A novel fixed point theorem, towards a replacement for replacement

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At a Category Theory conference

The other speakers have come here to show you their sophisticated categorical constructions.

I have some too: www.Paul Taylor.EU/ordinals/

- Well Founded Coalgebras and Recursion
  paper with referees,
  but Sections 2 & 8 have been rewritten since November
- Ordinals as Coalgebras
  replacing **Set** with **Pos** in the previous paper
  very rough early draft on the website
- Transfinite Iteration of Functors
  nowhere near ready for release
- slides for two recent seminars, plus this one
- historical material, including some translations, with more to come later.

At a Category Theory conference

Unfortunately, we will forget most of the details of each others' lectures about difficult constructions.

Instead I will give you a simple result about recursion,
mostly without diagrams,
that I needed for my categorical work.

It's (constructive and) much simpler than the classical catechism about Zorn's Lemma or counting past ∞.

It's so simple that you will be able to reconstruct it.

Your colleagues in many other subjects could use it.

But people will still insist that it must be proved using ordinals and wilfully mis-report the way that others have done it (for example Wikipedia on the "Bourbaki–Witt Theorem").

That is, even though Kazimierz Kuratowski gave us *Une Méthode d'élimination des Nombres Transfinis des Raisonnements Mathématiques* — 100 years ago!

Well-founded induction and recursion

One part of the machinery that would be needed for ordinal recursion is this:

A binary relation $\prec$ on a set A is a well founded relation if it obeys the induction scheme

$$\forall \phi{:}\Omega^A.\quad \frac{\forall a{:}A.\ \ (\forall b{:}A.\ \ b \prec a \Rightarrow \phi b) \Longrightarrow \phi a}{\forall a{:}A.\ \ \phi a}$$

From this we can derive well founded recursion:

For any $\theta : \mathcal{P}(\Theta) \to \Theta$,

$$r(a) \quad = \quad \theta\left(\{r(b) \mid b \prec a\}\right).$$

We will use the proof of this as our running example.

Initial segments and attempts

The proof is due to John (János Lajos) von Neumann, 1928.

The key ideas are:

Initial segments of the carrier set under the $<$ relation.

Attempts (partial solutions), defined on initial segments.

Successor initial segments and attempts.

Unions of initial segments and attempts.

The required total solution is

- the unique fixed point of the successor, which is
- the greatest initial segment and attempt.

This is a common situation in other forms of recursion
for example, well founded coalgebras.

Binary joins

Von Neumann's original proof (1928) considered
the union of all attempts.

This uses the well known fixed point theorem
for complete lattices that is mis-attributed to
Bronisław Knaster (1928) and Alfred Tarski (1955),
but was already well known in the early 20th century.

However, in many algebraic situations,
binary joins are at best unwieldy
but often don't exist.

So, classically, joins of chains were used to avoid them.

Constructively, we consider directed joins.

(I needed this to relax the requirements on the functor
in my 1990s work on well founded coalgebras.)

The Order-Theoretic Fixed Point Theorem

The underlying result that we require is this:

Let $(X, \leq)$ be an poset with

- least element $\bot$ and
- directed (instead of all) joins $\bigvee^{\uparrow}$, and
- an inflationary monotone endofunction $s : X \to X$.

Then s has a least fixed point.

(We will turn this conclusion into a sharper tool.)

Of course it is not legitimate to use "transfinite" recursion.

There is a classical proof,
but even when it is correctly stated,
it is mis-attributed
to Bourbaki (1949) and Ernst Witt (1951).

The classical credit belongs to Ernst Zermelo (1908).

Proof without excluded middle

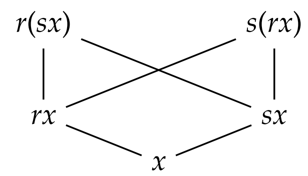
Dito Pataraia found a simple constructive proof in 1996/7,
based on a lemma that domain theorists (like me) missed:

Every dcpo (directed complete partial order) has
a greatest monotone inflationary endofunction.

Consider *all* the functions $r : X \to X$ that are

- monotone: $x \leq y \implies rx \leq ry$
- and inflationary: $x \leq rx$.

For any two functions $r, s : X \to X$ like this, and $x \in X$,



Therefore the poset of these functions is directed.

Since it also has directed joins, it has a top element t.

Discarding the rubbish

Before we can use Pataraia's Lemma
(or the Zermelo–Bourbaki–Witt one)
to find the least fixed point
(or use it to prove the recursion theorem),

we must cut down the original dcpo,
throwing out everything
above and beside the least fixed point,
since it is irrelevant to the problem.

Zermelo, Kuratowski, Bourbaki, Witt, Pataraia
and others all did this by considering
the subset generated by $\bot$, s and $\bigvee^{\uparrow}$.

But this uses second order logic.

It is a recursive *hors d'oevre*
before the main recursive meal.

Is there a better (first order) way of doing it?

My version of Pataraia's theorem

Suppose we have

- a partial order X with directed joins $\bigvee^{\uparrow}$ and least element $\bot$;
- an inflationary monotone endofunction $s : X \to X$;
- that satisfies $x = sx \;\le\; y = sy \Longrightarrow x = y$.

Then

- X has a top element $\top$;
- $\top$ is the unique fixed point $\top = s\top$;
- it obeys Pataraia induction:
  for any predicate Φ on X such that
  - $\Phi(\bot)$ holds;
  - $\Phi(x) \Longrightarrow \Phi(sx)$;
  - $\big(\forall x \in D.\, \Phi(x)\big) \Longrightarrow \Phi(\bigvee^{\uparrow} D)$ whenever $D \subset X$ is directed,

  we also have $\Phi(\top)$.

Well founded elements

Let $(A, \le)$ be any dcpo with $\bot$ and s.

Consider the $X \subset A$ defined by any of these conditions:

- $x \le sx$ and $\forall a.\, sa = a \Longrightarrow x \le a$
- $x \le sx$ and $\forall a.\, sa \le a \Longrightarrow x \le a$
- $x \le sx$ and $\forall a.\, sa \wedge x \le a \Longrightarrow x \le a$

Take your pick — they all do the same job.

The last is the poset translation of
the categorical definition of well founded coalgebra.

So we call the elements of X well founded elements.

Then $x, y \in X$ satisfy my "special condition"

$$x \;=\; sx \;\le\; y \;=\; sy \;\Longrightarrow\; x \;=\; y$$

(put $a \equiv x \in X$ in any of the properties of $y \in X$).

This is enough to turn the theorem into a sharper tool.

Characterising well founded elements

Whenever we have a dcpo with $\bot$ and a "successor" function
we can ask: what are its well founded elements?

For example,
any binary relation $\prec$ defines $s : \mathcal{P}(A) \to \mathcal{P}(A)$ by

$$sB \;\equiv\; \{c : A \mid \forall b{:}A.\;\; b \prec c \Longrightarrow b \in B\}.$$

Then $B \in \mathcal{P}(A)$ is a well founded element iff:
$B \subset A$ is an initial segment and
the restriction of $\prec$ to B is a well founded relation.

By Pataraia's Theorem, $(A, \prec)$ has
a greatest well founded initial segment.

It also admits Pataraia induction.

(Pataraia induction was defined on the previous slide,
but we will see a worked example shortly.)

Well founded attempts?

Returning to the proof of the recursion theorem,

Let $B \subset A$ be an initial segment of the relation $\prec$

and $A \supset B \xrightarrow{f} \Theta$ be an attempt with support B.

Then the successor attempt sf, with support $C \equiv sB$, is

$$sf(c) \quad = \quad \theta\{fb \mid b \in B \ \wedge \ b \prec c\}.$$

What are the well founded attempts?

They are more tricky to characterise,

but there's an easier way of doing it...

Functions and well founded elements

The hypotheses of the fixed point theorem $(\bot, s, \bigvee^{\uparrow})$
and the definition of well founded element

$$x \le sx \qquad \text{and} \qquad \forall a.\, sa \wedge x \le a \Longrightarrow x \le a$$

are algebraic, not logical ideas.

So we can ask about homomorphisms,
products, preservation, *etc.* of this structure.

For example,
the support function from attempts to initial segments
is a homomorphism.

What can we deduce from that?

Supports of attempts

In the traditional proof, the successor operation

$$sf(c) \quad = \quad \theta\{fb \mid b \in B \ \wedge \ b \prec c\}$$

lifts attempts with support B to support sB.

In fact, it defines a bijection,

so it proves $\Phi(B) \Longrightarrow \Phi(sB)$, where
$\Phi(B)$ says there is a unique attempt with support B.

Also $\Phi(\emptyset)$ holds and Φ preserves (directed) joins.

Hence $\Phi(A)$ holds for the unique fixed point,
i.e. for A, if it is well founded.

This is Pataraia induction,
first used constructively by Martín Escardó (2003)
but classically by Ernst Zermelo (1908).

Concluding the recursion theorem

We have proved that

In the dcpo $(\mathsf{Seg}, \subset)$ of initial segments,
well founded elements are well founded relations,
so my "special condition" in Pataraia's theorem is satisfied,
and the its successor has a unique fixed point.

The support function $\mathsf{supp} : \mathsf{Att} \longrightarrow \mathsf{Seg}$
is a homomorphism for all this structure,
each initial segment is the support of a unique attempt,
so supp is a bijection.

So we get a characterisation of well founded attempts for free.

Hence there is a unique solution to the recursion equation.

Other recursion theorems

We have dismantled von Neumann's recursion theorem.

The only thing left of it is the definition of successor.

Everything else has been transferred
to (my elaboration of) Pataraia's theorem.

For example,
initial segments of coalgebras are sub-coalgebras,
there is a successor functor and, in this structure,
well founded elements are well founded coalgebras.

This could be applied to more complicated situations,
maybe partial models of type theories.

We will now see some other applications of Pataraia's theorem...

Well (co-)powered categories

These methods about posets or propositions
can be transferred to categories or types.

A category is well (co-)powered if each object has
a set of incoming monos (outgoing epis).

For example, a topos, in both cases,
and lots of familiar "concrete" categories.

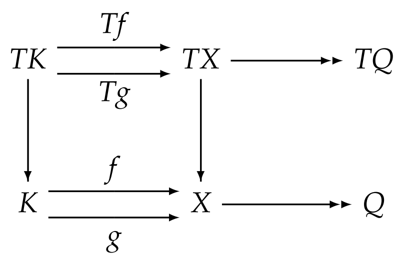
The "set" can be captured using fibred categories
(but we need a better textbook account of them).

We also need the cancellation properties of monos (epis)
and (co)filtered (co)limits of them.

Then the incoming monos and outgoing epis
form dcpos (up to equivalence).

Example: quotients of algebras

How do we construct the coequaliser of algebras?



Here Q is the coequaliser in the underlying category.

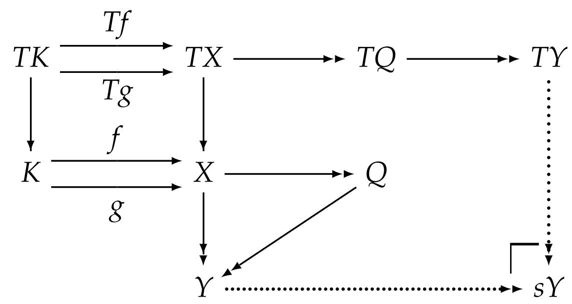
Some sort of iterative constuction is needed
to find the coqualiser of homomorphisms.

This is a classic supposed difficulty in our subject
that purportedly requires transfinite recursion.

Although we just consider T-algebras for a functor,
the method is easily adapted to monads.

Example: quotients of algebras

The successor $X \twoheadrightarrow Y \twoheadrightarrow sY$ is constructed as a pushout:



in which we always have $Y \leq sY$.
It's a T-algebra iff $sY \leq Y$ iff $Y \cong sY$ in the diagram.
It's the coequaliser of T-algebras iff

$$Y \leq sY \quad \wedge \quad (\forall A.\ sA \leq A \Longrightarrow Y \leq A)$$

for any homomorphism $K \rightrightarrows X \twoheadrightarrow A \hookrightarrow B$.

Example: quotients of algebras

This construction is carried out in the slice category of $\mathcal{C}$-epis $X \twoheadrightarrow Y$ that have equal composites from K.

This is because we need a preorder, for which we are using the cancellation property for epis.

So the functor $T : \mathcal{C} \to \mathcal{C}$ must preserve epis.

Filtered colimits provide directed joins.

The $\mathcal{C}$-coequaliser $X \twoheadrightarrow Q$ is the least element.

The coequalising homomorphism is the least fixed point.

Then Pataraia's Theorem says that this exists.

Mostowski's extensional quotient

A relation $\prec$ is extensional if

$$\forall\, x\, y. \quad (\forall z.\ z \prec x \iff z \prec y) \quad \Longrightarrow \quad x = y$$

so the coalgebra structure map $X \to \mathcal{P}(X)$ is mono.

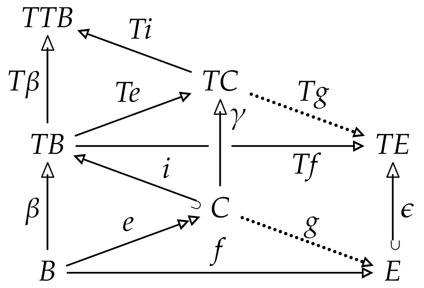
(Andrzej) Mostowski's theorem (using Replacement) says that every well ordering has an extensional quotient

but my 1996 JSL paper obtained this by an equivalence relation that was a bisimulation defined by co-recursion.

We can instead use epi–mono factorisation to define a successor and Pataraia's Theorem to give the quotient object.

Successor for the extensional quotient

Let $\beta : B \longrightarrow TB$ be a (well founded) coalgebra.
Form the epi–mono factorisation $\beta = e\,;i$:



The structure of C is given by appropriate composites, B is a fixed point iff it is extensional.

Restrict the dcpo using one of the versions of "well founded elements".
Then the fixed point exists and is the greatest element by Pataraia's theorem.

Successor for the extensional quotient

Now let $\epsilon : E \hookrightarrow TE$ be an extensional coalgebra and $f : B \to E$ a coalgebra homomorphism.
Then f factors through E by orthogonality of factorisation.

(Sorry, this was a last minute correction to the slides.
See Proposition 8.11 in the June 2023 version of *Well Founded Coalgebras and Recursion*.)

Generalised "Mostowski"

The idea of iterated factorisation of a coalgebra structure map is a purely categorical one.

So we can do it with any endofunctor of any category, equipped with any factorisation system.

It will converge, using Pataraia's theorem, if

- the "epis" have the usual cancellation property,
- they are well co-powered and
- have filtered colimits of that "size".

For example, using the lower sets functor on posets with inclusions with the restricted order, we obtain (one form of) ordinal rank.

Unfortunately, there are too many facts and fallacies to check for a short talk on ordinals as coalgebras.

In Memoriam Bill Lawvere 1937–2023

The historical research that I have done behind this work has shown me how badly set theory has swindled mathematics.

It is time to evict it from our Foundations.

We need a replacement for the axiom-scheme of Replacement.

Let us categorists speak our native language:

Bill taught us to use Adjointness in Foundations.

Let's frame the foundational question like this:

What functors can consistently be assumed to have adjoints?

Peter Freyd's "solution set condition"
just drags us back into set theory.
The point is to consider problems that
can be stated in a topos but not solved.

Transfinite iteration of functors

Alternatively, we could abandon convergence definable in the language of an elementary topos.

With a suitably constructed category of fibrations, transfinite iteration of functors is equivalent to a certain extensional reflection.

Transfinite iteration of functors is a procedure that is commonly used in various mathematical disciplines but goes beyond the logic of an elementary topos (or Zermelo set theory).

Are there other extensions that do not just use either universes or transfinite iteration of functors?