A computable axiomatisation of the topology of $\mathbb{R}$ and $\mathbb{C}$

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Categories, Logic and Foundations of Physics
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Foundations of Physics — a Disclaimer

When I was 18, I wanted to study General Relativity.
I was taught Quantum Mechanics by John Polkinghorne.
He was not very impressed with my efforts.
I haven't thought about physics since I graduated.

So, I don't come here "as a mathematician criticising physicists".

I just want to share some ideas about the foundations of mathematics with you because

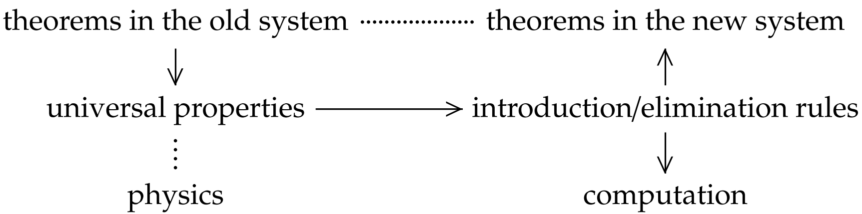
- I think we might agree on some of the methodology, and
- there may be some common ground that we could develop.

Using Categories and Logic in Foundations

Methodology:

- identify the structure that we believe physics, topology, *etc*. to have, as Axioms,
- reconstruct the mathematics,
- as a bonus, it's computable.

This can be done using category theory and symbolic logic.



What do we believe about $\mathbb{R}$ and $\mathbb{C}$?

There is a longstanding belief that $\mathbb{R}$ is for measurement.
Addition, multiplication, subtraction, division?, square roots?
...
Testing real numbers for strict order, equality? ...
Completion to solve equations.
All of this seems to be computable...

Computable real numbers

There are lots of constructions of the real line, especially in the theoretical computer science literature.

Let's just agree this extremely weak property:

> For $x : \mathbb{R}$ to be a recursive real number,
> there must be a program that,
> given $d, u : \mathbb{Q}$ as input,
> halts if $d < x < u$, but continues forever otherwise.

This is plainly not sufficient for practical purposes, because no *numbers* are ever output, but it's necessary.

If the program encodes a single genuine real number, the sets D and U of ds and us that cause the program to halt must satisfy certain consistency conditions.

In fact, they must define a Dedekind cut. Also, since they arise from programs, they are recursively enumerable.

One-way computability

Equality of real numbers is not computable.

Physicists, numerical analysts and constructive mathematicians know this.

Classical mathematicians deny it. God tells them the answer.

But, as Errett Bishop said, if God has mathematics that needs to be done, let Him do it Himself.

But, inequality of real numbers is computable.
You just have to wait for enough digits to be computed.
(That's not so obvious in Physics, but it's not for me to say.)

So the answer to a computational question is not Boolean.
It may be yes, or. . . . . . (maybe wait forever).
Alan Turing: there is no negation that swaps these.

Some types and algebraic operations

$\mathbf{0}, \mathbf{1}, \mathbf{2}, \ldots, \mathbb{N}, \mathbb{R}, \mathbb{C}, \mathbb{I} \equiv [0,1] \subset \mathbb{R}$.
$0, 1, +, \times$ on $\mathbb{N}$, $0, 1, +, -, \times$ $(\div?)$ on $\mathbb{R}$.

We can define $(+) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *etc*.

$=, \neq, <, >, \leq, \leqslant$ on $\mathbb{N}$ are decidable.
We can define $(=) : \mathbb{N} \times \mathbb{N} \to \mathbf{2}$ *etc*.

But $(<), (>), (\neq) : \mathbb{R} \times \mathbb{R} \to \Sigma$,
where Σ is the type of answers to computable questions,
$\top \equiv$ yes, $\bot \equiv$ wait.

Analogy between open and recursively enumerable subsets.

More logic on Σ

By considering sequential processes, we also have $(\wedge) : \Sigma \times \Sigma \to \Sigma$.

By considering parallel processes, we have $(\vee) : \Sigma \times \Sigma \to \Sigma$. Also $(\exists_{\mathbb{N}}) : \Sigma^{\mathbb{N}} \to \Sigma$.

However, we not have $\neg$, $\Rightarrow$ or $\forall_{\mathbb{N}}$.

'Tis a maxim tremendous but trite

Σ is the type with values

- $\top$ (a signal, "yes"), and
- $\bot$ ("wait").

Therefore a program $F : \Sigma \to \Sigma$ can only do one of three things:

- $Fx \equiv \bot$: never terminate;
- $Fx \equiv x$: wait for the incoming signal, do some internal processing and then output a signal; or
- $Fx \equiv \top$: (maybe do some internal processing and then) output a signal, without waiting for the input.

(As we have said, it cannot interchange $\top$ and $\bot$).

Since Σ is a lattice, this means that, for all $x \in \Sigma$,

$$Fx \;=\; F\bot \vee x \wedge F\top$$

So any $F : \Sigma \to \Sigma$ is a polynomial.

This is called the Phoa Principle. It turns out to be as important in topology as the distributive law is in arithmetic.

We'll come back to the polynomials later.

More operations on $\mathbb{R}$

Classically, any non-empty subset of $\mathbb{R}$ that is bounded above has a supremum.
(This is one way of stating Dedekind completeness.)

Now choose your favourite unsolved or unsolvable problem:

- let $g_n \equiv 0$ if there are prime numbers p and q such that $2n + 4 = p + q$, but $g_n \equiv 1$ if there aren't; or
- let $g_n \equiv 1$ if there is a proof that $0 = 1$ with at most n symbols in whatever logic you're using,
  but $g_n \equiv 0$ if there isn't.

So, we believe that $g_n = 0$ for all n, but cannot prove it.

More operations on $\mathbb{R}$

The sequence g_n only takes values 0 or 1,
so it is non-empty and bounded above,
We believe that $g_n = 0$ for all n.
What is its supremum, $a \equiv \sup g_n$?
Is it 0, 1 or something else?

Recall that, for $a : \mathbb{R}$ to be a recursive real number,
there must be a program that given $d, u : \mathbb{Q}$ as input,
halts if $d < a < u$, but continues forever if not.

So it must halt in exactly one of these two cases:

- $d \equiv -\frac{1}{3}$, $u \equiv +\frac{1}{3}$, if $a = 0$, or
- $d \equiv \frac{2}{3}$, $u \equiv +\frac{4}{3}$, if $a = 1$.

But if it does this, it has answered our unsolved problem
(the Goldbach conjecture or consistency of our ambient logic).

Hence a does not exist as a computable real number.

One-sided reals

We have shown that the value $a \equiv \sup g_n$ does not belong to $\mathbb{R}$.

In fact, there is no problem here.

This value can be given the type $\underline{\mathbb{R}}$
of ascending or lower reals.
Also, $-a \in \overline{\mathbb{R}}$ is a descending or upper real.
$\underline{\mathbb{R}}$ and $\overline{\mathbb{R}}$ have $\sup$, $\inf$ and $+$.

However, they don't have $-$.
They are to $\mathbb{R}$ as Σ is to $\mathbf{2}$ — much simpler.

In fact, many of the results that say
"such-and-such a real number is not computable"
just show that this number is an ascending or descending real.

Russian Recursive Analysis

Can we do analysis with the set $\mathbb{R}$ of recursive real numbers?

Yes, but it has some rather unpleasant features.

There is a singular cover of $\mathbb{I} \equiv [0, 1] \subset \mathbb{R}$.
This is a recursively enumerable sequence of intervals $(p_n, q_n) \subset \mathbb{R}$ with $p_n < q_n : \mathbb{Q}$ such that

- each recursive real number $a \in \mathbb{I}$ lies in some interval (p_n, q_n),
- but $\sum_n q_n - p_n < 1$.

There is no finite subcover of $\mathbb{I} \equiv [0, 1]$.

Measure theory also goes badly wrong.

One solution: Bishop's Constructive Analysis

Live without the Heine–Borel theorem.

Errett Bishop, *Foundations of Constructive Analysis*, 1967
He developed remarkably much of analysis in a "can do" way, without dwelling on counterexamples that arise from wrong classical definitions.

This theory has been developed by Douglas Bridges, Hajime Ishihara, Mark Mandelkern, Ray Mines, Fred Richman, Peter Schuster, ...

It is consistent with both Russian Recursive Analysis and Classical Analysis.

It uses Intuitionistic Logic (Brouwer, Heyting).

One solution: Bishop's Constructive Analysis

Compact = closed and totally bounded.

(X is *totally bounded* if, for any $\epsilon > 0$, there's a finite set $S_\epsilon \subset X$ such that for any $x \in X$ there's $s \in S_\epsilon$ with $d(x, s) < \epsilon$.)

Continuity on a *compact* interval (in this sense) is defined as uniform continuity.

Unfortunately, this makes it very difficult to say that $x \mapsto \frac{1}{x}$ is continuous on $(0, \infty)$.

Singular covers in physics

Consider a differential equation on a simply connected open domain D, such as $[0,1] \subset D \subset \mathbb{R}$.

Suppose that, given an initial condition at any point $x \in D$, we expect to be able to extend this to a unique local solution of the equation on some open $U_x \subset D$, with $x \in U_x$.

(For simplicity, we ignore the fact that U_x depends on the choice of the initial condition.)

Write $x \sim y$ if there is some patch U_x with $y \in U_x$.

By definition, $x \sim x$.

Also, by repeating (continuing) the solution of the equation,

$$x \sim y \Rightarrow y \sim x \quad \text{and} \quad x \sim y \sim z \Rightarrow x \sim z,$$

so $\sim$ is an equivalence relation.

Surely then $0 \sim 1$, the initial condition defines a global solution?

Using the Heine–Borel theorem, yes, but not in recursive analysis or Bishop's theory.

Another solution: Weihrauch's Type Two Effectivity

Consider all real numbers, not just recursive ones.
Represent them (for example) by signed binary expansions

$$a = \sum_{k=-\infty}^{+\infty} d_k \cdot 2^{-k} \qquad \text{with} \quad d_k \in \{+1, 0, -1\}.$$

Think of $\{\ldots, 0, 0, 0, \ldots, d_{-2}, d_{-1}, , d_0, d_1, d_2, \ldots\}$ as a Turing tape with finitely many nonzero digits to the left,
but possibly *infinitely* many to the right.

Do real analysis in the usual way.

Do computation with the sequences of digits.

Klaus Weihrauch, *Computable Analysis*, Springer, 2000.
Developed by Vasco Brattka, Peter Hertling, Martin Ziegler, ...

Disadvantages of these methods

Point-set topology and recursion theory separately are complicated subjects that lack conceptual structure.

Together, they give pathological results.

Intuitionism makes things even worse —
the natural relationship between open and closed subspaces is replaced by double negation.

We're still relying on Set Theory — the continuum as dust.

We said that we would only assume what we believe.

Category theory can do better than this!

Compactness and Scott continuity

A subset $\mathcal{U} \subset \mathcal{L}$ of a complete lattice is Scott open if any subset $\mathcal{S} \subset \mathcal{L}$ for which $\bigvee \mathcal{S} \in \mathcal{U}$
already has some *finite* $\mathcal{F} \subset \mathcal{S}$ with $\bigvee \mathcal{F} \in \mathcal{U}$.

The Scott open subsets $\mathcal{U} \subset \mathcal{L}$ form a topology.

Let $\mathcal{L}$ be the lattice of open subspaces of a space X.
Then a subset $K \subset X$ is compact iff the family

$$\mathcal{U}_K \equiv \{U \in \mathcal{L} \mid K \subset U\}$$

of open neighbourhoods of K is a Scott open subset of $\mathcal{L}$.

This is just the usual "finite open sub-cover" definition of compactness.

Higher-type operations in topology

We have already introduced the type Σ.

Continuous (computable) maps $\phi : X \to \Sigma$
correspond bijectively to
open (recursively enumerable) subsets $U \subset X$.

Hence Σ^X is the topology on X.

We give Σ^X the Scott topology.

If K is compact, $K \subset (-)$ is a Scott continuous map $\Sigma^X \to \Sigma$.

We write $\forall_K \phi$ or $\forall x : K.\ \phi x$ for $K \subset U$.

We can the use λ-calculus on types like $\Sigma^{\mathbb{R}}$ to write topological arguments.

The axioms so far

- Some base types, including $\mathbb{N}$ and $\mathbb{R}$, with arithmetic operations.
- The space Σ, with lattice operations.
- Relations such as $(<) : \mathbb{R} \times \mathbb{R} \to \Sigma$.
- Quantifiers such as $\exists_{\mathbb{N}} : \Sigma^{\mathbb{N}} \to \Sigma$ and $\exists_{[0,1]} : \Sigma^{\mathbb{N}} \to \Sigma$.
- In fact $\exists x : \mathbb{R}.\ \phi x \iff \exists x : \mathbb{Q}.\ \phi x$.
- λ-calculus to work with open subsets and higher types.

Computation with Dedekind reals

Any real-valued computation defines a pair of formulae $\delta, \upsilon : \Sigma^{\mathbb{Q}}$.

Implicitly, we have a proof that it is a Dedekind cut.

Computation to precision ϵ involves finding $d, u \in \mathbb{Q}$ with $\delta d \wedge \upsilon u$ and $u - d < \epsilon$.

This is a logic programming problem.

Methods such as constraint logic programming and interval-Newton can be used to solve it.

Efficient computation with Dedekind reals (Andrej Bauer).

The Dedekind reals

Recall that any computable real number corresponds to a recursively enumerable Dedekind cut (D, U).

In our λ-calculus we represent $D, U \subset \mathbb{Q}$ as

$$\delta, \upsilon : \Sigma^{\mathbb{Q}} \quad \text{where} \quad \delta d \iff (d < a) \quad \text{and} \quad \upsilon u \iff (a < u).$$

The condiditions for $(\delta, \upsilon) \in \Sigma^{\mathbb{Q}} \times \Sigma^{\mathbb{Q}}$ to form a Dedekind cut are equations,
so the Dedekind reals can be expressed as an equaliser,

$$\mathbb{R} \rightarrowtail \Sigma^{\mathbb{Q}} \times \Sigma^{\mathbb{Q}} \rightrightarrows (\text{a power of } \Sigma)$$

The universal property of the equaliser says that every recursively enumerable Dedekind cut defines a real number.

But there's more to it than this.

The topology of the Dedekind reals

When we apply $\Sigma^{(-)}$ to the diagram

$$\mathbb{R} \overset{i}{\rightarrowtail} \Sigma^{\mathbb{Q}} \times \Sigma^{\mathbb{Q}} \rightrightarrows \cdots$$

$$\Sigma^{\mathbb{R}} \; \begin{array}{c} \overset{\Sigma^i}{\twoheadleftarrow} \\ \underset{I}{\rightarrowtail} \end{array} \; \Sigma^{\Sigma^{\mathbb{Q}} \times \Sigma^{\mathbb{Q}}} \; \begin{array}{c} \longleftarrow \\ \longleftarrow \\ \longrightarrow \end{array} \; \cdots$$

we find that the Heine–Borel theorem
is equivalent to the existence of a map I with $\Sigma^i \cdot I = \mathsf{id}$.

In traditional notation, the map I takes
an open subspace $O \subset \mathbb{R}$ to the open subspace

$$\{(D, U) \in \Sigma^{\mathbb{Q}} \times \Sigma^{\mathbb{Q}} \mid \exists d, u : \mathbb{Q}.\ d \in D \wedge u \in U \wedge [d, u] \subset O\}.$$

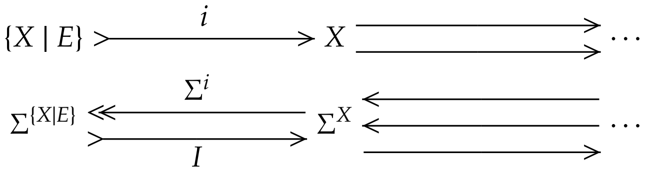
In our λ-calculus,

$$I\phi \;=\; \lambda\delta\upsilon.\ \exists d < u.\ \delta d \wedge \upsilon u \wedge \forall x : [d, u].\ \phi x.$$

The idempotent $\mathcal{E} \equiv I \cdot \Sigma^i$ on $\Sigma^{\Sigma^{\mathbb{Q}} \times \Sigma^{\mathbb{Q}}}$ can be defined just using rationals.

A type theory

Generate types from $\mathbf{1}$ and $\mathbb{N}$ using $\times$, $\Sigma^{(-)}$
and Σ-split subspaces



where $E : \Sigma^X \times \Sigma^X$ satisfies a suitable equation.

The object $\{X \mid E\}$ is defined
using an abstract type theory, not as a subset!

$\mathbb{R}$ can be constructed in this way, and
it satisfies the Heine–Borel theorem!
Andrej Bauer and Paul Taylor, *The Dedekind Reals in ASD*,
in *Mathematical Structures in Computer Science* (this month).

Any computably based locally compact space can be defined in this way.

Some bits of analysis

The quantifier $\forall x : [0,1]$ is derived from $\mathcal{E}$.
In fact, the ideas are equivalent to those in Interval Analysis.

Every function $f : \mathbb{R} \to \mathbb{R}$ is continuous in the ϵ–δ-sense:

$$\epsilon > 0 \;\Rightarrow\; \exists \delta > 0.\, \forall y : [x \pm \delta].\, \left(|fy - fx| < \epsilon \right).$$

Connectedness and the intermediate value theorem, in
A lambda calculus for real analysis (PT)

Every open subspace of $\mathbb{R}$ is uniquely expressible
as a countable disjoint union of open intervals.
("Countable" and "interval" have to be understood appropriately.)
The uniqueness fails in Bishop's theory.

Summary of the axioms on $\mathbb{R}$

- $\mathbb{R}$ is an overt space (it has $\exists$),
- it is Hausdorff, with an inequality or apartness relation, $\neq$,
- the closed interval $[0,1]$ is compact, with $\forall_{[0,1]}$,
- $\mathbb{R}$ has a total order: $(x \neq y) \Leftrightarrow (x < y) \vee (y < x)$,
- it is Dedekind-complete,
- it is a field, where x^{-1} is defined iff $x \neq 0$ in the above sense,
- and Archimedean: for $x, y : R$,

$$y > 0 \;\Rightarrow\; \exists n : \mathbb{Z}.\; y(n-1) < x < y(n+1).$$

These properties characterise $\mathbb{R}$ uniquely up to unique isomorphism.

Overtness

An English pun: "overt" means both "open" and "explicit".

I wish I could explain this you. Here are some related ideas:

Bishop's constructive analysis solves many of the problems of Russian Recursive Analysis by using total boundedness (finite nets of arbitrarily small mesh) and locatedness (another metrical property).

Recursive enumerability.

The lattice dual of compactness.

Surely it must be relevant in physics, but I don't know how.

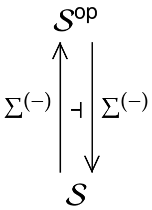
Where did the Σ-split subspaces come from?

Whenever the exponential $\Sigma^{(-)}$ exists,
this contravariant functor is self-adjoint:

Stone duality, abstractly:

The category of topologies is $\mathcal{S}^{\mathrm{op}}$,
the dual of the category $\mathcal{S}$ of "spaces".

It's also a category of algebras for a monad on $\mathcal{S}$.



This was my idea in 1993,
inspired by Robert Paré, *Colimits in topoi*, 1974:
the category of sets, or any topos, has this property.

Some generalised abstract nonsense

Jon Beck (1966) characterised monadic adjunctions:

- $\Sigma^{(-)} : \mathcal{S}^{\mathsf{op}} \to \mathcal{S}$ reflects invertibility,
  i.e. if $\Sigma^f : \Sigma^Y \cong \Sigma^X$ then $f : X \cong Y$, and
- $\Sigma^{(-)} : \mathcal{S}^{\mathsf{op}} \to \mathcal{S}$ creates $\Sigma^{(-)}$-split coequalisers.

Category theory is a strong drug —
it must be taken in small doses.

Beck's theorem says exactly that we may define Σ-split subspaces such as $\mathbb{R} \subset \Sigma^{\mathbb{Q}} \times \Sigma^{\mathbb{Q}}$.

The structure of the theory

Abstract Stone Duality was developed by
unwinding some abstract categorical ideas.

Abstractly, its axioms come in two parts:

There is the underlying structure of the category.
This is a way of defining types (spaces), terms (functions) and subtypes.
It ensures that

- the types and subtypes have the correct topology, and
- all functions are continuous.

It is based on an object Σ
but says nothing about its structure.

Then there are some simple lattice-theoretic axioms
that are specific to topology.

The structure of the theory

The axioms specific to topology were:

- Σ is a lattice ($\top$, $\bot$, $\wedge$, $\vee$, $\exists_{\mathbb{N}}$),
- the Phoa principle: every $F : \Sigma \to \Sigma$ is a polynmial,
  $Fx \;=\; F\bot \vee x \wedge F\top$,
- Scott continuity (actually implies Phoa).

The underlying abstract category theory does most of the work!

Could we use the underlying structure elsewhere?

Let $\mathcal{H}$ be the category of Hilbert spaces
and continuous linear maps.

$\mathcal{H}$ is a model of linear logic with $\oplus, \otimes, \dagger$.

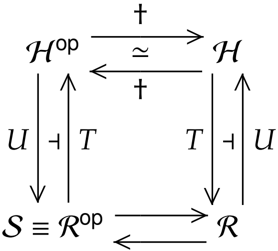
There is an of course comonad.

By a ring (for short) I mean a commutative monoid in $(\mathcal{H}, \otimes)$ with a unit. Let $\mathcal{R}$ be the category of rings and their homomorphisms.

There is a forgetful functor $U : \mathcal{R} \to \mathcal{H}$.
It has a left adjoint $T : \mathcal{H} \to \mathcal{R}$ called tensor algebra.

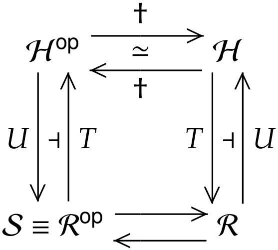
Σ and its powers

My model will be $\mathcal{S} \equiv \mathcal{R}^{\mathrm{op}}$
(the category of cocommutative coalgebras):



Let $\Xi, \Gamma \in \mathcal{S}$ be $X, G \in \mathcal{R}$.
Then there are correspondences amongst
$\Gamma \times \Xi \to \Sigma$ in $\mathcal{S}$
$TI \to G \otimes X$ in $\mathcal{R}$
$I \to UG \otimes UX$ in $\mathcal{H}$
${}^{\dagger}UX \to UG$ in $\mathcal{H}$
$T({}^{\dagger}UX) \to G$ in $\mathcal{R}$
$\Gamma \to \Sigma^X$ in $\mathcal{S}$,

Equalisers and coequalisers



I think (but need someone else to check) that
U preserves reflexive coequalisers, and
T preserves coreflexive equalisers
whilst both functors reflect invertibility.

This makes $\mathcal{H}^{op}$ a category of algebras over $\mathcal{S}$,
whilst we already knew that $\mathcal{R}$ is a category of algebras over $\mathcal{H}$.

$\mathcal{S}$ is cartesian closed (and a lot more).

Computable reaxiomatisation of Hilbert spaces

On the analogy of ASD in computable topology, we would expect:

$\mathcal{S}$ to have a simple axiomatisation consisting of
the underlying type theory with $\Sigma^{(-)}$ and equalisers
Σ to be an internal ring like $\mathbb{C}$,
all functions $\Sigma \to \Sigma$ are polynomials.

Over this, $\mathcal{H}^{\mathsf{op}}$ is a category of finitary algebras.

However, I need someone else to help me work this out.