Order-theoretic fixed point theorems: Some messy history of messy classical proofs and a simple constructive proof with applications to messy algebra

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Warning

I regard Set Theory as a religion

because it's not possible to have a rational discussion about it.

I am a devout atheist.

Before you burn me at the stake as
a skeptic (= someone who thinks, *σκεπτομαι*)
or a heretic (= one who chooses, *αιρεω*),

please hear me out.

As you will see, I actually identify
with Ernst Zermelo and other early set theorists.

And later in this lecture is a beautifully simple
and widely applicable theorem.

Because, above all, I believe G.H. Hardy's dictum that
there is no permanent place in the world for ugly mathematics.

(*A Mathematician's Apology*, 1940, §10.)

(Hardy was a militant atheist too.)

A well known fixed point theorem

Let X be a partial order
such that every subset $S \subset X$ has a meet $\bigwedge S \in X$.
Let $s : X \to X$ be a monotone (= order-preserving) function.
Then s has a least fixed point.
It is given by $\bigwedge \{x \mid sx \le x\}$.

This is impredicative, as is everything in this talk.

Who first proved it?

Alfred Tarski, *A Lattice-Theoretic Fixed Point Theorem and its Applications*, Pacific Journal of Mathematics, **5** (1955) 285–309.

No! The main idea was known much earlier than this!
To whom?

Knaster's fixed point theorem

Let X be a family of subsets of a set Y
such that for every sub-family $S \subset X$,
the intersection $\bigcap S$ is also in the family X.
Let $s : X \to X$ be a monotone (= order-preserving) function.
Then s has a least fixed point.
It is given by $\bigcap \{x \mid sx \subset x\}$.

Bronisław Knaster, *Un Théorème sur les Functions d'Ensembles*,
Comptes Rendues of a meeting of
the Polish Mathematical Society in Warsaw in 1927,
published in its Annals, **6** (1928) 133–4.

But all that appears in print is:
"$h(X)$ étant une fonction monotone d'ensembles
et A un ensemble tel que $h(A) \subset A$,
il existe un sous-ensemble D de A tel que $D = h(D)$"

Zermelo's fixed point theorem

The fixed point theorem
that was later attributed to Knaster and Tarski
is just assumed in passing
in Zermelo's second proof of the well ordering principle.

So it was probably well known long before.

Ernst Zermelo,
Neuer Beweis für die Möglichkeit einer Wohlordnung,
Mathematisches Annalen **65** (1908) 107–128.

English translation in pages 193–198 of
Jan van Heijenoort, *From Frege to Gödel: a Source Book in Mathematical Logic, 1879–1931*, Harvard University Press, 1967.

This also proves a less trivial result that we will see shortly,
which is why I'm citing the English translation in this case.

Maximal points, without binary joins

Usually in algebra general and binary joins are very complicated (or don't exist at all).

Let X be a partial order
such that every chain $C \subset X$ has a join $\bigvee C \in X$
(or just some upper bound).
In particular, with $C \equiv \emptyset$, there is least element $\bot$
(or just some element).

Then X has a maximal point.

Who first proved this (using the Axiom of Choice)?

Max Zorn, *A Remark on Transfinite Algebra*,
Bulletin of the American Mathematical Society **41** (1935).

Except that he denied responsibility for it:

Paul Campbell, *The Origin of Zorn's Lemma*,
Historia Mathematica **5** (1978) 77–89.
Campbell's earliest citations are to Felix Hausdorff, 1906.

A fixed point theorem without binary joins

Again let X be a partial order
such that every chain $C \subset X$ has a join $\bigvee C \in X$,
in particular X has $\bot$.

Let $s : X \to X$ be monotone (preserve order).

Then s has a least fixed point.

Ordinal (transfinite) recursion

Here is what you will find verydepressingly frequently:

Let:
- $x_0 \equiv \bot$,
- $x_{\alpha+1} \equiv s(x_\alpha)$,
- $x_\lambda \equiv \bigvee \{x_\alpha \mid \alpha \in \lambda\}$

where λ is a limit ordinal, which is a chain.

Then x_κ is the least fixed point.

But this is not a proof!!! (What's κ?)

Why isn't it a proof?

This is recursion,
whereas ordinals are defined in terms of induction,
so there is a theorem to be proved.

This theorem appears in most set theory textbooks,
usually without attribution, so who first proved it?

John von Neumann. Über die Definition durch transfinite Induktion und verwandte Fragen der allgemeinen Mengenlehre, *Mathematisches Annalen*, 99:373–393, 1928.

Also, ordinals form a proper class, so when do we stop?

Cesare Burali-Forti. Una questione sui numeri transfiniti. *Rendiconti del Circolo matematico di Palermo*, 11:154–164, 1897.

Who answered the *questione*?

Friedrich Hartogs. Über das Problem der Wohlordnung. *Mathematische Annalen*, 76:590–5, 1915.

More reasons why it's not a proof

Why is the stopping point a fixed point of s?

Identifying the reason for this
and for the uniqueness of any fixed point
may be a significant part of understanding your application.

Hint: algebras and coalgebras for functors.

Also, the traditional theory of ordinals makes very heavy use of excluded middle (and impredicativity).

Maybe we would like to develop some new theory that generalises the idea of ordinals in a non-obvious way (and doesn't use Excluded Middle).

Above all, transfinite recursion is
a huge piece of machinery that is very clumsy.

And it never was necessary...

Transfinite recursion is unnecessary

Casimir (Kazimierz) Kuratowski,
Une Méthode d'élimination des Nombres Transfinis des Raisonnements Mathématiques,
Fundamenta Mathematicae **3** (1922) 76–108.

He gave lots of examples from set theory, topology and measure theory that had been proved using ordinals and proved them more simply using closure operators.

Why did mathematicians get obsessed with ordinals and not follow Kuratowski's advice?

Sadly, he didn't even follow it himself:

His textbook, *Introduction to Set Theory and Topology*, 1961 contains the usual diet of set theory, cardinals and ordinals and no closure operators.

Use some subtlety!

Consider the subset $X_0 \subset X$ generated by

$$\bot, \qquad s, \qquad \bigvee_C$$

for all chains C. Then X_0 is already well ordered!

Who proved this and when?

Bourbaki, *Sur le théorème de Zorn*,
Archiv der Mathematik **2** (1949) 434–7.

Ernst Witt, *Beweisstudien zum Satz von M. Zorn*,
Mathematische Nachrichten **4** (1951) 434–8.

Who first proved Bourbaki–Witt?

Consider the subset $X_0 \subset X$ generated by

$$\bot, \qquad s, \qquad \bigvee_C$$

for all chains C. Then X_0 is already well ordered.

Wrong attribution again!!

As with the “Knaster–Tarski” theorem,
the argument is already in Zermelo’s second proof (1908).

Also, the Wikipedia page on the *Bourbaki–Witt Theorem*,
whilst giving the correct citations,
wrongly claims that it was proved using transfinite recursion.

Using the Zermelo–Bourbaki–Witt (ZBW) theorem

Suppose we have a construction whose completed form is difficult to describe.

- It belongs to some universe X of similar gadgets.
- X is closed under unions of chains.
- We have some notion of attempt at the construction.
- There is a basic attempt $\bot$.
- There is a construction $s : X \to X$ that improves attempts,
- such that the completed one is the least fixed point of s.

Then the ZBW subset $X_0 \subset X$ generated by $\bot$, s and $\bigvee_C$:

- has a greatest element;
- this is the unique fixed point in X_0;
- it’s the least fixed point in X;
- so it’s the completed construction that we require.

Induction with Zermelo–Bourbaki–Witt

Since ZBW Theorem gives a well-ordering, it gives induction:

Suppose we have some property Φ of members of X such that

- The basic attempt $\bot$ has $\Phi(\bot)$;
- if $\Phi(x)$ then $\Phi(sx)$; and
- if all members x of a chain C have $\Phi(x)$ then $\Phi(\bigvee C)$.

Then the complete construction $\top$ has $\Phi(\top)$.

Proof: the subset $Y \equiv \{y \mid \Phi(y)\}$
satisfies all of the requirements that we put on X,
so $X_0 \subset Y$.

Zermelo knew and used this in 1908.

This theorem tells you more about the construction
than “Zorn’s Lemma” does.

What about a constructive version?

The Zermelo–Bourbaki–Witt theorem

- does not depend on the Axiom of Choice,
- does not even depend on Excluded Middle
  for the argument itself, but
- proves “well-ordering” in Cantor’s original sense,
  for which induction uses Excluded Middle very heavily,
  whilst
- everything in this topic is Impredicative.

(Constructivity of the ZBW argument was shown by
Todd Wilson, *An intuitionistic version of Zermelo’s proof that every choice set can be well-ordered*, JSL 66 (2001) 1121–6.)

Is there an analogue in which induction is constructive,
i.e. without Excluded Middle?

Yes there is, and the proof is much easier!

Directed joins instead of chains

A subset $C \subset X$ is called directed if

- $\exists z.\, z \in C$ and
- $\forall x, y \in C.\, \exists z \in C.\, x \leq z \geq y$.

Finitary things preserve directed joins.

We use them in intuitionistic algebra instead of chains.

We also use them in domain theory for semantics of programming languages.

From now on, $(X, \leq)$ has joins $\bigvee^{\uparrow}$ of all directed subsets.

For the constructive difference between chains and directed sets, see Andrej Bauer, *On the failure of fixed-point theorems for chain-complete lattices in the effective topos*, Electronic Notes in Theoretical Computer Science, 249 (2009) 157–167.

Let's try intuitionistic ordinals

The key non-constructivity issue is the confusion between

- the well founded relation (membership) $\beta \in \alpha$, which must be irreflexive; and
- the reflexive containment relation $\beta \subset \alpha$.

Classically, $\beta \subset \alpha \iff \beta \in \alpha \vee \beta = \alpha$.
So the successor $\alpha + 1$ is $\quad \alpha \cup \{\alpha\} \;=\; \{\beta \mid \beta \subset \alpha\}$.

Intuitionistically, there are (at least) two different notions:

- thin ordinals have $\alpha + 1 = \alpha \cup \{\alpha\}$,
- plump ordinals have $\alpha + 1 = \{\beta \mid \beta \subset \alpha\}$.

In fact the plump ordinals grow very fat and need Replacement to construct them in (pre)sheaf toposes.

Paul Taylor, *Intuitionistic sets and ordinals*, Journal of Symbolic Logic **64** (1996) 705–744.

Intuitionistic ordinals, algebraically

Consider a universe V of sets or ordinals.
The free algebra for s and (all) $\bigvee$ such that:

no condition	sets
$x \leq sx$	thin ordinals
$x \leq y \Longrightarrow sx \leq sy$	plump ordinals
$s(x \vee y) = sx \vee sy$	directed ordinals

André Joyal and Ieke Moerdijk, *Algebraic Set Theory*, Cambridge University Press, LMS Lecture Notes 220, 1995.

Despite a lot of work developing these two approaches, Hartogs' Lemma is irretrievably classical, so neither method could prove the intuitionistic fixed point theorem.

(That is, without bringing a new axiom out of a hat.)

Functions instead of sets

Consider *all* the functions $r : X \to X$ that are

- monotone: $x \leq y \Longrightarrow rx \leq ry$
- and inflationary: $x \leq rx$.

This inherits the pointwise order and directed joins.

For any two functions $r, s : X \to X$ like this and $x \in X$,

$$\begin{tikzcd}[arrows=dash]
r(sx) \arrow[d] \arrow[drr] & & s(rx) \arrow[d] \arrow[dll] \\
rx \arrow[dr] & & sx \arrow[dl] \\
& x &
\end{tikzcd}$$

so the same happens in the pointwise order: $\mathsf{id} \leq r, s \leq r \cdot s, s \cdot r$.

Therefore the poset of these functions is directed.

Domain theorists knew this *ages* ago, but didn't spot...
Since it also has directed joins, it has a top element t.

Pataraia's fixed point theorem

Lemma Every poset with directed joins has
a greatest monotone inflationary endofunction.

Theorem Let $s : X \to X$ be a monotone endofunction
on a poset with least element $\bot$ and directed joins.

Then s has a least fixed point (without excluded middle).

Proof: Let $X_0 \subset X$ be generated by $\bot$, s and $\bigvee^{\uparrow}$
as in the Zermelo–Bourbaki–Witt theorem.
Let $t : X_0 \to X_0$ be as in the Lemma.
Then $t\bot$ is the least fixed point of s.

Dito Pataraia, 1997, but never published before he died in 2011.

In fact his original proof was more complicated,
and Alex Simpson simplified it.

Simplifying further

Everything in X outside X_0 is useless.
Can we cut down to X_0 or something similar
without using second order logic?

We will want the fixed point to be unique,

$$x = sx \quad \text{and} \quad y = sy \quad \Longrightarrow \quad x = y.$$

(Remember this from applying Hartogs' Lemma?)

But we can weaken this condition:

$$x = sx \quad \leq \quad y = sy \quad \Longrightarrow \quad x = y.$$

This is enough to prove a neater form of the theorem:

My version of Pataraia's theorem

Given

- a partial order X with directed joins $\bigvee^{\uparrow}$
  and least element $\bot$;
- a monotone endofunction $s : X \to X$;
- it satisfies my special condition

$$x = sx \quad \leq \quad y = sy \quad \Longrightarrow x = y.$$

Then

- X has a top element $\top$;
- $\top$ is the unique fixed point $\top = s\top$;
- it obeys Pataraia induction:
  for any predicate Φ on X such that
  - $\Phi(\bot)$ holds;
  - $\Phi(x) \Longrightarrow \Phi(sx)$;
  - $\big(\forall x \in D.\ \Phi(x)\big) \Longrightarrow \Phi(\bigvee^{\uparrow} D)$ whenever $D \subset X$ is directed,

  we also have $\Phi(\top)$.

Proof of Pataraia, in my version

Let $t : X \to X$ be the greatest inflationary monotone
endofunction.
Then

$$\forall x{:}X. \quad \bot \ \leq\ x \ \leq\ sx \ \leq\ tx \ =\ s(tx),$$

whence $\quad \forall x. \quad t\bot \ =\ s(t\bot) \ \leq\ s(tx) \ =\ tx \ \geq\ x,$

so the $\leq$ is equality by the special condition and
$t\bot$ is the greatest element ($\top$) and unique fixed point.

Beware that we have to cut down the original poset
using the special condition or otherwise first:
if there was *already* a top element, t just gives it us back.

The least fixed point is more easily derived from
my version than *vice versa*.

The induction principle was first used by
Martín Escardó in *Joins in the complete Heyting algebra of nuclei*,
Applied Categorical Structures, 11 (2003) 117–124.

Achieving the Special Condition

$$\forall x, y \in X_0.\quad y = sy \quad \leq \quad x = sx \quad \Longrightarrow \quad x = y \qquad (0)$$

holds in the following situations, with

$$(1) \Longrightarrow (3) \Longrightarrow (0) \qquad \text{and} \qquad (2) \Longrightarrow (3) \Longrightarrow (0).$$

If X_0 is generated by $\bot$, s and $\bigvee^{\!\uparrow}$. (1)

If X has meets $\wedge$ and X_0 consists of the well founded elements,

$$x \leq sx \quad \text{and} \quad \forall u{:}X.\ su \wedge x \leq u \Longrightarrow u \leq x. \qquad (2)$$

If X_0 consists of the recursive or tightly well founded elements,

$$x \leq sx \quad \text{and} \quad \forall a{:}X.\ sa \leq a \Longrightarrow x \leq a. \qquad (3)$$

(The strange names are the poset forms of the categorical properties in my paper *Well founded coalgebras and recursion*.)

Example: quotients of algebras

Let C be a well-co-powered category with set-indexed colimits and epi–mono factorisation (like many familiar categories) and let $T : C \to C$ be a functor that preserves epis.

(Other categorical technology handles the apparent set theory and the following also works for algebras for a monad.)

Then the category of T-algebras has coequalisers (and so all set-indexed colimits).

Let $K \rightrightarrows X$ be a parallel pair of T-algebra homomorphisms.

Consider the preorder of C-epis $X \twoheadrightarrow Y$ that have equal composites from K.

Filtered colimits provide directed joins.

The C-coequaliser $X \twoheadrightarrow Q$ is the least element.

The successor is more difficult...

Example: quotients of algebras

The successor $X \twoheadrightarrow Y \twoheadrightarrow sY$ is constructed as a pushout:

$$\begin{tikzcd}
TK \arrow[r, shift left, "Tf"] \arrow[r, shift right, "Tg"'] \arrow[d] & TX \arrow[r, two heads] \arrow[d] & TQ \arrow[r, two heads] & TY \arrow[dd, dotted] \\
K \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & X \arrow[r, two heads] \arrow[d, two heads] & Q \arrow[dl] & \\
& Y \arrow[rr, dotted, two heads] & & sY
\end{tikzcd}$$

in which we always have $Y \leq sY$.
It's a T-algebra iff $sY \leq Y$ iff $Y \cong sY$ in the diagram.
It's the coequaliser of T-algebras iff

$$Y \leq sY \quad \wedge \quad (\forall A.\ sA \leq A \Longrightarrow Y \leq A)$$

(using epi–mono factorisation).
Then Pataraia's Theorem says that this exists.

Well founded elements and relations

A binary relation $\prec$ on a set A is a well founded relation if

$$\forall U \subset A.\quad \frac{\forall a{:}A.\ (\forall b{:}A.\ b \prec a \Rightarrow b \in U) \Longrightarrow a \in U}{\forall a{:}A.\ a \in U}$$

Any binary relation $\prec$ defines $s : \mathcal{P}(A) \to \mathcal{P}(A)$ by

$$sB \equiv \{c : A \mid \forall b{:}A.\ b \prec c \Longrightarrow b \in B\}.$$

Then $B \in \mathcal{P}(A)$ is a well founded element iff
$B \subset A$ is an initial segment and
the restriction of $\prec$ to B is a well founded relation.

By Pataraia's Theorem, $(A, \prec)$ has
a greatest well founded initial segment.

Then $(A, \prec)$ also admits recursion...

Well founded (or ordinal) recursion again

Let X be any set with a function $\Theta : \mathcal{P}(X) \to X$.
Then there is a unique function $r : A \to X$ such that

$$r(a) \quad = \quad \Theta\{r(b) \mid b \prec a\}.$$

My proof: Consider attempts,
which are partial solutions defined on initial segments.

There is an empty (least) attempt.

Any directed union of attempts is also an attempt.

The successor attempt is defined on the successor initial segment:

$$sr(c) \quad = \quad \Theta\{r(b) \mid b \in B \ \wedge \ b \prec c\}.$$

An attempt is a well founded element of this poset iff
its support is well founded in the poset of initial segments.

Then there is a greatest attempt and by induction it is total.

The 20th century pure mathematical curriculum

Zermelo's 1908 axiom system for mathematical foundations wasn't a good one,

because it pre-dated the 1930s insights of

- Emmy Noether's Moderne Algebra,
- Gerhard Gentzen's formulation of predicate logic, and
- Alonzo Church's Simple Type Theory.

Nevertheless, it was a clearly formulated working system.

He, Gerhard Hessenberg, Friedrich Hartogs, Kazimierz Kuratowski and others built foundational structures using it.

As someone who has also tried to build aspects of mathematics in deliberately simplified systems, I identify with these people.

But then mathematical foundations turned into theology:
if you couldn't prove your theorem, just add another axiom!

What about the Zermelo–Bourbaki–Witt theorem?

The fixed point theorem has applications throughout mathematics.

The ZBW Theorem should have been in the core curriculum.

However, so far as I know, the only non-logic textbook that includes it is

Serge Lang, *Algebra*, first published 1965.

But Lang's proof is rambling and appears in
an appendix to the nth printing, where $1 < n \leq 9$.

Is there some textbook
(maybe for some branch of Algebra)
that states and proves the theorem in the first chapter
and then uses it systematically to develop the subject?

Proving the "Bourbaki–Witt" theorem

Why did it take so long
for Zermelo's argument to be presented as a theorem in itself?

Why didn't it become a key part of the curriculum?

Try proving it for yourself!

Define $\quad R(x, y) \ \equiv \ y \leq x \ \vee \ sx \leq y.$

This satisfies

$$xR\bot, \qquad xRy \Longrightarrow yR(sx),$$

$$(\forall y \in C.\, xRy) \Longrightarrow xR(\bigvee C)$$

for any chain C.

Since the induction step switches the arguments,

- it's quite difficult to find a proof,
- but then there are multiple strategies.

Walter Felscher made a historical survey in *Doppelte Hülleninduktion und ein Satz von Hessenberg und Bourbaki*, Archiv der Mathematik, 13 (1962) 160–5.

Pataraia should be in the curriculum!

Pataraia's Theorem can do everything
that the older theorems could do,
including transfinite recursion.
It is a drop-in replacement for the older results.

But my version of it is a precision tool.

This belongs in the core of the curriculum!

Students would like the fact that it is easy to prove,
unlike the Zermelo–Bourbaki–Witt theorem.

That is, now that I have told you the idea!
You should be able to re-construct it yourself.

(As a domain theorist, I should have found Pataraia's theorem.
I didn't, because I had been brainwashed with set theory.)

Replacement for Replacement?

There was one legitimate charge of heresy:

Von Neumann's proof was published in the setting of the Axiom-Scheme of Replacement and allows transfinite construction of sets.

Make that iteration of functors.

The next stage in this work is the theory of well founded coalgebras, in which Mostowski's extension reflection is understood using factorisation systems.

An application to that is a categorical account
of thin and plump ordinals.
As a further application, transfinite iteration of functors
can be expressed as a left adjoint.

This is now something in the mother tongue of category theory.

Foundational questions

Finally, to my colleagues in
category theory, type theory and proof theory:

Pataraia's Theorem is constructive,
in the sense that it doesn't use excluded middle.

But notice how simple it is (in my version).

It has no "controversial" assumptions
apart from the essential one of directed joins.

So, if you're interested in foundations,
such as im/predicativity or computability,
you can analyse what is needed for this.

Postscript, June 2025: even simpler proof

Just like in Galois theory (fields and groups), write

$$s \perp x \quad \text{for} \quad sx = x$$

and generalise this to subsets:

$$\mathcal{S}^{\perp} \equiv \{x \in \mathcal{X} \mid \forall s \in \mathcal{S}.\, s \perp x\} \equiv \mathbf{Fix}\,\mathcal{S}$$

$$^{\perp}\mathcal{A} \equiv \{\mathsf{id} \leq s : \mathcal{X} \to \mathcal{X} \mid \forall x \in \mathcal{A}.\, s \perp x\}$$

so that $\quad \mathcal{A} \subset \mathcal{S}^{\perp} \iff \mathcal{S} \perp \mathcal{A} \iff \mathcal{S} \subset {}^{\perp}\mathcal{A}$.

Then, since inflationary monotone endofunctions compose
(as domain theorists such as me should have noticed,
but Dito Pataraia had to point out to us),

${}^{\perp}\mathcal{A}$ is directed, so has a greatest element.

The greatest element of ${}^{\perp}(\mathcal{S}^{\perp})$ is the required closure operator m.

There is a categorical version of this using well pointed
endofunctors in place of inflationary monotone endofunctors