$$

Theorem: Every definable function $f : \mathbb{R} \to \mathbb{R}$ is continuous:

$$\epsilon > 0 \implies \exists \delta > 0.\ \forall y : [x \pm \delta].\ \left(\left|fy - fx\right| < \epsilon\right)$$

Proof: Put $\phi_{x,\epsilon} y \equiv \left(\left|fy - fx\right| < \epsilon\right)$, with parameters $x, \epsilon : \mathbb{R}$.

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Theorem: Every function f is uniformly continuous on any compact subspace $K \subset \mathbb{R}$:

$$\epsilon > 0 \Rightarrow \exists \delta > 0.\ \forall x : K.\ \forall y : [x \pm \delta].\ \left(\left|fy - fx\right| < \epsilon\right)$$

Proof: $\exists \delta > 0$ and $\forall x : K$ commute.

Some more challenging elementary analysis

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- connectedness.

Compact subspaces and necessity

The finite open sub-cover definition says that,
for a compact subspace K,
the predicate $K \subset U$ is Scott continuous in U.

Martín Escardó explained this in his lecture on Monday.

We have already written $\forall x : K.\ \phi x$ for $K \subset U$.

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We have already written $\forall x : K.\ \phi x$ for $K \subset U$.

We shall now write $\Box\, \phi$ for the same thing.

It defines the subspace K
(at least in an ambient Hausdorff space).

Properties of compact subspaces

$$\Box \top \Leftrightarrow \top \quad \text{and} \quad \Box(\phi \wedge \psi) \Leftrightarrow \Box \phi \wedge \Box \psi.$$

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Then $\Box$ defines a closed subspace,
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The direct image of $\Box$ under $f : X \to Y$ is also compact

$$\blacksquare \psi \equiv \Box(\phi \cdot f).$$

Overt subspaces and possibility

We wrote $\forall x : K.\ \phi x$ or $\Box\,\phi$ for $K \subset U$ (U covers K).
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Classically, for any set $S \subset X$ of points, write

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Forgetting the set S, we can consider any term $\Diamond : \Sigma^{\Sigma^X}$ that preserves disjunction like this.

We call $\Diamond$ an overt subspace.

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Overtness elsewhere

Open locales, *i.e.* those for which $X \to \mathbf{1}$ is an open map, were introduced by Peter Johnstone, André Joyal, Myles Tierney,...

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But it is computation that makes the need for this idea most apparent.

Why is overtness interesting computationally?

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It abstracts interval halving algorithms:
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Here constructive and numerical analysts are arguing at cross purposes.

There are other (logic programming) methods of finding solutions (members, accumulation points) of $\lozenge$ operators.

Accumulation points of a $\lozenge$ operator

Axiomatically, $\mathbb{N}$ is overt: it has $\exists_{\mathbb{N}}$.

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Suppose that $a : X$ satisfies $(\lambda\phi.\, \phi a) \le \Diamond$.
Let $\phi : \Sigma^X$ be a neighbourhood of a, so $\phi a \Leftrightarrow \top$.
Then

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In other words, some element of the sequence also belongs to ϕ, *i.e.* a is an accumulation point of the sequence.

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Some consequences of overtness of direct images.

Any overt subspace has the same $\Diamond$ operator as its (sequential) closure (if this exists).

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Is overtness the constructive content of this hypothesis?

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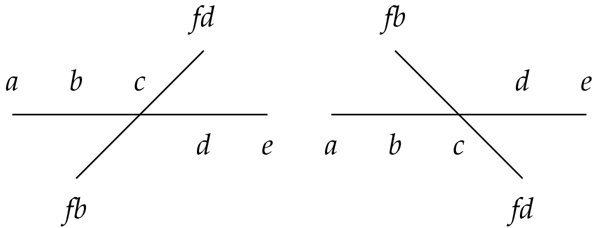
Is overtness the constructive content of this hypothesis?

Do all overt subspaces have dense subsequences?

Stable zeroes

Numerical algorithms find zeroes with this property:



Definition: $c : \mathbb{R}$ is a stable zero of f if

$$a, e : \mathbb{R} \vdash a < c < e \Rightarrow \exists bd. \quad (a < b < c < d < e) \\ \wedge \quad (fb < 0 < fd \ \vee fb > 0 > fd).$$

The subspace $Z \subset [0,1]$ of all zeroes is compact.
The subspace $S \subset [0,1]$ of stable zeroes is overt.

Straddling intervals

An open subspace $U \subset \mathbb{R}$ contains a stable zero $c \in U \cap S$ iff U also contains a straddling interval,

$$[b, d] \subset U \quad \text{with} \quad fb < 0 < fd \quad \text{or} \quad fb > 0 > fd.$$

[⇒] From the definitions. [⇐] The straddling interval is an intermediate value problem in miniature.

Straddling intervals

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[⇒] From the definitions. [⇐] The straddling interval is an intermediate value problem in miniature.

Notation: Write $\lozenge\, U$ if U contains a straddling interval.

$$\begin{aligned} \lozenge\,\phi \;\equiv\; \exists bd. \quad & (\forall x : [b, d].\ \phi x) \\ \wedge \quad & (fb < 0 < fd) \vee (fb > 0 > fd). \end{aligned}$$

Modal operators, separately

$\Box$ encodes the compact subspace $Z \equiv \{x \in \mathbb{I} \mid fx = 0\}$ of all zeroes.
$\Diamond$ encodes the overt subspace S of stable zeroes.

$$\Box\top \Leftrightarrow \top \qquad \Diamond\bot \Leftrightarrow \bot$$

$$\Box(\phi \wedge \psi) \Leftrightarrow \Box\phi \wedge \Box\psi \qquad \Diamond(\phi \vee \psi) \Leftrightarrow \Diamond\phi \vee \Diamond\psi$$

$$(Z \neq \emptyset) \quad \text{iff} \quad \Box\bot \Leftrightarrow \bot$$

$$(S \neq \emptyset) \quad \text{iff} \quad \Diamond\top \Leftrightarrow \top$$

Modal operators, together

In the intermediate value theorem
for functions that don't hover (*e.g.* polynomials):

- $S = Z$ in the non-singular case
- $S \subset Z$ in the singular case (*e.g.* double zeroes).

$\Diamond$ and $\Box$ for the subspaces $S \subset Z$ are related in general by:

$$\Box\phi \wedge \Diamond\psi \ \Rightarrow\ \Diamond(\phi \wedge \psi)$$

(this happens even when there are double zeroes and $S \neq Z$)

$S = Z$ (more precisely, S is dense in Z) iff

$$\Box(\phi \vee \psi) \ \Rightarrow\ \Box\phi \vee \Diamond\psi$$

Modal operators *versus* sets of zeroes

Example: cubic equation $x^3 + 3px + 2q = 0$

As p and q vary, the set of real zeroes goes from 3 to 2 to 1 and back.

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The modal operators $\Box$ and $\Diamond$ are (Scott) continuous throughout the paramater space.

Something must break at singularities: it is one of the mixed modal laws.

Compact overt subspaces

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Theorem: It is decidable whether such a subspace is

- empty, when $\Box \bot \Leftrightarrow \top$, or
- inhabited, when $\Diamond \top \Leftrightarrow \top$.

Proof:

$\Diamond \top \Leftrightarrow \bot$	empty	$\Box \bot \Leftrightarrow \top$
$\Diamond \top \Leftrightarrow \top$	inhabited	$\Box \bot \Leftrightarrow \bot$
$\Box \bot \vee \Diamond \top \Leftarrow$	complementary	$\Box \bot \wedge \Diamond \top \Rightarrow$
$\Box(\bot \vee \top) \Leftrightarrow \Box \top \Leftrightarrow \top$	(mixed)	$\Diamond(\bot \wedge \bot) \Leftrightarrow \Diamond \bot \Leftrightarrow \bot$

Non-empty compact and overt subspaces

An accumulation point $a : X$ of an overt subspace $\lozenge$ satisfies $\lambda\phi.\ \phi a \leq \lozenge$.

Then $\lozenge$ is inhabited, *i.e.* $\lozenge\top \Leftrightarrow \top$.

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Then $\Diamond$ is inhabited, *i.e.* $\Diamond \top \Leftrightarrow \top$.

A point $a : X$ of (the saturation of) a compact subspace $\Box$ satisfies $\lambda\phi.\ \phi a \ge \Box$.

Then $\Box$ is occupied, *i.e.* $\Box \bot \Leftrightarrow \bot$.

Example: any function $f : K \to \mathbb{R}$ on a compact space is bounded and attains its bounds and any given intermediate value on an occupied subspace.

Connectedness

Language and metalanguage

A simple counterexample concerning the intersection of two overt subspaces shows the importance of evidence.

Example: Let $g : \mathbb{R}$ such that neither $\vdash g = 0$ nor $\vdash g \neq 0$.

Let $K \equiv \{0\} \cap \{g\}$ and $U \equiv \mathbb{R} \setminus K$.
U is the open, overt subspace and K the closed, compact one defined by $\delta \vee \upsilon : \mathbb{R} \to \Sigma$, where

$$\delta d \equiv (d < 0 \vee d < g \vee g \neq 0) \quad \text{and} \quad \upsilon u \equiv (0 < u \vee g < u \vee g \neq 0)$$

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If $g = 0$ then $K = \{0\}$, which is compact and overt.

If $g \neq 0$ then $K = \emptyset$, which is compact and overt.

Is K compact? Yes: $\Box \phi \equiv \phi 0 \vee g \neq 0$, so $\Box \bot \Leftrightarrow (g \neq 0)$.

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No: the observation $\Diamond \top$ would allow us to detect $g = 0$.

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Compare this piece of classical hubris:
Are there irrational numbers a, b with a^b rational?
Yes: if $(\sqrt{2})^{(\sqrt{2})}$ is rational then let $a \equiv b \equiv \sqrt{2}$.
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However, U is not constructively connected.

Constructive (overt) connectedness

An overt subspace $I \subset X$ defined by $\Diamond : \Sigma^{\Sigma^X}$ is connected if

$$\Diamond \top \;\Leftrightarrow\; \top \quad \text{and} \quad \phi \vee \psi = \top_I \;\vdash\; \Diamond \phi \wedge \Diamond \psi \;\Rightarrow\; \Diamond(\phi \wedge \psi),$$

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where $\phi, \psi : \Sigma^X$, so whenever $I \subset U \cup V$ is covered by open inhabited subspaces, their intersection is inhabited.

Proposition: Any function $f : I \to \mathbb{R}$ that takes values both above $-\epsilon$ and below $+\epsilon$ also takes values within ϵ of zero:

$$\exists xz : I.\ (-\epsilon < fx) \wedge (fz < +\epsilon) \;\Rightarrow\; \exists y : I.\ (-\epsilon < fy < +\epsilon),$$

so the open, overt subspace $\{x : X \mid |fx| < \epsilon\}$ is inhabited.

Constructive (overt) connectedness

An overt subspace $I \subset X$ defined by $\Diamond : \Sigma^{\Sigma^X}$ is connected if

$$\Diamond\top \iff \top \quad \text{and} \quad \phi \vee \psi = \top_I \vdash \Diamond\phi \wedge \Diamond\psi \Rightarrow \Diamond(\phi \wedge \psi),$$

where $\phi, \psi : \Sigma^X$, so whenever $I \subset U \cup V$ is covered by open inhabited subspaces, their intersection is inhabited.

Proposition: Any function $f : I \to \mathbb{R}$ that takes values both above $-\epsilon$ and below $+\epsilon$ also takes values within ϵ of zero:

$$\exists xz : I.\ (-\epsilon < fx) \wedge (fz < +\epsilon) \Rightarrow \exists y : I.\ (-\epsilon < fy < +\epsilon),$$

so the open, overt subspace $\{x : X \mid |fx| < \epsilon\}$ is inhabited.

Proof: Let $\phi x \equiv (-\epsilon < fx)$ and $\psi x \equiv (fx < +\epsilon)$, so $\phi \vee \psi = \top$ and

$$\begin{aligned} \exists xz.\ (-\epsilon < fx) \wedge (fz < +\epsilon) &\equiv \Diamond\phi \wedge \Diamond\psi \\ &\Rightarrow \Diamond(\phi \wedge \psi) \equiv \exists y.\ (-\epsilon < fy < +\epsilon). \end{aligned}$$

Dually, compact connectedness

A compact subspace $K \subset X$ defined by $\Box : \Sigma^{\Sigma^X}$ is connected if

$$\Box\bot \;\Leftrightarrow\; \bot \quad \text{and} \quad \phi \wedge \psi = \bot_I \;\vdash\; \Box\phi \vee \Box\psi \;\Leftarrow\; \Box(\phi \vee \psi),$$

for $\phi, \psi : \Sigma^X$, so whenever $K \subset A \cup B$ is covered by closed occupied subspaces then their intersection is occupied.

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Proposition: Let $f : K \to \mathbb{R}$ such that both of the closed, compact subspaces $\{x : K \mid fx \geq 0\}$ and $\{x : K \mid fx \leq 0\}$ are occupied. Then so is $Z \equiv \{x : K \mid fx = 0\}$.

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Proof: Let $\phi x \equiv (0 < fx)$ and $\psi x \equiv (fx < 0)$, so $\phi \wedge \psi = \bot$. Then

$$\begin{aligned} (\forall x.\, 0 < fx) \vee (\forall x.\, fx < 0) \quad &\equiv \quad \Box\phi \vee \Box\psi \\ &\Leftarrow \quad \Box(\phi \vee \psi) \;\equiv\; (\forall x.\, fx \neq 0). \end{aligned}$$

Three definitions of connectedness?

Theorem: For any compact overt subspace, all three definitions are equivalent.

Proof: Using the mixed modal laws

$$\Diamond\phi \wedge \Box\psi \;\Rightarrow\; \Diamond(\phi \wedge \psi) \quad \text{and} \quad \Diamond\phi \vee \Box\psi \;\Leftarrow\; \Box(\phi \vee \psi)$$

and the Gentzen-style rules

$$\frac{\sigma \Leftrightarrow \top \;\vdash\; \alpha \;\Rightarrow\; \beta}{\vdash \sigma \wedge \alpha \;\Rightarrow\; \beta} \qquad \frac{\sigma \Leftrightarrow \bot \;\vdash\; \alpha \;\Rightarrow\; \beta}{\vdash \alpha \;\Rightarrow\; \beta \vee \sigma}$$

connectedness may be expressed in four equivalent ways:

$$\begin{array}{lcl} \phi \vee \psi = \top & \vdash & \Diamond\phi \wedge \Diamond\psi \;\Rightarrow\; \Diamond(\phi \wedge \psi) \\ \phi \wedge \psi = \bot & \vdash & \Box\phi \vee \Box\psi \;\Leftarrow\; \Box(\phi \vee \psi) \\ \Box(\phi \vee \psi) & \Rightarrow & \Box\phi \vee \Box\psi \vee \Diamond(\phi \vee \psi) \\ \Diamond(\phi \wedge \psi) & \Leftarrow & \Box(\phi \vee \psi) \wedge \Diamond\phi \wedge \Diamond\psi \end{array}$$

The interval $[0, 1]$ is connected (usual proof)

Any closed subspace of a compact space is compact.
Any open subspace of an overt space is overt.

Any clopen subspace of an overt compact space is overt compact, so it's either empty or has a maximum.

Since the clopen subspace is open, its elements are interior, so the maximum can only be the right endpoint of the interval.

Any clopen subspace has a clopen complement.

- They can't both be empty, but
- in the interval they can't both have maxima (the right endpoint).

Hence one is empty and the other is the whole interval.

Converse

Theorem: Any compact overt connected subspace $K \subset \mathbb{R}$ is an interval $[d, u]$.

Proof: If K is compact overt,

- either $K \cong \emptyset$, which is forbidden by either definition of connectedness;
- or it has $e \equiv \mathsf{min}\, K$ and $t \equiv \mathsf{max}\, K$.

We want to show that $\omega x \equiv (\forall y : K.\ x \neq y)$ is $\delta x \vee \upsilon x$, where (δ, υ) is the pseudo-Dedekind cut
$\delta d \equiv (d < e), \quad \upsilon u \equiv (t < u)$.

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This should be impossible.
Imagine arriving from a hike at an isolated bus stop to find the timetable obliterated.
The one daily bus is not there now (ωx).
How can you decide whether you should wait for it (δx) or if it's already gone (υx)?

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How can you decide whether you should wait for it (δx) or if it's already gone (υx)?

You can't. But we can still prove a more general theorem...

Compact intervals

Theorem: Any compact connected subspace $K \equiv (\Box, \omega) \subset \mathbb{R}$ is a closed interval $[\delta, \upsilon]$, *i.e.* it is co-classified by $\delta \vee \upsilon$ where (δ, υ) is a rounded, bounded and disjoint pseudo-cut.

Proof: For $x : \mathbb{R}$, let $\phi_x y \equiv (y < x)$ and $\psi_x y \equiv (x < y)$, so $\phi_x \wedge \psi_x = \bot$. By compact connectedness,

$$\omega x \;\equiv\; \Box(\lambda y.\, x \neq y) \;\equiv\; \Box(\phi_x \vee \psi_x) \;\Leftrightarrow\; \Box\,\phi_x \vee \Box\,\psi_x,$$

so $\delta d \equiv \Box\,\phi_d$ and $\upsilon u \equiv \Box\,\psi_u$.

(Either the bus hasn't come yet, or it will never come again.)

δ and υ are disjoint because

$$\begin{aligned} \delta x \wedge \upsilon x \quad &\equiv \quad \Box(\lambda y.\, y < x) \wedge \Box(\lambda y.\, x < y) \\ &\Leftrightarrow \quad \Box(\lambda y.\, y < x \wedge x < y) \;\Leftrightarrow\; \Box\,\bot \;\Leftrightarrow\; \bot. \end{aligned}$$

Open intervals

Theorem: Let $U \subset \mathbb{R}$ be open, classified by $\alpha : \Sigma^{\mathbb{R}}$.
This is overt connected iff $\alpha = \delta \wedge \upsilon$,
where (δ, υ) is an overlapping pseudo-cut.

Proof [$\Rightarrow$]: For $x : \mathbb{R}$, let $\phi x \equiv \alpha x \wedge (x < y)$ and $\psi x \equiv \alpha x \wedge (x > y)$, so $\lozenge(\phi \wedge \psi) \Leftrightarrow \bot$. Then

$$\alpha x \;\Rightarrow \alpha x \wedge (\alpha y \vee x \neq y) \;\Rightarrow\; \alpha y \vee \phi x \vee \psi x.$$

If $\alpha y \Leftrightarrow \bot$, this says that $\alpha \leq \phi \vee \psi$, so by overt connectedness,

$$\begin{aligned} \alpha y \Leftrightarrow \bot \;\vdash\; \alpha x \wedge (x < y < z) \wedge \alpha z \;&\Rightarrow\; \phi x \wedge \psi z \;\Rightarrow\; \lozenge\,\phi \wedge \lozenge\,\psi \\ &\Rightarrow\; \lozenge(\phi \wedge \psi) \;\Rightarrow\; \bot \end{aligned}$$

Hence $\alpha x \wedge (x < y < z) \wedge \alpha z \Rightarrow \alpha y$, so $\alpha = \delta \wedge \upsilon$, where

$$\delta d \;\equiv\; \exists e.\, d < e \wedge \alpha e \quad \text{and} \quad \upsilon u \;\equiv\; \exists t.\, \alpha t \wedge t < u.$$

The main case for the defence

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It is also our counter-claim against Bishop.

Open sets as unions of intervals

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For any open $U \subset \mathbb{R}$ classified by $\phi : \Sigma^{\mathbb{R}}$, define

$$\begin{aligned}(x \approx_\phi y) &\equiv \big([x,y] \subset \phi\big) \wedge \big([y,x] \subset \phi\big) \\ &\equiv \big(x > y \vee \forall z : [x,y].\ \phi z\big) \ \wedge \ \big(x < y \vee \forall z : [y,x].\ \phi z\big),\end{aligned}$$

which is a partial equivalence relation on $\mathbb{R}$
that is reflexive exactly on U: $\phi x \Leftrightarrow (x \approx x)$.

We shall show that the equivalence classes of $\approx$
are the required intervals.

Open relations on the closed interval

Lemma: Let $\sim$ be an open reflexive relation on $\mathbb{I} \equiv [0,1]$:

$$\cdots,\ x, y : \mathbb{R} \vdash x \sim y : \Sigma \quad \text{such that} \quad \forall x \colon [0,1].\ x \sim x.$$

Then $\sim$ is represented by finitely many dyadic rationals:

$$\begin{aligned} & \forall x \colon [0,1].\ x \sim x \\ & \qquad \Rightarrow \quad \forall x.\ \exists p, q \colon \mathbb{R}.\ p < x < q \wedge \forall y \colon [p,q].\ x \sim y \\ & \qquad \Rightarrow \quad \forall x.\ \exists k, m \colon \mathbb{N}.\ 0 \le m \le 2^k \wedge x \sim \tfrac{m}{2^k} \\ & \qquad \Rightarrow \quad \exists k.\ \forall x.\ \exists m.\ 0 \le m \le 2^k \wedge x \sim \tfrac{m}{2^k} \end{aligned}$$

Graph-theoretic connectedness

Lemma: Let $\theta_0, \ldots, \theta_{n-1}$ be open subsets of a space X that

- are each inhabited in $(\Diamond\, \theta_i)$ and
- together cover $(\exists i < n.\ \theta_i) = \top_I)$

an overt connected subspace $I \subset X$ defined by $\Diamond$.

Then the overlaps of the θ_i define a connected graph,
in the sense that there is some permutation $p : \mathbf{n} \cong \mathbf{n}$ for which

$$\forall 1 \leq i < n. \quad \exists 0 \leq j < i. \quad \Diamond(\theta_{p(i)} \wedge \theta_{p(j)}).$$

Because of the infinitary lemma,
we want $n : \mathbb{N}$ to be a parameter,
so "fill in" $\theta_i \equiv \top$ for $i \geq n$.

Graph-theoretic connectedness

Proof: We prove by induction on $1 \leq m \leq n$ that

$$\exists p : \mathbf{n} \cong \mathbf{n}.\ \left\{ \begin{array}{rl} & \forall 1 \leq i < m.\ \exists 0 \leq j < i.\ \Diamond(\theta_{p(i)} \wedge \theta_{p(j)}) \\ \wedge & \forall m \leq i < n.\ p(i) = i, \end{array} \right.$$

where the initial case $m \equiv 1$ is satisfied by $p \equiv \mathsf{id}$
and the final one $m \equiv n$ gives the required result.

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Assume the induction hypothesis for some $1 \le m < n$ and put

$$\phi x \;\equiv\; \exists 0 \le j < m.\ \theta_{p(j)} x \quad \text{and} \quad \psi x \;\equiv\; \exists m \le i < n.\ \theta_i x.$$

Then $\phi \vee \psi = \top_I$, $\top \Leftrightarrow \lozenge\,\theta_0 \Rightarrow \lozenge\,\phi$ and $\top \Leftrightarrow \lozenge\,\theta_{n-1} \Rightarrow \lozenge\,\psi$.

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Since $\Diamond$ is overt connected and preserves joins, we deduce

$$\Diamond(\phi \wedge \psi) \ \equiv\ \exists m \le i < n.\ \exists 0 \le j < m.\ \Diamond(\theta_i \wedge \theta_{p(j)}).$$

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Let $s : \mathbf{n} \cong \mathbf{n}$ be the swap (m, i), and $p' \equiv s \cdot p$.
Then p' satisfies the induction hypothesis for $m + 1$ in place of m.

Equivalence relations

Theorem: Any open equivalence relation $\sim$ on $\mathbb{I} \equiv [0, 1]$ is indiscriminate, *i.e.* $\forall x, y : \mathbb{I}.\ x \sim y$, and in particular $0 \sim 1$.

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Proof: Using k from the infinitary lemma,
put $n \equiv 2^k + 1$ and $\theta_i x \equiv (x \sim i \cdot 2^{-k})$ in the finitary one.
Then $\sim$ is connected in the graph-theoretic sense.
As it is also symmetric and transitive, $0 \sim 1$,
and more generally $\forall xy : [0, 1].\ x \sim y$.

Local connectedness with equivalence relations

Corollary: Any open partial equivalence relation $\sim$ on $\mathbb{R}$ satisfies

$$\left(\forall y : [x, z].\ y \sim y\right) \ \Rightarrow\ x \sim z.$$

Corollary: Any function $f : X \to N$ with N discrete is constant, where $X \equiv \mathbb{I}$, $\mathbb{R}$, (d, u) or (v, δ).

Proof: The open equivalence relation $(x \sim y) \equiv (fx =_N fy)$ is indiscriminate.

Witness: Andrej Bauer

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Example: Let $n : \mathbb{N} \vdash \theta_n$ be a singular cover of $[0, 1]$ in recursive analysis, *i.e.* one with no finite subcover.

Define the reflexive relation $\sim$ by

$$(x \sim z) \;\equiv\; \exists n.\, \forall y : [x, z].\; \theta_n y.$$

Then its symmetric transitive closure has infinitely many equivalence classes in $[0, 1]$.

A universal property

Proposition: $\approx$ is an open partial equivalence relation on $\mathbb{R}$ that is reflexive on the open subspace classified by ϕ:

$$\phi x \Rightarrow (x \approx x), \qquad x \approx y \Rightarrow y \approx x \quad \text{and} \quad x \approx y \approx z \Rightarrow x \approx z.$$

The classes are disjoint in the sense that
if any two overlap, they actually coincide.
Each of these classes is open and connected.

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if any two overlap, they actually coincide.
Each of these classes is open and connected.

It is the sparsest such relation:
any other one, $\sim$, satisfies $(x \approx y) \Rightarrow (x \sim y)$.

Finally, $(x \approx x) \Rightarrow \phi x$.

Countably many intervals

Lemma: From any open subspace $U \subset \mathbb{R}$
there is an open surjection $U \twoheadrightarrow N/\approx$ with open connected fibres
onto an overt discrete space.

Proof: Let U be classified by $\phi : \Sigma^{\mathbb{R}}$.

Let $N \equiv \{q : \mathbb{Q} \mid \phi q\} \subset \mathbb{Q}$,
which is an open subspace of an overt discrete space.

Then $\approx$ restricts to a (total) open equivalence relation on N,

so the quotient $N/\approx$ is an overt discrete space.

Since $\mathbb{Q} \subset \mathbb{R}$ is dense, $N \twoheadrightarrow U$ is epi

so there is an open surjection $U \twoheadrightarrow N/\approx$.

Geometric and Higher Order Logic in terms of ASD,
Theory and Applications of Categories, **7** (2000) 284–338.

Order on the intervals

Lemma: The relation $\leq$ on $N/\approx$ defined by

$$[x] \leq [y] \;\equiv\; \exists p, q : \mathbb{Q}.\ x \approx p \leq q \approx y$$

is a total order, in the sense that

$$[x] \leq [x], \quad [x] \leq [y] \leq [x] \Rightarrow [x] \leq [z],$$

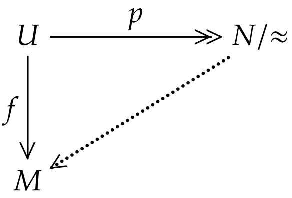
$$[x] \leq [y] \leq [x] \Rightarrow (x \approx y), \quad [x] \leq [y] \vee [y] \leq [x].$$

Example: It need not be decidable:
the open complement U of $\{0\} \cap \{g\}$ has $\{-1, +1\}/\approx$ components,
where $(-1 \approx +1) \equiv g \neq 0$.

Re-stating the universal property categorically

Theorem: Every open subspace $U \subset \mathbb{R}$ is locally connected:

- there is a map $p : U \twoheadrightarrow N/\approx$ with $N/\approx$ discrete;
- any map $f : U \to M$ to a discrete space factors uniquely as



- $N/\approx$ is overt and p is an open surjection;
- this representation is unique up to unique isomorphism.

Proof: $(x \sim y) \equiv (fx =_M fy)$ is an open partial equivalence relation on $\mathbb{R}$ with $\phi x \Rightarrow x \sim x$, so $x \approx y \Rightarrow x \sim y$.
Hence f factors uniquely through the quotient.

Witnesses: Richard Dedekind, Eduard Heine and Emile Borel

$\mathbb{R}$ is

- overt, with $\exists$;
- Hausdorff, with $\neq$;
- totally ordered, *i.e.* $(x \neq y) \Leftrightarrow (x < y) \vee (y < x)$;
- a field, where x^{-1} is defined iff $x \neq 0$;
- Dedekind complete; and
- Archimedean;
- and the closed interval is compact, with $\forall$.

Practical defence: these axioms are natural, necessary and complete for analysis.

Formalist defence: they have a recursive model (ASD).