Ordinals as Coalgebras

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Symposium for Andy Pitts
University of Cambridge
Wednesday, 23 August 2023

`www.paultaylor.eu/ordinals/`

In honour of

Andy Pitts (retiring), who
supervised (tutored) me for Set Theory and Logic in 1981
and
organised (some conference) in 1992,
where I presented my early work on this topic.

and

Peter Aczel (passed away this month), who was
the editor at the Journal of Symbolic Logic for
my 1996 paper *Intuitionistic Sets and Ordinals*.

Categorical Set Theory

$\in$ and $\{\ \}$
are completely inappropriate as foundations for mathematics

because

they pre-date the 1930s revolutions in

- Algebra (Emmy Noether and Bartel van der Waerden),
- Logic (Kurt Gödel and Gerhard Gentzen) and
- Computability (Alonzo Church and Alan Turing).

But they're an interesting mathematical structure.

Let's do a categorical investigation of this structure!

Beware that each and every detail
that you know or remember about Set Theory
will make it more difficult to follow this lecture!

Objectives

To destroy the idea that
ordinals are totally/linearly ordered.

To find a replacement for Replacement
in the native language of category theory,
i.e. adjointness in foundations.

To extend the idea that sets ($\in$-structures)
provide partial models for
the non-extistent free algebra for the powerset functor
to more complex structures, such as type theories.

Categorical Set Theory

Gerhard Osius (1974)

represented any relation $(\prec)$ as a coalgebra $X \longrightarrow \mathcal{P}(X)$

and subsets (in the sense of set theory) as
coalgebra homomorphisms.

Dimitry Mirimanoff had recognised them
as bisimulations ("isomorphismes") in 1917,
although his wording was rather vague.

Osius re-constructed Zermelo set theory
within any elementary topos.

(Osius became a Professor of Statistics and died in 2019.)

Intuitionistic Ordinals, first attempt

Robin Grayson (1977) considered relations that are

- transitive,
- extensional and
- well founded.

The successor defined by $\alpha^+ \equiv \alpha \cup \{\alpha\}$ satisfies

$$\beta^+ \in \alpha^+ \iff (\beta^+ \in \alpha \ \vee \ \beta^+ = \alpha) \implies \beta^+ \subset \alpha \iff \beta \in \alpha$$

We will call these thin ordinals,
α^+ the thin successor and
$(\beta \leq \alpha) \ \equiv \ (\beta^+ \prec \alpha \ \vee \ \beta^+ = \alpha)$ the thin order.

Can we fix the one-way implication?

(Grayson couldn't get an academic job,
so became a schoolteacher and then a priest,
not far from the places where I grew up.)

Plump Ordinals

Trying to get

$$\beta^+ \in \alpha^+ \iff \beta^+ \subset \alpha \iff \beta \in \alpha$$

PT (1996)
first defined the fat successor as $\{\beta \mid \beta \subset \alpha\}$.

But this is too fat!

The plump successor required a difficult recursion
("recursively plump") to define it.

But that was in the Journal of Symbolic Logic —
Category theory will do it much more neatly!!

Algebraic Set Theory

André Joyal and Ieke Moerdijk (1994)
adapted the fibred category theory of open maps
to model (large) families of small sets.

They considered the free algebra X for
small joins $\bigvee$ and a successor $s : X \to X$ such that

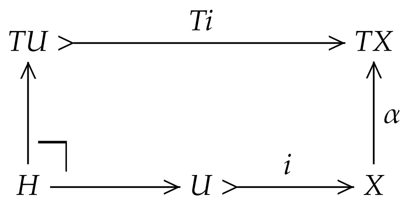
- no condition: "sets" ($\in$-structures);
- $x \leq sx$: thin ordinals;
- $x \leq y \Rightarrow sx \leq sy$: plump ordinals; or
- $s(x \vee y) = sx \vee sy$: directed ordinals,

where $\leq$ is the order defined from $\bigvee$.

I avoid large objects (universes) altogether
(treating them as schemes whenever possible)
but these conditions help us to understand
the different constructive systems of ordinals.

Well Founded Coalgebras

A coalgebra $\alpha : X \to TX$ for an endofunctor T
is well founded if any pullback



has $i : U \cong X$.

My book (1999) & earlier unpublished work
assumed that T preserves inverse images.

Well Founded Coalgebras and Recursion (2019–23)
only assumes that T preserves monos.

But we have nothing new to say about well-foundedness today.

Extensionality

The first of Zermelo's axioms of set theory:

$$(\forall z.\quad z \prec x \iff z \prec y) \quad\implies\quad x = y$$

or $$\{z \mid z \prec x\} = \{z \mid z \prec y\} \quad\implies\quad x = y$$

So, the structure map of the coalgebra is 1–1:

$$\mathcal{P}(X) \xleftarrow{\ \alpha\ } X$$

An innocent axiom?

Far from it!

We will examine two major theorems
about extensional well founded coalgebas.
(Proved in *Well Founded Coalgebras and Recursion*.)

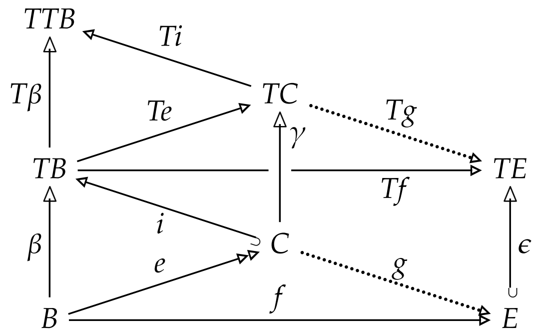
First theorem

For any functor $T : \mathbf{Set} \to \mathbf{Set}$ that preserves monos,
the category of
extensional well founded T-coalgebras
and coalgebra homomorphisms
is like the "von Neumann hierarchy" in set theory:

- it is a preorder
  (at most one morphism between any two objects);
- the underlying function of that morphism is 1–1;
- the preorder has binary meets
  like set-theoretic intersection,
  given by "zipping together" the coalgebras,
  whose underlying functions form a pullback
  if T preserves inverse images;
- $\emptyset$ is the least element;
- there are filtered/directed unions; and
- there are also binary joins like set-theoretic union.

Second theorem

We can turn any (well founded) coalgebra into an extensional
one by repeatedly factorising its structure map:



This does not require T to preserve monos.

It does require that epis
- have the cancellation property and
- be well co-powered.

This is our version of Mostowski's extensional quotient.

Let's pretend

that the category of posets $(X, \sqsubseteq)$ is a topos.

What could possibly go wrong?

Instead of the full powerset $\mathcal{P}$,
we use the covariant down-sets functor $\mathcal{D} : \mathbf{Pos} \to \mathbf{Pos}$:

$$
\begin{array}{lcl}
\mathcal{D}(X, \sqsubseteq) & \equiv & \{U \mid U \subset_{\downarrow} X\} \\
\mathcal{D}fU & \equiv & \{y \in Y \mid \exists x \in U.\, y \sqsubseteq_Y f(x)\} \\
U \subset_{\downarrow} X & \equiv & \forall x, y \in X.\, x \sqsubseteq y \in U \Rightarrow x \in U
\end{array}
$$

The purpose of a categorical investigation
is that you can do the same for other categories.

We use constructive reasoning and category theory
so that we understand the foundations of an argument
and can re-deploy elsewhere.

Lots of different kinds of monos!

There are (at least) three factorisation systems on **Pos**:

$\mathcal{I}$	monos	1–1 on points, need not reflect order;
$^{\perp}\mathcal{I}$	regepis	image generates the order on target;
$\mathcal{R}$	regmonos	inclusions with the induced order;
$^{\perp}\mathcal{R}$	epis	onto on points;
$\mathcal{L}$	lower	lower subset inclusions;
$^{\perp}\mathcal{L}$	cofinal	$\forall y.\, \exists x.\, y \sqsubseteq fx.$

Can we use these in place of 1–1 and onto functions
in the two theorems about extensionality?

Properties of the factorisation systems

Pretending that **Pos** is a topos,
how well can we use the factorisation systems
like 1–1 and onto functions?

There are lots of facts and fallacies to check:

	$\mathcal{I}$	$^{\perp}\mathcal{I}$	$\mathcal{R}$	$^{\perp}\mathcal{R}$	$\mathcal{L}$	$^{\perp}\mathcal{L}$
$\mathcal{D}$ preserves	N		Y		Y	
inverse images exist	Y		Y		Y	
$\mathcal{D}$ pres inv image	N		N		Y	
cancellation	Y	Y	Y	Y	Y	N
well (co)powered	Y	Y	Y	Y	Y	N
nice pushouts	N		N		Y	

So for each of $\mathcal{I}$, $\mathcal{R}$ and $\mathcal{L}$, some things work, others don't.

(In fact there are lots more than this!)

Coalgebras for the lower sets functor

A coalgebra $\alpha : X \to \mathcal{D}X$ in **Pos**

has two order relations $(\sqsubseteq)$ and $(\prec)$:

- $(X, \sqsubseteq)$ is the underlying poset and
- $y \prec x \iff y \in \alpha(x)$, as for $\mathcal{P}$.

They must be compatible:

- $z \sqsubseteq y \prec x \Longrightarrow z \prec x$ because $\alpha(x)$ is lower; and
- $z \prec y \sqsubseteq x \Longrightarrow z \prec x$ because α preserves order.

We write (X, α) and $(X, \sqsubseteq, \prec)$ interchangeably.

$(\sqsubseteq)$ could just be $(=)$,
so the discrete $(\mathcal{P})$ case is embedded.

Extensionality for $\mathcal{I}$

Let's see what happens when we use
the three classes $\mathcal{I}$, $\mathcal{R}$ and $\mathcal{L}$
instead of 1–1 functions in the two theorems.

$\mathcal{I}$-extensionality is the same as the traditional notion.

But $\mathcal{D}$ doesn't preserve $\mathcal{I}$.

So the first theorem fails: morphisms needn't be 1–1.

But the second (extensional quotient) is still valid.

Extensionality for $\mathcal{R}$

A coalgebra $(X, \sqsubseteq, \prec)$ is $\mathcal{R}$-extensional iff

$$\forall yz. \quad \big(\forall x.\, x \prec y \Longrightarrow x \prec z\big) \iff (y \sqsubseteq z).$$

so $(\sqsubseteq)$ is set-theoretic inclusion, renamed $(\subseteq)$.

But for compatibility we require meta-transitivity:

$$\forall w, x, y. \quad (\forall z.\, z \prec y \Rightarrow z \prec x) \;\wedge\; (x \prec w) \Longrightarrow (y \prec w).$$

Any well founded meta-transitive relation is transitive
in the usual sense, but not conversely.

So this looks a bit like the popular definition of ordinal
as a transitive, extensional well founded relation.

Extensionality for $\mathcal{L}$: Plump Ordinals

A coalgebra $(X, \sqsubseteq, \prec)$ is $\mathcal{L}$-extensional iff

every subset $U \subset X$ that is

- $(\prec)$-bounded above, $\exists y \in X.\, \forall u \in U.\, u \prec y$, and
- $(\sqsubseteq)$-lower, $\forall y \in X.\, \forall u \in U.\, y \sqsubseteq u \Longrightarrow y \in U$

is represented by some unique $x \in X$: $U = \{u \mid u \prec x\}$.

A well founded $\mathcal{L}$-extensional coalgebra
is called a plump ordinal.

This is much simpler than the 1996 symbolic definition,
because we have treated the two relations independently.

The first theorem is valid (plump ordinals form a preorder)
but the second (extensional quotient) fails for them.

Coalgebra homomorphisms

Recall that a function $f : (Y, \prec_Y) \to (X, \prec_X)$ is
a $\mathcal{P}$-coalgebra homomorphism iff it's a bisimulation

$$\forall x{:}X.\ \forall y{:}Y. \quad x \prec_X fy \iff \exists y'{:}Y.\ x = fy' \;\wedge\; y' \prec_Y y.$$

A function $f : (Y, \sqsubseteq_Y, \prec_Y) \to (X, \sqsubseteq_X, \prec_X)$ is
a $\mathcal{D}$-coalgebra homomorphism iff instead

$$\forall x{:}X.\ \forall y{:}Y. \quad x \prec_X fy \iff \exists y'{:}Y.\ x \sqsubseteq_X fy' \;\wedge\; y' \prec_Y y$$

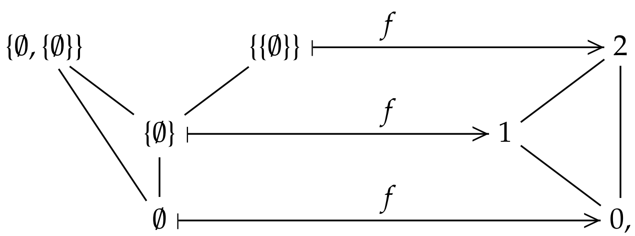
and $\qquad \forall yy'{:}Y. \quad y' \sqsubseteq_Y y \Longrightarrow fy' \sqsubseteq_X fy.$

(You might like to make a note of this one formula.)

There is no forgetful functor from
$\mathcal{D}$-coalgebra homomorphisms to $\mathcal{P}$-coalgebra homomorphisms.

A $\mathcal{D}$- but not $\mathcal{P}$-homomorphism

The rank function of the von Neumann hierarchy:



To make this a $\mathcal{D}$-homomorphism, we need $0 \sqsubseteq 1$.

The two coalgebras are $\mathcal{I}$-extensional.
But the function f between them is not in $\mathcal{I}$ (1–1).
So $\mathcal{I}$-extensional $\mathcal{D}$-coalgebras fail the first theorem.

Nevertheless, this is the universal way of
making the $\mathcal{D}$-coalgebra on the left
transitive, meta-transitive or $\mathcal{R}$-extensional.

Another $\mathcal{D}$- but not $\mathcal{P}$-homomorphism

Here is another natural and informative example.

The map $f : \mathbf{3} \equiv \{0, 1, 2\} \to \mathcal{D}\Omega$ by

$$0 \mapsto \emptyset, \qquad 1 \mapsto \{\emptyset\}, \qquad 2 \mapsto \Omega$$

is the universal way of making $\mathbf{3}$ into a plump ordinal.

It too is a $\mathcal{D}$- but not $\mathcal{P}$-homomorphism:

Let $\xi \in \Omega$ be an undecidable truth value,
for example in the presheaf topos $\mathbf{Set}^{\to}$.

Then

$$x \equiv \{\emptyset \mid \xi\} \quad \prec_{\mathcal{D}\Omega} \quad \Omega \equiv f2$$

so $x \subseteq f1 \equiv \{\emptyset\}$ but $x \neq fy'$ for any $y' \in \mathbf{3}$.

Slim and plump ordinals form preorders

Recall the first theorem:
extensional well founded $\mathcal{P}$-coalgebras are "sets"
and form a preorder, like the "von Neumann hierarchy".

If a well founded $\mathcal{D}$-coalgebras is
- $\mathcal{R}$-extensional (meta-transitive), we call it slim;
- $\mathcal{L}$-extensional, we call it plump.

$\mathcal{R}$ and $\mathcal{L}$ satisfy the conditions for this first theorem.

So slim and plump ordinals form preorders.

NB slim is not the same as thin!

Since (one way of seeing) thin ordinals is as extensional
well founded $\mathcal{P}$-coalgebras, they too form a preorder.

Under excluded middle, thin, slim and plump ordinals
all agree with the traditional notion.

Extensional quotient

Recall the second theorem:

Amongst well founded $\mathcal{P}$-coalgebras,
extensional ones form a reflective subcategory.

This was our version of the Mostowski extensional quotient.

The classes $\mathcal{I}$ and $\mathcal{R}$ also satisfy the conditions for this theorem.

So slim ($\mathcal{R}$) ordinals form a reflective subcategory.

The slim reflection is (one version of) the ordinal rank.

The version using $\mathcal{I}$ is part of the construction of the thin
ordinal rank; the transitive closure is needed too.

However, $\mathcal{L}$ fails the conditions: there is no plump rank
in just the logic of an elementary topos or Zermelo set theory.

Slim and thin ordinals

The theory of slim (meta-transitive) ordinals
looks promising from a categorical point of view.
It is the natural adaptation using $\mathcal{R}$ instead of 1–1 functions.

But now it breaks down.

$\mathcal{R} \subset$ **Pos** has badly behaved pushouts.
So slim ordinals have badly behaved binary joins.

The one-point successor, *cf.* $\alpha \cup \{\alpha\}$,
preserves both thin and slim ordinals.

However, the notion of global element,

$$\beta \in \alpha \iff \beta^+ \subset \alpha,$$

works nicely for thin but not slim ordinals.

I therefore see no way of obtaining transfinite recursion
(with successors and limits) for slim ordinals.

Thin ordinals

We originally defined a thin ordinal to be
an extensional well founded coalgebra $(X, =, \prec)$
where $(\prec)$ is transitive and the poset order is discrete.

However, to make the rank function a $\mathcal{D}$-homomorphism
we need to use an order in which $y \prec x \Rightarrow y \sqsubseteq x$.

If $(\sqsubseteq)$ is $(\subseteq)$ we get meta-transitivity, which doesn't work.

So try the thin order, $\quad y \preceq x \quad \equiv \quad y \prec x \ \vee \ y = x.$

Isn't this just classical recidivism?

No, because every $\mathcal{D}$-homomorphism

$$f : (Y, \preceq_X, \prec_X) \to (X, \preceq_X, \prec_X)$$

is actually a $\mathcal{P}$-homomorphism if X is well founded.

Moreover, it's a lower inclusion (in $\mathcal{L}$),
even though the structure map α is not in $\mathcal{L}$.

We can't do thin ordinals properly today

A transitive coalgebra $(X, \sqsubseteq, \prec)$ has $(\prec) \subset (\sqsubseteq)$.

This is $\alpha \leq \eta$, which is a natural thing to do
for a KZ-monad (T, η, μ).

The "transitive closure", forcing $\alpha \leq \eta$,
ought to be a 2-categorical colimit.

But when I asked a leading 2-categorist about it,
he couldn't identify this as a known construction.

Maybe there is a factorisation system for T-homomorphisms,
rather than for morphisms of the underlying category **Pos**,
that does this.

Slice categories (preorders)

The preorders Thin and Plump are "large".

I treat them as schemes —
what it is to be a thin, plump ordinal, *etc.*

But the slices are essentially small, because

$$\mathsf{Thin}/X \simeq \mathcal{D}(X) \qquad \text{and} \qquad \mathsf{Plump}/X \simeq \mathcal{D}(X),$$

simply because all homomorphisms
between thin (or plump) ordinals are lower inclusions.

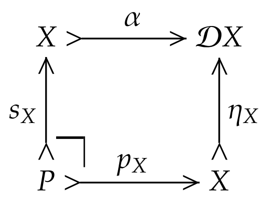
Beware that, for thin as well as plump ordinals, the morphisms
are lower inclusions and the order on $\mathcal{D}(X)$ is $(\subseteq)$, not $(\preceq)$.

Therefore the large preorders Thin and Plump are unions.

Functions Thin $\to \Theta$ and Plump $\to \Theta$ are cocones of
legimate functions, so long as the slices are compatible.

Successors

The generic definition is given by the pullback



For $\mathbb{N}$, $sn = n + 1$ and $pm = m - 1$ for $m \neq 0$.

If we do this for $\mathcal{D}X$ instead of X itself,
we recover three familiar constructions
when the poset order is

discrete	$(=)$	$\{x\} \in \mathcal{D}(X) \equiv \mathcal{P}(X)$	no condition
thin	$(\preceq)$	$\{y \mid y \preceq x\} \equiv \alpha(x) \cup \{x\} \in \mathcal{D}(X)$	$x \subseteq sx$
plump	$(\subseteq)$	$\{y \mid y \subseteq x\} \in \mathcal{D}(X)$	s preserves $(\subseteq)$

Successors

The successor is related to the two orders by

$$\forall y{:}X. \quad y \prec sx \iff y \sqsubseteq x.$$

and conversely this property characterises sx.

For any homomorphism $f : Y \to X$,
if $s_Y y$ is defined then so is $f(s_Y y)$ and

$$f(s_Y y) \;=\; s_X(fy).$$

$s_X(fy)$ may be defined even when $s_Y y$ is not,
indeed there is always a homomorphism that does this.

Hence we make define successor as an endofunction

- s : Thin $\to$ Thin, with id $\subseteq s$;
- s : Plump $\to$ Plump that preserves $(\subseteq)$.

Every thin or plump ordinal is
the join of the successors of its elements.

Transfinite recursion

We recover the Joyal–Moerdijk "large free algebra" result:

Given any $\mathcal{D}$-algebra ($\bigvee$-semilattice) Θ
with an endofunction $s : \Theta \to \Theta$,

if s is inflationary, id $\leq s$, but not necessarily monotone,
there is a unique function r : Thin $\to \Theta$
that preserves $\bigvee$ and s.

if s is monotone,
there is a unique function r : Plump $\to \Theta$
that preserves $\bigvee$ and s.

Plumper and plumper!

In the presheaf topos $\mathbf{Set}^{\to}$,
where **Set** is classical Zermelo set theory,
plump $\omega \cdot 2$ does not exist.

Probably plump ω doesn't exist in $\mathbf{Set}^{\omega}$.

Transfinite iteration of functors can be encoded
as an example of our "generalised Mostowski theorem".

I propose this idea as a replacement for replacement.

What about making successor preserve binary joins?

Repeat the construction using binary semilattices
(with an operation $\vee$ but not a constant $\bot$)
instead of posets.

And lots of other categories besides these!
That's why we do category theory, not symbolic logic!