# Equideductive Categories and their Logic

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## The take-home message from this lecture

(The abstract is much too ambitious for half an hour.)

Sober topological spaces form an equideductive category.
Equideductive logic is a logic for them.
This generalises ASD for locally compact spaces.

It is a logic of deductions about equations.
Its predicates have $\forall$, $\Rightarrow$ and $=$.

An "existential quantifier" $\ni$ is definable.
This corresponds to the epis in the category.

A generalisation of Scott's equilogical spaces
can be also defined using it, but we shall not have time for this.

## Not the definition of a topos

The wider aim: generalise Abstract Stone Duality from locally compact spaces to axioms for a CCC with equalisers and an object $\Sigma$ (Sierpiński space).

Take inspiration from the category of sets:

A topos

- has an internal Heyting algebra $\Omega$; and
- is cartesian closed, with equalisers as well as products, and all powers, in particular of $\Omega$.

Even though this is much weaker than the correct definition, these two ideas are surprisingly powerful.

First we study the interaction between equalisers and exponentials as a generalisation of the categorical part of ASD.

The lattice structure on $\Sigma$ will fit in later (not today).

## Equalisers of exponentials

Working with nested equalisers and exponentials is clumsy.

We want to write

$$E \equiv \{x \mid \forall y.\ \alpha x y = \beta x y\} \rightarrowtail X \underset{\beta}{\overset{\alpha}{\rightrightarrows}} \Sigma^Y$$

This can be justified both categorically and symbolically.

Warning: this quantifier $\forall$ belongs to the purely categorical structure. It can range over any object, but it yields a new kind of predicate or general subspace.

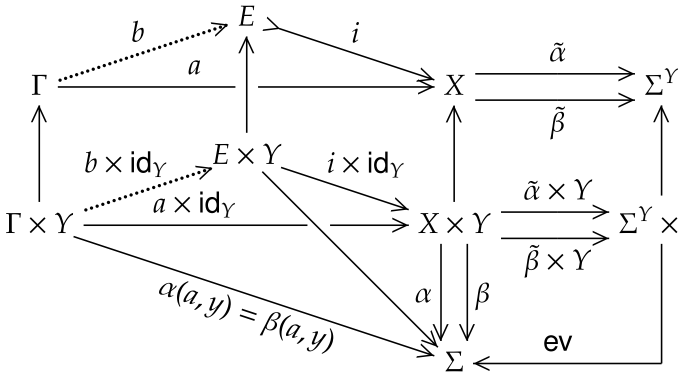
It is not the same as $\boldsymbol{\forall}$ in the lattice structure, which can only range over a compact space $K$, but is a term of type $\Sigma^{\Sigma^K}$.

The idea (later) is that $K$ will be compact iff

$$\forall x : K.\ (\phi x = \star) \quad \dashv\vdash \quad (\boldsymbol{\forall} x : K.\ \phi x) = \star.$$

# Partial products

(The formal category theory starts here.)
The universal property of an equaliser targeted at an exponential can be stated without mentioning $\Sigma^Y$ as a universal property called a partial product.




(NB: this is not quite the same as the usual notion, but partial equaliser would be an even worse name.)

# Working in a smaller category

Slogan: an equideductive category sits nicely within its cartesian closed extensions such as the Yoneda embedding.

If $X$ lies in the subcategory of the CCC then the equaliser

$$E \rightarrowtail X \rightrightarrows \Sigma^Y$$

also has $E$ within the subcategory.

The CCC will be "not much more complicated" than the subcategory.

Loose analogy: $\mathbb{R}$ within $\mathbb{C}$.

This lecture is about the subcategory, not the CCC.

# Which categories have partial products like this?

- Any topos, as equalisers of powersets.
- Richard Wood's $\mathbf{CCD}^{\mathsf{op}}$, for the same reason.
- Topological spaces in the traditional sense.
- The same, but sober and/or countably based.
- Probably not locales.
- Affine varieties over a field.
- Probably not affine varieties over a general ring.

(Even in the extremely unlikely event that locales have partial products, they will still fail the additional requirements for an equideductive category.)

# Constructing partial products in these categoriesof spaces

Let $\mathcal{Q}$ be any category with finite limits such that each object $Y \in \mathcal{Q}$ has some object $SY \in \mathcal{Q}$ and a natural 1–1 function between hom-sets

$$\mathcal{Q}(- \times Y, \Sigma) \rightarrowtail \mathcal{Q}(-, SY).$$

Then $\mathcal{Q}$ has partial products.

This function is a bijection iff $SY \cong \Sigma^Y$ in $\mathcal{Q}$, but this is not needed.
For topological spaces, $SY$ is the topology of $Y$, equipped with the Scott topology.

## Constructing partial products of varieties

Let $Q$ be any category with finite limits such that each object $Y \in Q$ has some object $SY \in Q$ and a natural 1–1 function between hom-sets

$$Q(- \times Y, \Sigma) \rightarrowtail Q(-, SY).$$

Then $Q$ has partial products.

For vector spaces (modules over a field), there is 1–1 function

$$U \otimes V \longrightarrow \hom(V^\perp, U), \quad \text{where} \quad V^\perp \equiv \hom(V, K),$$

that is linear and natural in $U$.

Hence for an affine variety $Y$ corresponding to a ring $V$,

$$SY \equiv T(V^\perp)$$

has the require property, where
$T(V^\perp)$ is the tensor algebra on $V^\perp$ *quâ* vector space.

## The equideductive universal quantifier

Here is the first part of the symbolic calculus.
The symbolic rules for $\forall \Rightarrow$ are as you would expect:

$$\frac{\Gamma,\ x : A,\ \mathfrak{p}(x)\ \vdash \qquad \alpha x = \beta x}{\Gamma \qquad \vdash \forall x : A.\ \mathfrak{p}(x) \Rightarrow \alpha x = \beta x}\ \forall I$$

$$\frac{\Gamma \vdash a : A,\ \mathfrak{p}(a) \qquad \Gamma \vdash \forall x : A.\ \mathfrak{p}(x) \Rightarrow \alpha x = \beta x}{\Gamma \vdash \alpha a = \beta a}\ \forall E$$

Although $\forall \Rightarrow$ fundamentally has an equation on the right, we may define

$$\forall y.\ \big(\mathfrak{p}(y) \Rightarrow \forall z.\ (\mathfrak{q}(z) \Rightarrow \alpha xyz = \beta xyz)\big)$$

as
$$\forall yz.\ \big(\mathfrak{p}(y)\ \&\ \mathfrak{q}(z) \Rightarrow \alpha xyz = \beta xyz\big).$$

## More categorical properties of sober spaces

The category $Q \equiv \mathbf{Sob}$ of sober topological spaces has finite limits and partial products.

Partial product inclusions ($\mathcal{M}$) are exactly subspace inclusions.

There is a full subcategory $\mathcal{A} \equiv \mathbf{LKSp} \subset Q$ for which
- for each object $A \in \mathcal{A}$, the exponential $\Sigma^A$ exists;
- such exponentials $\Sigma^A$ are injective with respect to $\mathcal{M}$-maps;
- every object $X \in Q$ has an $\mathcal{M}$-map $X \rightarrowtail \Sigma^A$ to an injective.

$\mathcal{A}$ may consist of either all locally compact spaces or just algebraic lattices with the Scott topology.

The functor $\Sigma^{(-)} : \mathcal{A}^{\mathsf{op}} \to \mathcal{A}$ reflects isomorphisms.
If $\mathcal{A}$ consists of all locally compact spaces then this adjunction $\Sigma^{(-)} \dashv \Sigma^{(-)}$ is monadic.

Abstractly, we call the objects of $\mathcal{A}$ urtypes.

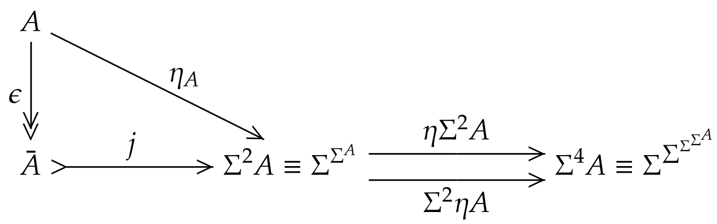
All objects of $Q$ respect these universal properties.

## The Definition

Putting these ideas together,
an equideductive category consists of
- a category $Q$ with all finite limits;
- a pointed object $\star : \mathbf{1} \to \Sigma$ in $Q$; and
- a full subcategory $\mathcal{A} \subset Q$ of urspaces; such that
- $\mathcal{A} \subset Q$ is closed under products;
- $\Sigma$ and all powers $\Sigma^A$ for $A \in \mathcal{A}$ exist in $Q$ and belong to $\mathcal{A}$;
- $Q$ has partial products based on the object $\Sigma$;
- $\Sigma$ is injective with respect to all of the maps in the class $\mathcal{M}$;
- there are enough injectives, *i.e.* every $Q$-object $X$ is the source of some $\mathcal{M}$-map $X \rightarrowtail \Sigma^A$ with $A \in \mathcal{A}$; where
- $\mathcal{M}$ is the class of monos defined by the partial products and intersections;
- all objects $\Gamma \in Q$ respect the universal properties mentioned.

## Sobriety in equideductive categories

Following a theme in synthetic domain theory (Phoa, Hyland, me), let $A \in \mathcal{A}$ and consider the equaliser




Then

- $j : \bar{A} \rightarrowtail \Sigma^2 A$ is a partial product inclusion ($\mathcal{M}$) and
- $\epsilon \times Y : A \times Y \to \bar{A} \times Y$ is epi for any object $Y$ ($\mathcal{E}$).

For the second, we need to be careful because the exponentials $\Sigma^n A$ are assumed to exist, but those of $\bar{A}$ and $Y$ are not.

## Sobriety in equideductive categories

The classes

- $\mathcal{E}$ of epis that are stable under $- \times Y$ and
- $\mathcal{M}$ of partial product inclusions

are orthogonal in the sense of a factorisation system.

Since every object $X$ has an $\mathcal{M}$-map $X \rightarrowtail \Sigma^B$, we have:

Theorem: Every exponentiable object $A$ is sober:
$\epsilon : A \cong \bar{A}$ and $A \rightarrowtail \Sigma^2 A \rightrightarrows \Sigma^4 A$ is an equaliser.

In fact, if $\mathcal{A}$ consists of all objects $A$ for which $\Sigma^A$ exists then the adjunction $\Sigma^{(-)} \dashv \Sigma^{(-)}$ between $\mathcal{A}^{\mathsf{op}}$ and $\mathcal{A}$ is monadic.

## Sobriety in equideductive logic

We add (the restricted $\lambda$-calculus and) sobriety to the symbolic calculus.

We write $\mathsf{prime}(P)$ for the predicate

$$\forall \Phi : \Sigma^3 A.\ \ \Phi P \;=\; P\big(\lambda x.\, \mathcal{F}(\lambda \phi.\, \phi x)\big)$$

To capture sobriety in an equideductive category we add this rule to equideductive logic:

$$\frac{\Gamma \quad \vdash \quad P : \Sigma^{\Sigma^X}, \quad \mathsf{prime}(P)}{\Gamma \quad \vdash \quad \mathsf{focus}\, P : X, \quad \forall \phi.\ \phi(\mathsf{focus}\, P) = P\phi}$$

In equideductive logic, primality can be conditional on other hypotheses and combined with other predicates.

## Equideductive logic

Like predicate calculus, a two-level theory.

The object language has

- urtypes with $\times$ and $\Sigma^{(-)}$
- terms from the restricted $\lambda$-calculus
- with focus, possibly formed conditionally on predicates as hypotheses.

The predicates are built from

- equality of terms;
- conjunction (intersection of subobjects); and
- quantified implication.

We combine $\forall$ and $\Rightarrow$ because of the variable-binding rule:
in the expression $\forall \vec{y}.\ \mathfrak{p}(\vec{y}) \implies \alpha\vec{x}\vec{y} = \beta\vec{x}\vec{y}$,
all of the variables on the left of $\Rightarrow$ must be bound by $\forall$.

This is because the target of the equaliser was $\Sigma^Y$,
not a dependent type.

## Predicates are not terms

In a topos, all of the logic (for sets) is within Ω:

$$\forall_X : \Omega_X \to \Omega \qquad \exists_X : \Omega_X \to \Omega.$$

In equideductive logic (for topology) we distinguish

- terms of type $\Sigma^A$, which classify open subspaces, from
- predicates, which describe general subspaces.

## Equideductive translation of rules

An algebraic theory may be presented using judgements

$$x : X,\ y : Y,\ \ldots,\ a = b,\ c = d,\ \ldots \quad \vdash \quad e = f$$

which we re-write in equideductive logic as

$$\forall x : X. \forall y : Y. \ldots\ a = b \ \&\ c = d \ \&\ \ldots \quad \Longrightarrow \quad e = f,$$

in which all of the variables are bound by ∀.

Then a rule

$$\frac{x : X,\ y : Y,\ \ldots,\ a = b,\ c = d,\ \ldots \quad \vdash \quad e = f}{u : U,\ v : V,\ \ldots,\ g = h,\ k = \ell,\ \ldots \quad \vdash \quad m = n}$$

is re-written as

$$\big(\forall x : X. \forall y : Y. \ldots\ a = b \ \&\ c = d \ \&\ \cdots \Rightarrow e = f\big)$$

$$\Longrightarrow \big(\forall u : U. \forall v : V. \ldots\ g = h \ \&\ k = \ell \ \&\ \cdots \Rightarrow m = n\big).$$

## Apply this translation to the rule for ∃

Recall that the ∋-elimination rule is

$$\frac{y : Y, \quad x : X, \quad \mathfrak{p}(x, y) \quad \vdash \quad (\phi y = \psi y)}{y : Y, \quad \ni x \colon X.\ \mathfrak{p}(x, y) \quad \vdash \quad (\phi y = \psi y)}$$

where it will be necessary to be more explicit about the context (parameters) $\Gamma \equiv [y : Y]$.

In the case $\mathfrak{q}(y) \equiv (\phi x = \psi y)$ this translates to

$$\vdash \quad \big(\forall y'.\ \forall x'.\ \mathfrak{p}(x', y') \Rightarrow (\phi y' = \psi y')\big)$$

$$\Longrightarrow \big(\forall y.\ \big(\ni x \colon X.\ \mathfrak{p}(x, y)\big) \Rightarrow (\phi y = \psi y)\big).$$

We may re-arrange this into We turn this into the definition of ∋

$$y : Y, \quad \vdash \quad \ni x \colon X.\ \mathfrak{p}(x, y) \quad \vdash \ \equiv$$

$$\big(\forall y'.\ \forall x'.\ \mathfrak{p}(x', y') \Rightarrow (\phi y' = \psi y')\big) \Longrightarrow (\phi y = \psi y).$$

## Does ∋ satisfy the rules?

The introduction rule is easily satisfied:

$$x : A,\ y : B, \quad \mathfrak{p}(x, y) \quad \vdash \quad \ni y.\ \mathfrak{p}(x, y).$$

The elimination rule

$$\frac{x : A,\ y : B, \quad \mathfrak{p}(x, y) \quad \vdash \quad \mathfrak{r}(x)}{x : A, \quad \ni y \colon B.\ \mathfrak{p}(x, y) \quad \vdash \quad \mathfrak{r}(x)}$$

is provable, so long as

- the parameters $x$ belong to urtypes $A$ with no other predicates and
- $\mathfrak{r}(x)$ obeys the variable-binding rule.

We have the weak Frobenius law

$$\mathfrak{q}(z) \ \&\ \ni y.\ \mathfrak{p}(x, y) \quad \dashv\vdash \quad \ni y.\ \mathfrak{q}(z) \ \&\ \mathfrak{p}(x, y),$$

so long as $x$ and $z$ are distinct variables (or disjoint sets of them).

## Making a category from the logic

Objects: $\{\phi : \Sigma^A \mid \mathfrak{p}(\phi)\}$
(exponentials are injective, which makes the definition of morphisms simpler)

Morphisms $\{\phi : \Sigma^A \mid \mathfrak{p}(\phi)\} \to \{\psi : \Sigma^B \mid \mathfrak{q}(\psi)\}$
are $F : \Sigma^A \to \Sigma^B$
such that $\phi : \Sigma^A,\ \mathfrak{p}(\phi) \quad \vdash \quad \psi(F\phi)$
subject to an equivalence relation that $F \sim G$ if
$\phi : \Sigma^A,\ \mathfrak{p}(\phi) \quad \vdash \quad F\phi = G\phi$.

The definitions of the objects and morphisms of the CCC are similar to this but complicated by another predicate on $A$ itself that is used to define a quotient of $\Sigma^A$.

## Epis and monos in topology

In suitable categories, any map $f$ factorises as both

- $f = m \cdot e'$ with $m$ mono and $e'$ regular epi, and
- $f = m' \cdot e$ with $m'$ regular mono and $e$ epi.

Both factorisations are important.

For sober topological spaces,

- a mono is 1–1 on points;
- a regular mono has the subspace topology;
- an epi need not be surjective on points ($\mathbb{N} \twoheadrightarrow \varpi$);
- a regular epi has the quotient topology.

There are irreversible implications

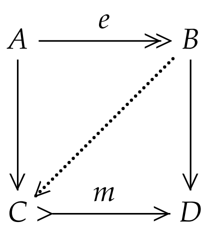
$$\text{regular epi} \Longrightarrow \text{surjective} \Longrightarrow \text{epi} \Longrightarrow \text{dense image}$$

$$\mathbb{R} \to S^1 \qquad \mathbf{2} \to \Sigma \qquad \mathbb{N} \to \varpi \qquad \mathbb{Q} \to \mathbb{R}$$

## Universal property

A factorisation system is a pair $(\mathcal{E}, \mathcal{M})$ of classes of maps with

- closure under composition and isomorphisms;
- factorisation of any map as $m \cdot e$;
- orthogonality: in any commutative square




there is a unique fill-in that makes both squares commute.

Then there is the question of whether $\mathcal{E}$ (or the factorisation) is preserved by pullbacks or products.

## Existential quantification and factorisation

For intuitionistic set theory, we have an agreement amongst

| idiom | types | categories |
|---|---|---|
| hide a known witness | introduce | the epi |
| pretend have witness | eliminate | orthogonality |
| substitute | commute CUT | pullback-stable |

Martin Hyland and Andrew Pitts, *The Theory of Constructions: Categorical Semantics and Topos-Theoretic Models*, 1989.

Also PT, *Practical Foundations of Mathematics*, Section 9.3.

## What might happen in topology?

Since epis are not stable under pullback, we have bad news:

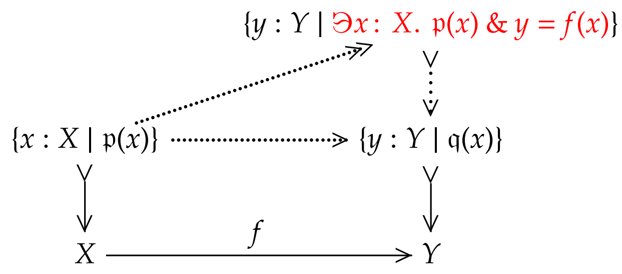
| idiom | types | categories |
|---|---|---|
| hide known witness | introduce | the epi |
| pretend have witness | eliminate | orthogonality |
| ~~substitute~~ | ~~commute CUT~~ | ~~pullback-stable~~ |

For sober topological spaces, epis are stable under product, but not pullback.

Even that fails for locales.
The epi $\mathbb{Q} \twoheadrightarrow QE$ between the rationals with the discrete and Euclidean topologies is not preserved by the product $(-) \times QE$.

## What about epi–mono factorisation?

CanWhen we factorise the morphism




by injectivity, inclusions compose,
so the intermediate object (the image)
must be another subspace of $Y$.

What is the predicate on $Y$? We guess the quantifier.
This works, with orthogonality too!

## The quantifier and coproducts

Suppose that the diagram of sets and functions

$$X \xrightarrow{f} Z \xleftarrow{g} Y$$

is a coproduct. Then it is the union ($\vee$) of the images ($\exists$),

$$z : Z \quad \vdash \quad (\exists x.\, z = fx) \quad \vee \quad (\exists y.\, z = gy),$$

and there are lots of uses of this property in category theory, combinatorics, programming, *etc*.
Coproducts may also be constructed in equideductive logic using the same idea,

$$\{A \mid \mathfrak{p}\} \xrightarrow{f} \{\Sigma^{\Sigma^A \times \Sigma^B} \mid \cdots\} \xleftarrow{g} \{A \mid \mathfrak{q}\}$$

where the predicate $\cdots$ is

$$(\exists x.\, \mathfrak{p}(x) \,\&\, z = fx) \quad \vee \quad (\exists y.\, \mathfrak{q}(y) \,\&\, z = gy).$$

Then the category is extensive:
it has coproducts that are stable and disjoint.

## Idiomatic use of the existential quantifier

In the middle of a mathematical argument (in any discipline), we might invoke some axiom or theorem that says $\exists x.\, \mathfrak{p}(x)$.

After this, we argue as if we have an $x$ with $\mathfrak{p}(x)$.

What does this mean?

Hilbert wrote $\epsilon x.\, \mathfrak{p}(x)$ and Bourbaki wrote $\square x.\, \mathfrak{p}(x)$ for $x$.

They thought that some choice of $x$
needed to be (and could be) derived from $\mathfrak{p}$.

In fact, there is no need for this.
The proof-theoretic rules agree with the idiom.
See *Practical Foundations of Mathematics*, Section 1.6.

## Extending the idiom to Ǝ

Recall that the weak Frobenius law

$$\mathfrak{r}(z) \;\&\; Ǝx.\, \mathfrak{p}(x, y) \quad \dashv\vdash \quad Ǝx.\; \mathfrak{r}(z) \,\&\, \mathfrak{p}(x, y),$$

holds so long as $y$ and $z$ are distinct variables (or disjoint sets of them).

This means that we can only use Ǝ$E$
(the “there exists” idiom)
when there are no other constraints on the parameters.

This restriction is in the same spirit as
- only $\lambda$-abstract (logical) terms of type $\Sigma$;
- the variable-binding rule;

*etc.*

We accept restrictions such as these on the use of ordinary logic in return for obtaining topological results instead of set-theoretic ones.

## Pretending that we have a witness in analysis

Given two maps $f, g : \mathbb{R} \to \mathbb{R}$,

$$\mathbb{Q} \twoheadrightarrow \mathbb{R} \overset{f}{\underset{g}{\rightrightarrows}} \mathbb{R}$$

to show that $f = g$, it is enough to check on the rationals.

This is because $\mathbb{Q} \to \mathbb{R}$ has dense image and $\mathbb{R}$ is $T_1$.

In order to prove $f(x) = g(x)$ for arbitrary real $x$,
we may pretend that $x$ is rational.

## Pretending that we have a witness in domain theory

Consider the epi $\mathsf{List}(\mathbb{N}) \twoheadrightarrow \Sigma^{\mathbb{N}}$,
where $\mathsf{List}(\mathbb{N})$ is the discrete space of finite lists
and $\Sigma^{\mathbb{N}}$ is full powerset with the Scott topology.

We may pretend that any subset is a finite list,
when we want to deduce equations between Scott-continuous functions.

So this is a way of formulating the Scott principle.

Similarly, the Phoa principle can be stated as

$$x : \Sigma \quad \vdash \quad x = \bot \quad \vee \quad x = \top.$$

Further axioms, intended for example to control the topology on $\mathbb{N}^{\mathbb{N}^{\mathbb{N}}}$, could be perhaps be stated like this.

## Conclusion

Equideductive logic provides (the first part) of
a “predicate calculus” with =, &, ∨, ∀, Ǝ
for sober topological spaces.
In many ways it is similar to the one for sets.

The existential quantifier Ǝ may be suggestive of some interesting conjectures.

An equideductive category is not a CCC,
but a CCC like equilogical spaces can be built around it.