Equideductive Logic and CCCs with Subspaces

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Thursday 5 February 2009

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Equideductive logic

- is a *very* simple predicate calculus with $\forall$ and $\Rightarrow$ over a simply typed λ-calculus
- is the logic of regular monos (equalisers) in any cartesian closed category with all finite limits
- captures judgements and proof rules for equations
- has arbitrarily nested implication for induction
- is the metalanguage for Dana Scott's equilogical space construction
- needs an extra axiom
- has a realisability interpretation in itself that we might use to show conservativity (but I need help doing this).

Abstract Stone Duality

ASD's axiomatisation of general topology consists of

- a lattice part: $\top$, $\bot$, $\wedge$, $\vee$ for open sets, $=$ for discrete spaces, $\neq$ for Hausdorff, $\Cup$ for compact and $\exists$ for overt ones (we'll see the reason for the new symbol $\Cup$ in place of $\forall$);
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But the categorical part only handles locally compact spaces.

It needs to be generalised.

Abstract Stone Duality – generalisation

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It needs to be generalised.

We will get a CCC, but that's not important, because

- the exponential Y^X is tested by incoming maps,
- but its topology by outgoing ones.

Not the definition of a topos

We certainly need products, $\Sigma^{(-)}$ and equalisers.

A topos

- has an internal Heyting algebra Ω; and
- is cartesian closed, with equalisers as well as products, and all powers, in particular of Ω.

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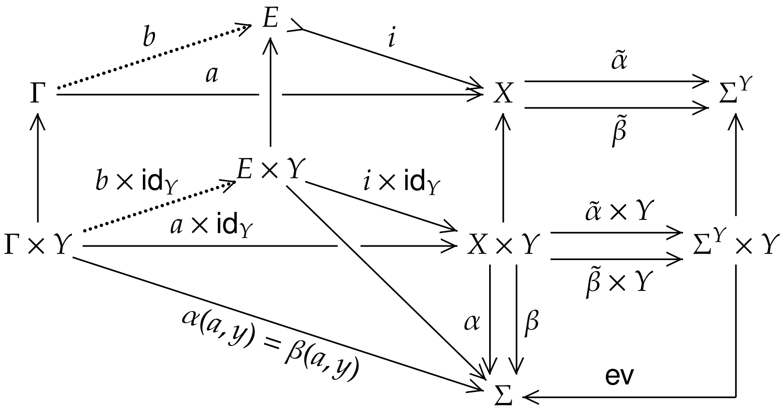
Even though this is much weaker than the correct definition, these two ideas are surprisingly powerful.

First we look at CCCs with equalisers.

Then we see how a special semilattice Σ sits in this.

CCCs with all finite limits

Working with nested equalisers and exponentials is clumsy.
Want to write $E = \{x \mid \forall y.\ \alpha x y = \beta x y\}$.



This can be stated without mentioning Σ^Y
as a universal property called a partial product.

Equideductive logic

The symbolic rules for $\forall \Rightarrow$ are as you would expect:

$$\frac{\Gamma,\ x : A,\ \mathfrak{p}(x)\ \vdash\ \alpha x\ =\ \beta x}{\Gamma\ \vdash\ \forall x : A.\ \mathfrak{p}(x)\ \Rightarrow\ \alpha x\ =\ \beta x}\ \forall I$$

$$\frac{\Gamma\ \vdash\ a : A,\ \mathfrak{p}(a) \qquad \Gamma\ \vdash\ \forall x : A.\ \mathfrak{p}(x)\ \Rightarrow\ \alpha x\ =\ \beta x}{\Gamma\ \vdash\ \alpha a\ =\ \beta a}\ \forall E$$

Of course, we need substitution (cut) for the free variable x.
It is given by a small change to the partial product diagram.

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This logic also has conjunction, with

$$\vdash\ \top \qquad \mathfrak{p},\ \mathfrak{q}\ \vdash\ \mathfrak{p}\&\mathfrak{q} \qquad \mathfrak{p}\&\mathfrak{q}\ \vdash\ \mathfrak{p} \qquad \mathfrak{p}\&\mathfrak{q}\ \vdash\ \mathfrak{q},$$

given by equalisers targeted at products. So, although $\forall \Rightarrow$ fundamentally has an equation on the right, we may define

$$\forall y.\ \big(\mathfrak{p}(y)\ \Rightarrow\ \forall z.\ (\mathfrak{q}(z) \Rightarrow \alpha xyz = \beta xyz)\big)$$

as

$$\forall yz.\ (\mathfrak{p}(y)\ \&\ \mathfrak{q}(z)\ \Rightarrow\ \alpha xyz = \beta xyz).$$

The variable-binding rule

In the expression $\forall \vec{y}.\ \mathfrak{p}(\vec{y}) \implies \alpha\vec{x}\vec{y} = \beta\vec{x}\vec{y}$,
all of the variables on the left of $\Rightarrow$ must be bound by $\forall$.

This is because the target of the equaliser was Σ^Y,
not a dependent type.

Not all dependent types

Maybe we can add some dependent types later, but
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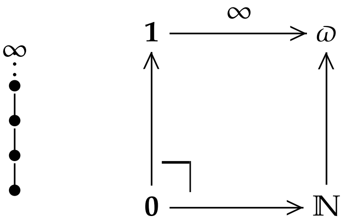
Write ϖ for the ascending natural number domain,



Then $\mathbb{N} \to \varpi$ is epi but not surjective,
since ∞ has no inverse image, *i.e.* its pullback is the initial object.

Therefore, a category of "sober" spaces and Scott-continuous functions cannot be locally cartesian closed.

Equideductive translation of rules

An algebraic theory may be presented using judgements

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Then a rule

$$\frac{x : X,\ y : Y,\ \ldots,\ a = b,\ c = d,\ \ldots \quad \vdash \quad e = f}{u : U,\ v : V,\ \ldots,\ g = h,\ k = \ell,\ \ldots \quad \vdash \quad m = n}$$

is re-written as

$$\big(\forall x : X. \forall y : Y. \ldots\ a = b \;\&\; c = d \;\&\; \cdots \Rightarrow e = f\big)$$

$$\Longrightarrow \big(\forall u : U. \forall v : V. \ldots\ g = h \;\&\; k = \ell \;\&\; \cdots \Rightarrow m = n\big).$$

Induction

Let's play with the logic a bit more.

Since $\Rightarrow$ can be nested arbitrarily deeply, we can write induction as

$$\forall n.\ \ \mathfrak{p}(0) \,\&\, \Big(\forall m.\, \mathfrak{p}(m) \ \Rightarrow\ \mathfrak{p}(m+1)\Big) \implies \mathfrak{p}(n).$$

A “double negation” property

If $\mathfrak{p}(a)$ is $\quad \top, \quad \mathfrak{q}(a) \& \mathfrak{r}(a) \quad$ or $\quad \forall y.\, \mathfrak{q}(y) \Rightarrow \alpha a y = \beta a y \quad$ then

$$\mathfrak{p}(a) \quad \dashv\vdash \quad \forall \phi \psi.\ (\forall a'.\, \mathfrak{p}(a') \Rightarrow \phi a' = \psi a') \implies \phi a = \psi a$$

where $a : A$ and $\phi, \psi : \Sigma^A$.

Disjunction and existential quantification

Using $\mathfrak{p}(a) \quad \dashv\vdash \quad \forall\phi\psi.\ (\forall a'.\, \mathfrak{p}(a') \Rightarrow \phi a' = \psi a') \implies \phi a = \psi a$
we may also define $(\mathfrak{p} \vee \mathfrak{q})(a)$ as

$$\begin{aligned}\forall\phi\psi. \qquad & (\forall a'.\, \mathfrak{p}(a') \Rightarrow \phi a' = \psi a')\ \& \\ & (\forall a''.\, \mathfrak{q}(a'') \Rightarrow \phi a'' = \psi a'') \ \Rightarrow\ \phi a = \psi a\end{aligned}$$

and $(\exists x.\ \mathfrak{p})(a)$ as

$$\forall\phi\psi.\, (\forall a' x.\ \mathfrak{p}(x, a') \Rightarrow \phi a' = \psi a') \ \Rightarrow\ \phi a = \psi a.$$

However, this only satisfies the distributive and Frobenius laws

$$(\mathfrak{p} \vee \mathfrak{q})(\vec{a})\&\mathfrak{r}(\vec{b}) \ \dashv\vdash\ \Big((\mathfrak{p}\&\mathfrak{r}) \vee (\mathfrak{q}\&\mathfrak{r})\Big)(\vec{a}, \vec{b})$$

and

$$(\exists x.\ \mathfrak{p})(\vec{a})\&\mathfrak{r}(\vec{b}) \ \dashv\vdash\ \Big(\exists x.\ (\mathfrak{p}\&\mathfrak{r})\Big)(\vec{a}, \vec{b})$$

when $\vec{a}$ and $\vec{b}$ are disjoint sets of variables.

From the logic back to the category

Given a CCC $\mathcal{S}$ with all finite limits,
its regular monos obey equideductive logic.

Inside $\mathcal{S}$ there are non-CCC subcategories $\mathcal{L}$
that still obey equideductive logic,
with $\forall \Rightarrow$ defined using partial products.

Other categories $\mathcal{L}$ might have these properties too.

Conversely, from the logic or another category $\mathcal{L}$
we can define a CCC $\mathcal{S}$ with all finite limits.

Equilogical spaces

Dana Scott introduced equilogical spaces.
They are given by partial equivalence relations
on (the sets of points of) algebraic lattices.

They provide a cartesian closed extension
of the textbook category of topological spaces.

There are many variations, including
Martin Hyland's filter spaces and Alex Simpson's QCB.

Giuseppe Rosolini related these categories to presheaves on,
and exact completions of, the textbook category.

However, these categories include many objects
that owe more to set theory than to topology.

Equideductive spaces

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that are restrictions of congruences.

So we replace one, two-argument partial equivalence relation
with two one-argument predicates ($\mathfrak{p}$ and q).

Also, instead of set theory, we use equideductive logic,
possibly with some other interpretation.

What other interpretation?
That's a question for you — at the end of this lecture!

Equideductive spaces

Urtypes: generated from $\mathbf{0}$, $\mathbf{1}$ and $\mathbb{N}$ by $+$, $\times$ and $((-) \to \Sigma)$.
Combinators, including

$$\mathbb{I} : (A \to \Sigma) \to A \to \Sigma, \qquad \mathbb{K} : (A \to \Sigma) \to B \to A \to \Sigma,$$

$$\mathbb{C} : \big((B \to \Sigma) \to (C \to \Sigma)\big) \to \big((A \to \Sigma) \to (B \to \Sigma)\big) \to (A \to \Sigma) \to C \to \Sigma$$

$$\mathbb{T} : \mathbf{1}, \qquad \nu_0 : A \to (A + B), \qquad \nu_1 : B \to (A + B),$$

$$\pi_0 : \big((A + B) \to \Sigma\big) \to A \to \Sigma, \qquad \pi_1 : \big((A + B) \to \Sigma\big) \to B \to \Sigma,$$

$$\langle\rangle : \big((C \to \Sigma) \to A \to \Sigma\big) \to \big((C \to \Sigma) \to B \to \Sigma\big) \to (C \to \Sigma) \to (A{+}B) \to \Sigma.$$

$$\mathbb{A} : \big(((A \to \Sigma) + A) \to \Sigma\big) \to \mathbf{1} \to \Sigma,$$

$$\mathbb{L} : \big(((A + B) \to \Sigma) \to \mathbf{1} \to \Sigma\big) \to (A \to \Sigma) \to (B \to \Sigma) \to \Sigma.$$

with appropriate equational axioms, such as
$\forall MN\phi c.\ \mathbb{C}NM\phi c \;=\; N(M\phi)c$, without $\Rightarrow$.

Equideductive spaces

An equideductive space X is $(A, \mathfrak{p}, \mathfrak{q})$ where A is an urtype, $\mathfrak{p}$ is a predicate on Σ^A and $\mathfrak{q}$ one on A, for which

$$\phi, \psi : \Sigma^A, \quad \mathfrak{p}(\phi), \quad \forall a : A.\ \mathfrak{q}(a) \Rightarrow \phi a = \psi a \quad \vdash \quad \mathfrak{p}(\psi).$$

This rule is important in the construction.
It can be tightened to ensure that all spaces are definable using exponentials and equalisers.

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A morphism $M : X \equiv (A, \mathfrak{p}, \mathfrak{q}) \to Y \equiv (B, \mathfrak{r}, \mathfrak{s})$ is an urterm $M : (A \to \Sigma) \to B \to \Sigma$ such that

$$\phi : \Sigma^A, \quad \mathfrak{p}(\phi) \quad \vdash \quad \mathfrak{r}(M\phi)$$

$$\phi, \psi : \Sigma^A, \quad \mathfrak{p}(\phi), \quad \forall a.\ \mathfrak{q}(a) \Rightarrow \phi a = \psi a \quad \vdash \quad \forall b.\ \mathfrak{s}(b) \Rightarrow M\phi b = M\psi b,$$

where $M_1 = M_2$ if

$$\phi : \Sigma^A, \quad \mathfrak{p}(\phi) \quad \vdash \quad \forall b : B.\ \mathfrak{s}(b) \Rightarrow M_1\phi b = M_2\phi b.$$

The type structure

$\mathbf{1} \equiv (\mathbf{0}, \top, \top), \quad \Sigma \equiv (\mathbf{1}, \top, \top).$

The product is $(A, \mathfrak{p}, \mathfrak{q}) \times (B, \mathfrak{r}, \mathfrak{s}) \;\equiv\; \big(A + B, (\mathfrak{p} \cdot \pi_0 \& \mathfrak{r} \cdot \pi_1), [\mathfrak{q}, \mathfrak{s}]\big).$

The equaliser is

$$E \equiv (A, \mathfrak{t}, \mathfrak{q}) \overset{I}{\rightarrowtail} (A, \mathfrak{p}, \mathfrak{q}) \underset{N}{\overset{M}{\rightrightarrows}} (B, \mathfrak{r}, \mathfrak{s})$$

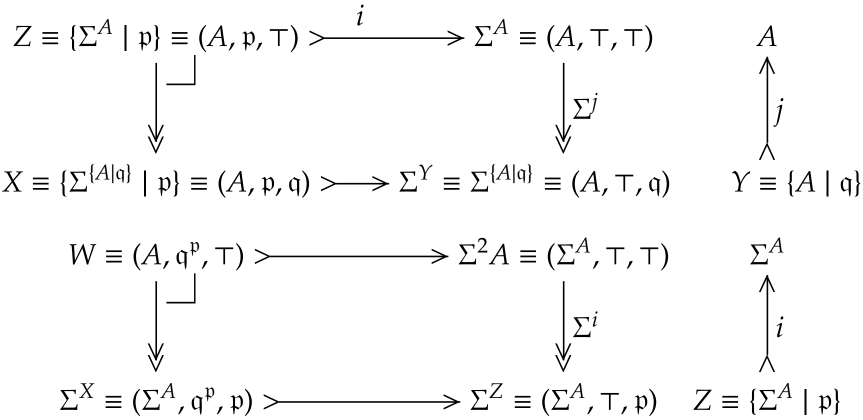
$$\mathfrak{t}(\phi) \quad \equiv \quad \mathfrak{p}(\phi) \quad \& \quad \forall b : B.\ \mathfrak{s}(b) \Rightarrow M\phi b = N\phi b,$$

The exponential of $X \equiv (A, \mathfrak{p}, \mathfrak{q})$ is $\Sigma^X \equiv (\Sigma^A, \mathfrak{q}^{\mathfrak{p}}, \mathfrak{p})$, where

$$\mathfrak{q}^{\mathfrak{p}}(F) \;\equiv\; \forall \phi, \psi : \Sigma^A.\ \mathfrak{p}(\phi) \,\&\, (\forall a : A.\ \mathfrak{q}(a) \Rightarrow \phi a = \psi a) \;\Rightarrow\; F\phi = F\psi,$$

cf. the "double negation" property earlier.
(The modulation $\mathfrak{p}(\phi) \& \cdots$ is the source of many difficulties.)

An exactness property



This property is a special case of that enjoyed by any topos. However, it is not strong enough to simplify equideductive expressions in the way that we would like, because it is "rooted" at the special object $(A, \top, \top)$.

This is the only show in town

Let $\mathcal{S}$ be a cartesian closed category that has all finite limits, this exactness property and all objects definable.

Consider the equideductive logic that $\mathcal{S}$ obeys, including any judgements (inclusions between regular monos) that it happens to obey "accidentally" .

Then $\mathcal{S}$ is equivalent to the category of equideductive spaces in this logic.

(Actually, there's an extra syntactic condition on $(A, \mathfrak{p}, \mathfrak{q})$ to ensure that it is definable using equalisers.)

So, the only way to get a CCC with subspaces and (some stronger version of) this exactness property is to use my generalisation of Scott's construction.

The structure on Σ

So far, we have assumed nothing special about the object Σ.

In the context of Abstract Stone Duality,
the old theory of locally compact spaces
was based on an underlying abstract categorical structure,
namely the monadic adjunction.

We have replaced one abstract categorical theory with another.

(It's *not* actually a *generalisation*,
but we leave this problem aside for the moment.)

Now we consider how the topological super-structure
can be rebuilt on the new theory.

We shall need Σ to be, at least, a distributive lattice:
$(\Sigma, \top, \bot, \wedge, \vee)$.

Classifying open subsets

Recall the motivation provided by the definition of a topos.

We want Σ to be a dominance (Giuseppe Rosolini again):

$$\begin{array}{ccc} U & \longrightarrow & \mathbf{1} \\ {\scriptstyle i}\big\downarrow & \lrcorner & \big\downarrow{\scriptstyle \top} \\ X & \xrightarrow{\ \chi_U\ } & \Sigma \end{array}$$

- If $U \cong V$ then $\chi_U = \chi_V$ (*pace* Per Martin-Löf);
- id_X is a pullback of $\top : \mathbf{1} \to \Sigma$ (along $\lambda x.\,\top$);
- If $U \hookrightarrow V$ and $V \hookrightarrow W$ are pullbacks of $\top : \mathbf{1} \to \Sigma$ then so is their composite $U \hookrightarrow W$;
- i is Σ-split: there is $\exists_i : \Sigma^U \to \Sigma^X$ with $\Sigma^i \cdot \exists_i = \mathrm{id}_{\Sigma^U}$ and $\exists_i \cdot \Sigma^i = (-) \wedge \chi_U \leqslant \mathrm{id}_{\Sigma^X}$, so $\exists_i \dashv \Sigma^i$.

When is the lattice Σ a dominance?

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This happens (and is given by the same urterm as id_{Σ^X})
iff $\Rrightarrow$ and $\Rightarrow$ are related by the Euclidean principle
in the form

$$\sigma = \top \;\Rrightarrow\; \alpha = \beta \quad \vdash \quad \sigma \wedge \alpha = \sigma \wedge \beta.$$

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This is the translation of the Gentzen-style rule

$$\frac{\sigma = \top \;\vdash\; \alpha = \beta}{\vdash\; \sigma \wedge \alpha = \sigma \wedge \beta}$$

Interaction of $\Rightarrow$ with $\Rightarrow$ and & with $\wedge$

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Hence it is natural to read

$$\begin{array}{lll} \sigma : \Sigma & \text{as} & \sigma = \top \\ \phi : \Sigma^X & \text{as} & \forall x.\, \phi x = \top \end{array}$$

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Then we have, as observed by Matija Pretnar,

$$\alpha = \top \mathbin{\&} \beta = \top \quad \dashv\vdash \quad \alpha \wedge \beta = \top$$

making $\wedge$ a special case of &.

The Phoa principle

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$$\forall a.\, \phi a \Rightarrow \psi a \quad \vdash \quad F\phi \Rightarrow F\psi$$

for $\phi, \psi : \Sigma^A$ and $F : \Sigma^A \to \Sigma$.

The dual Euclidean principle is

$$\sigma = \bot \;\Rightarrow\; \alpha = \bot \quad \dashv\vdash \quad \sigma \Leftarrow \alpha,$$

cf. the contrapositive in classical logic.

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Then the lattice-theoretic $\vee$ and $\exists$
are special cases of those defined earlier using $\forall \Rightarrow$ from the
categorical structure (??).

Interaction with topological structure

Similarly, equality $=_N$ in a discrete space N is a special case of general equality of terms:

$$n = m \;\dashv\vdash\; (n =_N m) = \top, \quad \text{whilst} \quad h = k \;\dashv\vdash\; (h \neq_H k) = \bot$$

in a Hausdorff space H.

The universal quantifier $\forall$ in a compact space is related to $\forall$:

$$(\forall x.\, \phi x = \top) \;\dashv\vdash\; (\forall x.\, \phi x) = \top$$

(Existential quantifiers in an overt space too???)

A more complicated example

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We write the property that ϕ intersects I as $\lozenge\,\phi$.

The $\lozenge$ operator defines an overt subspace by

$$I \;\equiv\; \{x : X \mid \forall \theta.\theta x \Rightarrow \lozenge\,\theta\}.$$

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Then $I \subset X$ is connected if

$$\Diamond\,\top \;\Leftrightarrow\; \top \quad \text{and} \quad \ldots,\ \phi,\psi : \Sigma^X,\ \phi\lor\psi = \top_I \vdash \Diamond\,\phi\land\Diamond\,\psi \;\Rightarrow\; \Diamond(\phi\land\psi).$$

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$$I \;\equiv\; \{x : X \mid \forall\theta.\theta x \Rightarrow \Diamond\,\theta\}.$$

Then $I \subset X$ is connected if

$$\Diamond\top \;\Leftrightarrow\; \top \quad \text{and} \quad \ldots,\ \phi,\psi : \Sigma^X,\ \phi\vee\psi = \top_I \vdash \Diamond\,\phi\wedge\Diamond\,\psi \;\Rightarrow\; \Diamond(\phi\wedge\psi).$$

The second clause of connectedness is

$$\forall\phi,\psi.\Big(\forall x.\ (\forall\theta.\theta x \Rightarrow \Diamond\,\theta) \;\Rightarrow\; \phi x\vee\psi x\Big) \Longrightarrow \Big(\Diamond\,\phi\wedge\Diamond\,\psi \Rightarrow \Diamond(\phi\wedge\psi)\Big).$$

A new language for topology

Since $\Rightarrow$, $\wedge$, $\vee$ (in Σ) and $=_N$, $\biguplus$, $\exists$ (discrete, compact, overt) are special cases of $\Rightarrow$, &, $\vee$, $=$, $\forall$, $\exists$ we can just use the traditional symbols.

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- The inner one provides the terms of type Σ, which are observable properties or open subspaces; computably continuous functions are derived from these.
- The outer one is the logic of provable properties and general subspaces.

We may form $=$, $\neq$, $\forall$ or $\exists$ within the inner calculus so long as the relevant space is discrete, Hausdorff, compact or overt, as in the old calculus.
The other cases, including $\Rightarrow$, take us to the outer calculus.

Two languages in a new theory of topology

- All maps are automatically continuous and computable.
- They represent computationally observable properties.
- Subspaces represent provable properties.
- Define subspaces as mathematicians (not set theorists) use set theory, *e.g.* $K \equiv \{x : X \mid \forall\phi.\ \Box\phi \Rightarrow \phi x\}$.
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But, as it stands we do not necessarily have the "correct" topology (whatever that is, which I shall not discuss now) or all of the exactness properties (of ASD) that we would like.

We need some extra axioms

Writing

$$\phi \overset{\mathfrak{q}}{\sim} \psi \;\equiv\; \forall a.\, \mathfrak{q}(a) \Rightarrow \phi a = \psi a,$$

I would like to simplify the expression

$$\mathfrak{q}^{\mathfrak{p}}(F) \;\equiv\; \forall \phi\psi.\, \mathfrak{p}(\phi)\ \&\ (\phi \overset{\mathfrak{q}}{\sim} \psi) \implies F\phi = F\psi,$$

which occurs in $\Sigma^2(A, \mathfrak{p}, \mathfrak{q}) = (\Sigma^2 A, \mathfrak{p}^q, \mathfrak{q}^{\mathfrak{p}^{\mathfrak{q}}})$, to

$$\exists G.\, F \overset{\mathfrak{p}}{\sim} G\ \&\ \mathfrak{q}^{\top}(G).$$

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Could it be consistent using the weak $\exists$ that we defined using equideductive logic?

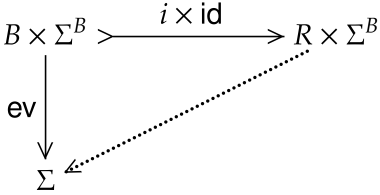
A critical example

$B \equiv \mathbb{N}^{\mathbb{N}}$ is not locally compact,

so $i : B \equiv \mathbb{N}^{\mathbb{N}} \rightarrowtail R$ (where $R \equiv \Sigma^{\mathbb{N}\times\mathbb{N}}$ or $\mathbb{N}_{\perp}^{\mathbb{N}}$) is not Σ-split,

i.e. there is no $I : \Sigma^B \to \Sigma^R$ with $\Sigma^i \cdot I = \mathrm{id}$.

Hence there is no diagonal fill-in



so $\Sigma^{i\times\mathrm{id}}$ is not surjective.

($(-) \times \Sigma^B$ is crucial to this counterexample.)

Conjecture: $\Sigma^{i\times\mathrm{id}}$ could still be regular epi.

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Previous work in ASD (and other topological approaches to computation) has shown that we have the right terms and equations for types $\mathbb{N}$, $\Sigma^{\mathbb{N}}$, *etc*.
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Idea: interpret proofs in the extended (equideductive) logic in the terms of the basic calculus.

Use realisability to do this.

Equideductive realisability calculus

We use judgements of the form

$$\vec{x} : \vec{X}, \quad \vec{\xi} \Vdash \vec{\mathfrak{p}} \quad \vdash \quad \Phi \Vdash \mathfrak{q}.$$

Realisers for $\forall\!\Rightarrow$ are given by λ-terms:

$$\frac{\vec{x} : \vec{X}, \quad \vec{\xi} \Vdash \vec{\mathfrak{p}}, \quad \vec{y} : \vec{Y}, \quad \vec{\zeta} \Vdash \vec{\mathfrak{q}} \quad \vdash \quad \Phi \Vdash \mathfrak{r}}{\vec{x} : \vec{X}, \quad \vec{\xi} \Vdash \vec{\mathfrak{p}} \quad \vdash \quad \lambda \vec{y} \vec{\zeta}.\, \Phi \Vdash \forall \vec{y}.\, \vec{\mathfrak{q}} \Rightarrow \mathfrak{r}} \ \forall I$$

$$\frac{\begin{array}{c} \vec{x} : \vec{X},\, \vec{\xi} \Vdash \vec{\mathfrak{p}} \quad \vdash \quad \Theta \Vdash \forall \vec{y}.\, \vec{\mathfrak{q}} \Rightarrow \mathfrak{r} \\ \vec{x} : \vec{X} \vdash \vec{b} : \vec{Y} \\ \vec{x} : \vec{X},\, \vec{\xi} \Vdash \vec{\mathfrak{p}} \quad \vdash \quad \vec{\Psi} \Vdash [\vec{b}/\vec{y}]^* \vec{\mathfrak{q}} \end{array}}{\vec{x} : \vec{X}, \quad \vec{\xi} \Vdash \vec{\mathfrak{p}} \quad \vdash \quad \Theta \vec{b} \vec{\Psi} \Vdash [\vec{b}/\vec{y}]^* \mathfrak{r}} \ \forall E$$

Compact hypothesis principle

Suppose that (we have a proof that)

$$x : X, \quad \mathfrak{q}(x) \equiv \forall y : Y.\ \mathfrak{p}(x,y) \Rightarrow \phi xy \quad \vdash \quad \psi x.$$

Topologically, this means that

$$\text{if} \quad x \in X \quad \text{satisfies} \quad Z_x \subset U_x \quad \text{then} \quad x \in V,$$

where

$$Z_x \equiv \{y : Y \mid \mathfrak{p}(x,y)\}, \quad U_x \equiv \{y : Y \mid \phi xy\}, \quad V \equiv \{x : X \mid \psi x\}.$$

Then there should be an X-indexed family of compact subspaces $K_x \subset Y$ with necessity operator $Ax\theta$ (or a proper map $W \to X$) such that

$$x : X, \quad \theta : \Sigma^Y, \quad \forall y : Y.\ \mathfrak{p}(x,y) \Rightarrow \theta y \quad \vdash \quad Ax\theta$$

and

$$x : X, \quad Ax(\lambda y.\ \phi xy) \vdash \psi x.$$

Topologically, these statements mean that

$$K_x \subset Z_x \qquad \text{and} \qquad \text{if} \quad K_x \subset U_x \quad \text{then} \quad x \in V.$$

An adjoint function conjecture

Ideally, we would like

$$\mathfrak{q}^{\mathfrak{p}}(G) \;\dashv\vdash\; \exists\!\!\exists F.\, \mathfrak{q}^{\top}(F) \,\&\, \forall\phi.\, \mathfrak{p}(\phi) \Rightarrow (F\phi \Leftrightarrow G\phi),$$

Such an F would be given by either of the adjoints to the inclusion j in

$$\{F \mid \mathfrak{q}^{\top}(F)\} \overset{j}{\rightarrowtail} \{G \mid \mathfrak{q}^{\mathfrak{p}}(G)\} \rightarrowtail \Sigma^{\Sigma^X},$$

which would exist if we were working in a *topos*, because j preserves meets and joins.

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$$\{F \mid \mathfrak{q}^{\top}(F)\} \xrightarrow{\;j\;} \{G \mid \mathfrak{q}^{\mathfrak{p}}(G)\} \longrightarrow \Sigma^{\Sigma^X},$$

which would exist if we were working in a *topos*, because j preserves meets and joins.

In equideductive logic, we can *simulate* their construction by

$$\mathfrak{r}(G,\phi) \;\equiv\; \exists F.\, (\forall\theta.\, \phi \overset{\mathfrak{q}}{\sim} \theta \Rightarrow F\phi = F\theta) \;\&\; (\forall\theta.\, F\theta \Rightarrow G\theta) \;\&\; F\phi$$

$$\mathfrak{l}(G,\phi) \;\equiv\; \forall F.\, (\forall\theta.\, \phi \overset{\mathfrak{q}}{\sim} \theta \Rightarrow F\phi = F\theta) \;\&\; (\forall\theta.\, G\theta \Rightarrow F\theta) \;\Rightarrow\; F\phi,$$

although $\mathfrak{l}(G,-)$ and $\mathfrak{r}(G,-)$ are predicates and not terms of type Σ^{Σ^X}.

I would then like to use the compact hypothesis principle to replace the equideductive quantifier $\forall$ with a compact one $\forall\!\!\!\!\forall$.

Where do I stand with this?

- I need help from a proof theorist with the realisability. (There are some notes available for private circulation.)
- Maybe this would suggest a different, weaker but useful axiom.
- Equideductive logic as it stands, together with the lattice structure, Euclidean, Phoa and Scott principles, is valid in currently studied categories such as Simpson's QCB.
- Unlike the definition of QCB, it is entirely computable, with no underlying set theory.
- It is already a pretty good approximation to topology, at least for familiar spaces.
- In particular, it obeys the Heine–Borel theorem.
- I believe that a new axiom will be needed to eliminate similar pathologies at higher types.