Well Founded Coalgebras

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Tallinn and Birmingham
2. märts 2023

www.PaulTaylor.EU/ordinals/

Thank you

Tere.

Tänan teid kutse eest esineda sellel seminaril Tallinnas.

(Courtesy of DeepL.com)

In particular, thank you to Tarmo Uustalu and Varmo Vene for their interest and support in this subject.

Also, sorry to Niccolò Veltri for reading 14:00 as 4pm and thanks for changing your seminar time to fit in with Birmingham and me!

NB: Unlike the previous one about order-theoretic fixed point theorems, this seminar will assume some proficiency in category theory.

Categorical Set Theory, at first

After Bill Lawvere and Miles Tierney
had introduced elementary toposes in 1970,
it was necessary to prove that
they could do everything that set theory could do,
(at least the things that ordinary mathematicians care about).

Barry Mitchell, Jean Bénabou and others
showed how to interpret higher order logic in toposes.

Gerhard Osius re-constructed models of set theory
($\in$-structures) in an elementary topos.

Christian Mikkelsen showed how to do well founded recursion.

Mikkelsen and his supervisor Anders Kock also simplified the axioms for an elementary topos by constructing exponentials and finite colimits from powersets and pullbacks.

For bibliographic references, see my paper.

Categorical Set Theory, the future

It is now beyond doubt that category theory and type theory provide the foundations for established pure mathematics.

We do not need to justify ourselves using set theory or to set theorists. (That's a waste of effort anyway.)

But (albeit in its own obscure language)
set theory may be the source of
some useful or interesting mathematical ideas.

In particular, here we will study
well founded induction, recursion and iteration
and extensionality.

Then we want to do some good mathematics, in the native language of category theory (universal properties).

We do not intend to mimic set theory and
it doesn't matter if our results don't exactly match it.

Well-founded induction and recursion

A binary relation $\prec$ on a set A is a well founded relation if it obeys the induction scheme

$$\forall U \subset A. \quad \frac{\forall a{:}A.\ \ (\forall b{:}A.\ \ b \prec a \Rightarrow b \in U) \Longrightarrow a \in U}{\forall a{:}A.\ \ a \in U}$$

or

$$\forall \phi{:}\Omega^A. \quad \frac{\forall a{:}A.\ \ (\forall b{:}A.\ \ b \prec a \Rightarrow \phi b) \Longrightarrow \phi a}{\forall a{:}A.\ \ \phi a}$$

This is of course the intuitionistic definition.
See my paper for some of the ancient history.

The principal theorem, due to John von Neumann 1928, is to derive well founded recursion for $\theta : \mathcal{P}(\Theta) \to \Theta$:

$$r(a) \quad = \quad \theta\left(\{r(b) \mid b \prec a\}\right).$$

We will build up to a (categorical) proof of this.

Extensionality

A binary relation $\prec$ on a set A is extensional if

$$\forall a, b{:}A. \quad \frac{\forall c{:}A.\ \ c \prec a \iff c \prec b}{a = b}$$

Any well founded relation has an extensional quotient, popularly known as Mostowski's Theorem.

This may look innocent.

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But that is far from being the case! (Even in Set Theory.)

It will be the most powerful part of this subject,
as my replacement for (the Axiom-Scheme of) Replacement.

Expressing these ideas in category theory

There are many ways to encode a binary relation $\prec$ on A, for example $(\prec) \hookrightarrow A \times A$.

We choose a coalgebra for the covariant powerset:

$$\alpha : A \longrightarrow \mathcal{P}(A) \equiv \Omega^A,$$

where

$$\alpha a \;\equiv\; \{b : A \mid b \prec a\} \;\equiv\; \lambda b{:}A.\, b \prec a.$$

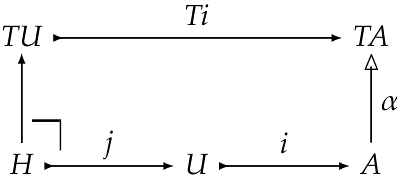
Then $(A, \prec)$ is extensional iff α is mono:

$$a = b : A \qquad \Longleftarrow \qquad \{c \mid c \prec a\} = \{c \mid c \prec b\} : \mathcal{P}(A).$$

Then, as categorists, we may (in principle)
replace $\mathcal{P}$ by any endofunctor T of any category,
and mono by any suitable class of maps.

Well founded coalgebras

A coalgebra $\alpha : A \longrightarrow TA$ is well founded
if in any pullback diagram of the form



the maps i and therefore j are necessarily isomorphisms.

Well founded coalgebras for $\mathcal{P}$

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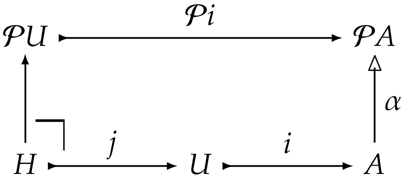

the maps i and therefore j are necessarily isomorphisms.

The pullback $H \subset A \times \mathcal{P}U$ consists of $a : A$, $V \subset U \subset A$ such that

$$\alpha(a) \equiv \{x : A \mid x \prec a\} = V.$$

So V is unique,

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$$\begin{array}{ccccc}
\mathcal{P}U & & \xrightarrow{\quad \mathcal{P}i \quad} & & \mathcal{P}A \\
\uparrow & & & & \uparrow \alpha \\
H & \xrightarrow{\; j \;} & U & \xrightarrow{\; i \;} & A
\end{array}$$

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So V is unique, but for $(a, V) \in H$, we need $V \subset U$, *i.e.*

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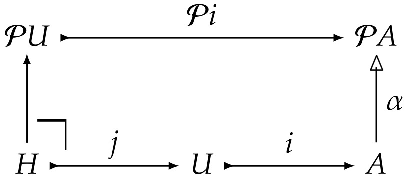

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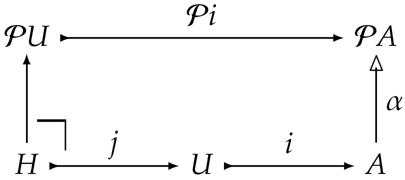

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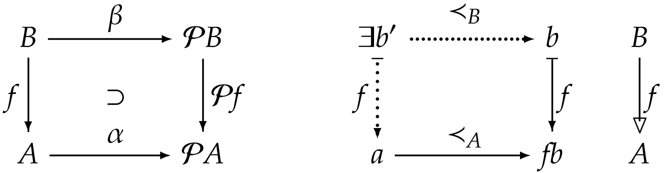
so H corresponds to the induction hypothesis.
Then the inclusion $H \subset U$ expresses the induction premise.
That $i : U \cong X$ is the induction conclusion.

Coalgebra homomorphisms as simulations

A function $f : (B, \prec_B) \to (A, \prec_A)$ is a homomorphism of $\mathcal{P}$-coalgebras



iff it is strictly monotone, *i.e.* it preserves the binary relation

$$\forall b_1, b_2 : B. \quad b_1 \prec_B b_2 \implies fb_1 \prec_A fb_2,$$

and a simulation

$$\forall a' : A.\ \forall b : B. \quad a' \prec_A fb \implies \exists b' : B.\ a' = fb' \wedge b' \prec_B b.$$

Then the relation $(a \sim b) \equiv (a = fb)$ is actually a bisimulation.

Coalgebra homomorphisms in nature

(Bi)simulations are nowadays well known in process algebra.

People in this audience know far more about this than me.

Who first recognised them in process algebra and when?

(Categorical) Set Theory probably got there much earlier:

Osius (1974) saw that $\mathcal{P}$-coalgebra monomorphisms characterise the set-theoretic subset relation.

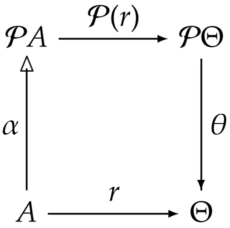
Dimitri Mirimanoff (1917–19) had already recognised that subsets were characterised by bisimulations, which he called isomorphismes.

Coalgebra-to-algebra homomorphisms

If $(A, \prec)$ is a (well founded) relation or $\mathcal{P}$-coalgebra and $\theta : \mathcal{P}(\Theta) \to \Theta$ is any $\mathcal{P}$-algebra then the recursion equation

$$r(a) \quad = \quad \theta\left(\{r(b) \mid b \prec a\}\right).$$

is a coalgebra-to-algebra homomorphism:



Reversing the arrow is not unfamiliar:
recall that, for the initial algebra or final coalgebra of a functor, the structure map is invertible.

Partial algebras for functors

You and I do category theory and (various kinds of) algebra.
Surely set theory is redundant nowadays?

Why should we be interested in set-theoretic structures?

Since $\mathcal{P}$ has no initial algebra,

extensional well founded relations,
also called transitive sets or ϵ-structures, provide
approximations to the (non-existent) initial algebra.

So we want to generalise to other functors T, whose
initial algebras may not exist or be very complicated.

Later, I hope, to consider other algebraic structures
that are not just defined by a single functor,
such as type and proof theories.

Von Neumann's Recursion Theorem

For any well founded relation $(A, \prec)$ and $\theta : \mathcal{P}(\Theta) \to \Theta$,

$$r(a) \quad = \quad \theta\left(\{r(b) \mid b \prec a\}\right)$$

has a unique solution $r : A \to \Theta$.

Idea of the proof: partial solutions (attempts),
defined on initial segments: subsets $B \subset A$ such that

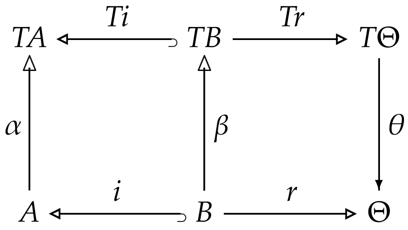
$$\forall bc{:}A.\;\; c \prec b \in B \Longrightarrow c \in B.$$

We need to consider

- the least attempt;
- the successor of an attempt;
- the union of attempts;
- uniqueness of solutions; and
- totality.

Categorical attempts

Putting the ideas about coalgebra homomorphisms together, an attempt is a coalgebra-to-algebra homomorphism defined on a sub-coalgebra:



What properties must the functor T and category C have to re-produce von Neumann's proof using attempts?

We have already assumed that T preserves monos.

The simple categorical version

At first we let the category C be **Set** or any elementary topos.

Then $\emptyset \to T\emptyset$ is a well founded coalgebra,
it is initial and extensional and
$\emptyset \to \Theta$ is the least attempt.

For unions of partial maps,
the unions of supports agree with
the colimits of functions.

Colimits of coalgebras are formed using colimits in C.
Colimits of well founded coalgebras are again well founded.
Similarly with unions of (well founded) sub-coalgebras.

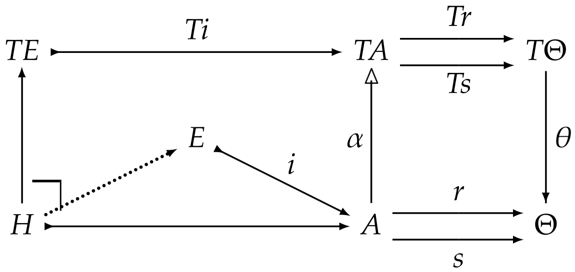
Unions of subobjects in **Set** are indexed by the powerset,
so it is legitimate to form the union of all of them.

But there are some other issues to consider...

Uniqueness using equalisers

In **Set** (or a topos) we also have equalisers.

So if $r, s : A \to \Theta$ are two solutions of the recursion equation, we apply well-foundedness of $\alpha : A \to TA$ to the equaliser $i : U \hookrightarrow A$ of r and s:



This is valid by some easy diagram-chasing, so by well-foundedness i is invertible and $r = s$.

However, in generalisations, equalisers may not be available.

A basic fact about well-foundedness??

In the simpler version of the proof, any initial segment of a well founded coalgebra is well founded.

This is an instance of the general result that underlies "induction on size", where "size" might be length, depth or some other measure:

If $(A, \prec)$ is well founded and
$f : (B, <) \to (A, \prec)$ preserves the order
then $(B, <)$ is also well founded.

The proof of this for relations is trivial classically.
It is more difficult intuitionistically and categorically.
It holds for coalgebras for $T : \mathbf{Set} \to \mathbf{Set}$ so long as
T preserves inverse images (pullbacks of monos).
See Section 9 of my paper.

We would like to weaken this assumption.

Another problem: binary unions

In von Neumann's original proof, we form all unions:
empty set, binary unions and directed unions.

However, for T-coalgebras to have well behaved binary unions,

- T must preserve inverse images, and
- the category C must have binary unions that behave like those in **Set**.

See Section 10 of my paper.

We would like to get a more general theorem,
avoiding these requirements.

Can we prove it without using binary/general unions?

The Order-Theoretic Fixed Point Theorem

The underlying result that we require is this:

Let $(X, \leq)$ be an poset with

- least element $\bot$ and
- directed (instead of all) joins $\bigvee^{\uparrow}$, and
- a monotone endofunction $s : X \to X$.

Then s has a least fixed point.

My previous seminar outlined the history of the classical proofs of this, in particular the Bourbaki–Witt theorem (1949/51), the key point of which is that

$$\forall x, y \colon X_0. \quad y \leq x \ \vee \ sx \leq y,$$

where $X_0 \subset X$ is smallest subset closed under $\bot$, s and $\bigvee^{\uparrow}$.

The Bourbaki–Witt proof was actually used by Ernst Zermelo in his second proof of well-ordering, 1908.

Pataraia's Theorem

The first proof of the order-theoretic fixed point theorem without using Excluded Middle was found by Dito Pataraia in 1996 and simplified by Alex Simpson.

Any Domain Theorist should have seen his key idea long ago!

They (I) didn't because, as maths students, we were taught that subsets were basic.

Like computer science students, Pataraia considered functions instead.

The (blindingly simple) idea is that, for any dcpo X_0, the directed-complete poset of all inflationary monotone functions $X_0 \to X_0$ is directed, because

$$x \quad \leq \quad s_1 x, \ \ s_2 x \quad \leq \quad s_1(s_2 x).$$

Hence there is a greatest such function, t.

Then $t\bot \in X_0 \subset X$ is the least fixed point of $s : X \to X$.

Practical use of Pataraia's Theorem

Both the Zermelo–Bourbaki–Witt and Pataraia proofs begin by cutting the original dcpo X down to the subset $X_0 \subset X$ generated by $\bot$, s and $\bigvee^{\uparrow}$.

The least fixed point of s in X is the unique fixed point in X_0 and is also the top element there.

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Can we obtain something similar in a simpler (first order) way that is more natural for the problem?

The Special Condition

Suppose that X is such that $s : X \to X$ satisfies

$$\forall xy{:}X. \quad x = sx \ \leq \ y = sy \quad \Longrightarrow \quad x = y.$$

Without $\leq$, this would be the standard definition of uniqueness. The antecedent is stronger, so the whole statement is weaker.

It is also weaker than requiring $X_0 = X$:
there may still be stuff that is not generated from $\bot$, s and $\bigvee^{\!\!\uparrow}$.

But the Special Condition is enough to deduce that

- X has a greatest element, which we call $\top$;
- $\top$ is the unique fixed point of s;
- if $\bot$ satisfies some predicate that is preserved by s and directed joins then this also holds for $\top$ (induction).

This constructive induction principle was first exploited by Martín Escardó in 2003.

In case you missed that

It is a common situation to have some universe of partial constructions, where it is easy to construct

- an empty, basic or smallest version,
- directed unions, and
- a one-step improvement.

Typically, such partial constructions are too complicated to see how to find

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It also provides a method of proof by induction for properties of this fixed or largest version.

Achieving the Special Condition

How, without using recursion or second order logic, can we cut down a more general dcpo X with $s : X \to X$ to one that obeys the special condition?

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The subset of those x that satisfy

$$x \le sx \quad \text{and} \quad \forall a{:}X.\ \ sa \le a \implies x \le a$$

does this, where $\forall$ ranges *â priori* over the original dcpo.

If X has binary meets as well as directed joins,
the subset of well founded elements, *i.e.* those x with

$$x \le sx \quad \text{and} \quad \forall u{:}X.\ \ su \wedge x \le u \implies x \le u,$$

also satisfies the special condition.

Well founded elements of structures

The definition of well founded *element* is the poset form of our notion of well founded *coalgebra*.

But the three well founded notions agree as a theorem:

Any binary relation $\prec$ defines $s : \mathcal{P}(A) \to \mathcal{P}(A)$ by

$$sB \;\equiv\; \{c : A \mid \forall b{:}A.\;\; b \prec c \Longrightarrow b \in B\}.$$

Then $B \in \mathcal{P}(A)$ is a well founded element iff
$B \subset A$ is an initial segment and
the restriction of $\prec$ to B is a well founded relation.

There is a similar result for (well founded) coalgebras (three slides on).

We would hope that the same idea could be used to generalise our results about partial algebras for functors to more complex structures.

What is the new fixed-point idiom?

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The foregoing remarks already prove that any binary relation $(A, \prec)$ has a greatest well founded initial segment.

Sometimes the natural proof uses the special condition directly.

In the paper, the recursion theorem for coalgebras
uses the direct definition of well founded coalgebras
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The labour is concentrated in the definition and properties of the successor operation. Pataraia does everything else!

The recursion theorem for coalgebras

It is probably more important to spread the message about Pataraia's Theorem and the Special Condition

than to give the details of the proof of the recursion theorem for well founded coalgebras.

So I will just show a couple of ideas

and leave you to study the paper in your own time.

This will allow time here to give you a picture of how the subject could be developed in future.

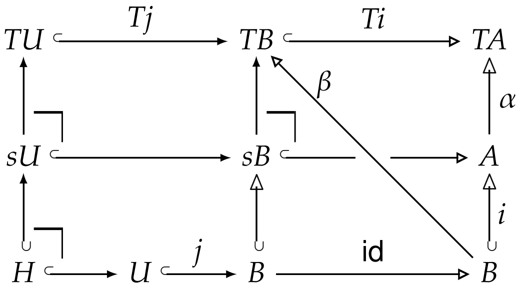
Sub-coalgebras that are well founded elements

Recall the definition of a well founded element b:

$$b \le sb \quad \text{and} \quad \forall u.\, su \wedge b \le u \implies b \le u.$$

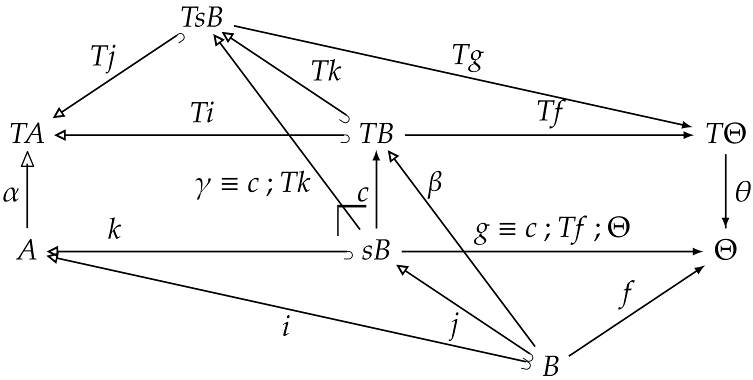
Then B is a well founded element
of the lattice of sub-objects of A
with respect to the successor s
defined by the top two pullbacks ...

iff it is a well founded coalgebra.



Recursion for well founded coalgebras

The successor of an attempt is given by the diagram



There is a bijection between attempts with support B and sB.

This lifts to filtered colimits by their universal property.

Then by Pataraia induction, there is a unique attempt whose support is the greatest well founded initial segment.

Extensional well founded coalgebras

The recursion theorem for these
is the analogue of “zipping up” two classical ordinals.

It is technically more complicated, so please see the paper.

It was to prove this result
that I had to re-think Pataraia’s Theorem.

Mostowski's extensional quotient

The Set Theory books say that this requires Replacement,

but my 1996 JSL paper used the quotient by an equivalence relation that was a bisimulation defined by (co?)recursion.

In the generalised categorical version,
epi–mono factorisation defines a successor (next slide),
which satisfies the Special Condition,
so Pataraia's Theorem gives the quotient object.

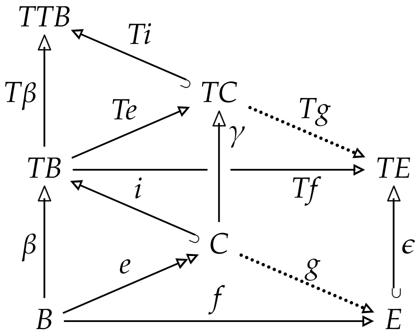
Just as the category had to be
well powered for the recursion theorem,
it must be well co-powered for the extensional quotient.

That is, the monos into an object and the epis out of it
must be indexed by a dcpo in an elementary topos,
so that we can apply Pataraia's Theorem.

Successor for the extensional quotient

Let $f : B \to E$ be a coalgebra homomorphism, where
$\beta : B \longrightarrow TB$ is a well founded coalgebra
and $\epsilon : E \hookrightarrow TE$ is an extensional coalgebra.

Form the epi–mono factorisation $\beta = e \,;\, i$:



The structure of C is given by appropriate composites
and f factors through it by orthogonality of factorisation.
B is a fixed point iff it is extensional.
The fixed point follows from direct use of the Special Condition.

Categorical generalisations

The purpose of category theory
is to understand the essentials of an argument
in a familiar setting like **Set**,
and then re-apply in other categories.

The functor T only needed to preserve monos
(apart from in a handful of expendable results).

Monos (and epis) can be replaced by a factorisation system.

Directed unions and colimits need to behave like in **Set**.

We briefly consider $\mathbf{Set}^{\mathsf{op}}$ and **Pos**.

Badly behaved colimits

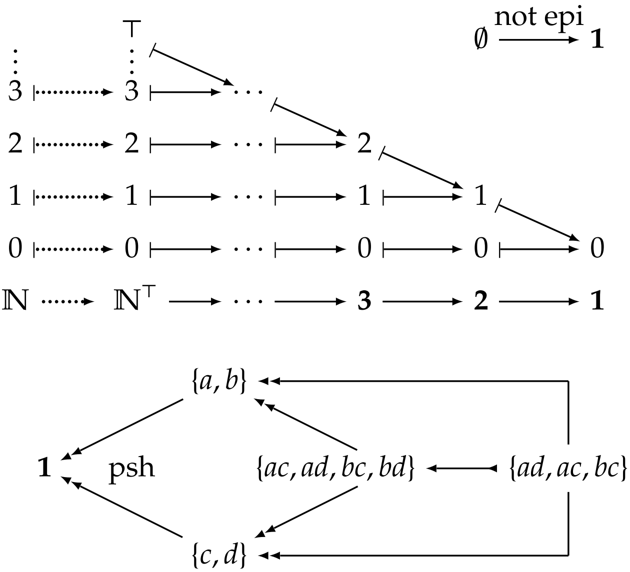
The proof of the recursion theorem used directed unions of partial functions. For this,
unions of subobjects must agree with colimits of functions.

In particular, mediators from colimits of monos must be mono. This is a *non-trivial* property of **Set**, because it *fails* in $\mathbf{Set}^{\mathsf{op}}$: mediators to limits in **Set** need not be epi.

For limits of chains, the counterexample (on the next slide), due to Venanzio Capretta, Tarmo Uustalu and Varmo Vene, is essentially based on the ascending natural numbers object, with or without $\top \equiv \infty$.

To find a categorical home for both recursion and co-recursion, apparently we need domain theory, sober topological spaces or locales. These are beyond what I have done.
Do systems with both recursion and co-recursion necessarily also have Scott-style fixed points?

Non-epi mediators to limits of epis in **Set**



Ordinals

The obvious first application should be **Pos**.

In the 1990s, André Joyal and Ieke Moerdijk,
and independently I, investigated intuitionistic ordinals.

We learned that there are many different kinds of them,
and that $\in$ and $\subset$ must be treated separately, because

$$\beta \subset \alpha \quad \text{is not the same as} \quad \beta \in \alpha \ \vee \ \beta = \alpha.$$

Considering extensional well founded coalgebras in **Pos**
is the natural way to understand these things:
the poset models $\subset$ and the coalgebra $\in$.

But there are many facts and fallacies about posets
that need to be checked.
I haven't yet done that as carefully as is needed.

Notions of sub-posets

The power of this theory will come from
the choice of factorisation systems
to replace 1–1 and onto functions between sets.

- $\mathcal{I}$: ***monos***, injective functions: subsets with a possibly sparser order; these are the ***monos*** in **Pos**;
- $\mathcal{R}$: ***full subsets***: arbitrary subsets, but equipped with the restricted order relation; these are the ***regular monos*** in **Pos**;
- $\mathcal{L}$: ***lower subsets***: if $x \leq y$ in X with $y \in U \subset X$ and $U \in \mathcal{L}$ then $x \in U$, where U carries the restriction of the order relation on X.

These all have partner classes forming factorisation systems.

In fact $\mathcal{I}$ is not closed under the "down-sets" functor
that we use as the analogue of the powerset,
but is useful as a way of embedding "sets" in the system.

Generalised Mostowski

It turns out that it doesn't matter which class we use in the definition of well-founded coalgebra.

Generalised Mostowski

It turns out that it doesn't matter which class we use in the definition of well-founded coalgebra.

However, the classes give very different notions of extensionality
and so our categorical version of the extensional quotient has quite different forms:

- with $\mathcal{I}$ it is the traditional set-theoretic extensional quotient;
- with $\mathcal{R}$ it gives the ordinal rank of a well founded relation;
- with $\mathcal{L}$ it does not exist in an elementary topos: (part of) the Axiom-Scheme of Replacement is needed.

Transfinite iteration of functors

In category theory, we solve problems by inventing new categories.

Using our categorical definition of ordinals, we may define
- a category whose objects are fibrations over ordinals; and
- a factorisation system whose monos are pullbacks.

Then the extensional quotient exists in this setting iff transfinite iterates of functors exist.

These do not exist as a Theorem: this is a characterisation.

Such iterated functors are often used, but traditionally their existence was justified using the Axiom-Scheme of Replacement, thereby relapsing into set theory.

Using our characterisation, the Axiom for the existence of iterated functors is stated instead in the language of Bill Lawvere's *Adjointness in Foundations*.

Bigger and bigger ordinals

In a factorisation system $(\mathcal{E}, \mathcal{M})$,

as we we require the $\mathcal{M}$-subobjects
to be closed under more and more structure,
so the class $\mathcal{M}$ becomes smaller,

its partner $\mathcal{E}$ becomes larger:
its maps are only "surjective" in the most tenuous sense
(rather, their image generates the target)
and the class $\mathcal{E}$ is no longer well-co-powered.

Then asserting the corresponding Mostowski theorem
(as an Axiom)
becomes more and more powerful logically.

So this could give a natural categorical way
of expressing "large cardinal" axioms from set theory.

MathOverflow

Nowadays, after whatever service, however trivial, customers are asked to post a review on some website.

For over 100 years, mathematicians have been indoctrinated that all recursion is done using classical ordinals.

Kuratowski tried to show otherwise in 1922.

The (Zermelo–)Bourbaki–Witt theorem is a simpler proof, but Wikipedia mis-represents it as transfinite recursion.

In this seminar I have shown you a new, subtle and widely applicable idiom of proof by induction and recursion.

But the suppression of heterodox ideas continues to this day on MathOverflow, where the bullies are piling in on me.

Please up-vote my Questions there:
mathoverflow.net/questions/441882