A Fixed Point Theorem for Categories

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Transfinite (ordinal) recursion

Friends, Logicians, Colleagues, lend me your ears!
I come to bury the Ordinals, not to praise them!

Transfinite recursion over the classical ordinals is apparently a primordial reflex of mathematicians when faced with any fixed point problem.

But I'll skip my rant about it this time round.
See my papers and the slides for other talks on my webpage.

This lecture is primarily about a neat finitary argument in category theory.

But more widely, I want to begin the study of the intrinsic structure of recursive and fixed point situations.
The result might be to obtain new notions of "ordinal" that are more naturally applicable to complex structures, such as in Type Theory and Proof Theory.

Fixed points and free models

We will consider the construction and properties of

- the least fixed point of a monotone endofunction of a poset with least element and joins of a certain specific directed diagram, and
- the initial algebra for an endofunctor of a small category with initial object and colimits of a certain specific directed diagram.

This proof naturally falls into two parts:

- a specific finitary one and
- a general infinitary one.

The (very) long term goal is to generalise the finitary part as the system of partial constructions of a recursively defined model to yield its intrinsic system of ordinals.

Bourbaki–Witt theorem 1949/51

This is actually due to Ernst Zermelo, 1908.

Consider the subset $W_0 \subset \mathcal{X}$ generated by $\bot, s, \bigvee^{\uparrow}$.

It satisfies $\forall x, y \in W_0. \quad x \leq y \quad \vee \quad sy \leq x.$

(The proof requires a tricky double induction, and Excluded Middle.)

It follows that W_0 is (classically) well ordered,

so we can do induction and recursion over W_0.

Unfortunately, this never got into mainstream pure mathematics textbooks, except in an *appendix* to a *reprinting* of Serge Lang's *Algebra*.

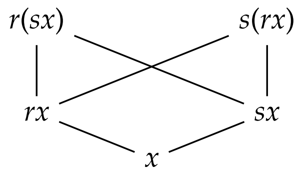
Nevertheless, we keep the idea that W_0 is a system of partial solutions.

Dito Pataraia, 1996/7

Abandon Set Theory, ordinals and transfinite recursion!

Use functions instead (like a good Computer Scientist!)

The inflationary monotone endofunctions of any dcpo form a directed set $\mathcal{F}$, with



If ($\mathcal{X}$ and so) $\mathcal{F}$ are directed-complete then there is a greatest such function.

From this we may deduce the fixed point theorem.

It's constructive — no Axiom of (Choice or) Excluded Middle and much easier than the classical proof!

Two parts to Pataraia's Theorem

Any dcpo has a greatest inflationary monotone endofunction.

For example, consider the three-point dcpo like a V: its greatest inflationary monotone endofunction is the identity. (Not much help!)

So there must be something special about our dcpo W_0 so that the greatest inflationary monotone endofunction $t : W_0 \to W_0$ yields the greatest element of W_0.

The mysterious special condition that does the job is

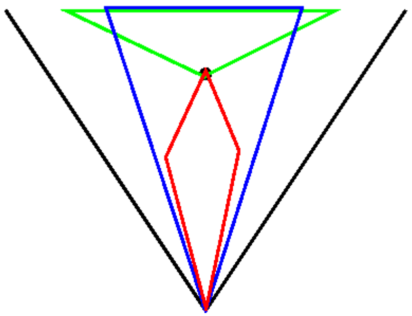
$$\forall x, y \in W_0. \quad x \;=\; sx \;\leq\; y \Longrightarrow x = y.$$

Then, since $t\bot \;=\; s(t\bot) \;\leq\; s(tx) \;\geq\; x$, $t\bot$ is the greatest element of W_0.

If you've got a fixed point, there's nothing more beyond it.

But where does this "special condition" come from?

The general scheme for fixed point problems



The ambient set/type/category $\mathcal{X}$ for the construction.
Those $x \in \mathcal{X}$ for which $x \leq sx$ (coalgebras).
Those $x \in \mathcal{X}$ for which $sx \leq x$ (algebras).
Partial solutions.
Everything outside these areas is useless to the problem.
How to define the red area is the subject of this lecture.
(Usage of pre- and post- fixed points is ambiguous.)

Well founded (or recursive) elements

It is enough to use the subset

$$W \;\equiv\; \{x \in \mathcal{X} \mid x \leq sx \wedge \forall a.\, sa \leq a \Rightarrow x \leq a\}.$$

instead of W_0 (although there are several variations on this).

This subset is closed under $\bot$, s and any joins that exist.

So it contains the subset W_0 generated by $\bot$, s and $\bigvee^{\uparrow}$.

But it's defined in a finitary, first order, or predicative way.

More importantly, it is defined using the idioms of order theory, not logic.

And it satisfies the special condition,

$$\forall x, y \in W. \quad x \;=\; sx \;\leq\; y \Longrightarrow x = y,$$

so it's good enough to use in Pataraia's theorem.

Characterising W in applications

I advocate doing this
in each specific inductive or recursive situation.

For example:

Let $(A, <)$ be any set with a binary relation.

The full powerset $\mathcal{P}A$ has a least element $\emptyset$ and directed unions.

Consider the operation $s : \mathcal{P}A \to \mathcal{P}A$ by

$$sX \quad \equiv \quad \{a : A \mid \forall b{:}A.\ \ b < a \Longrightarrow b \in X\}.$$

Then any subset $X \subset A$ is

- a well founded element iff
- it is an initial segment on which $(<)$ is a well founded relation.

Categorical Pataraia

The analogue of the poset F of
inflationary monotone endofunctions of a poset W.

Consider the category $\mathcal{F} \equiv \mathsf{id} \downarrow [\mathcal{W} \to \mathcal{W}]$
of pointed endofunctors (R, ρ) of $\mathcal{W}$, so $R : \mathcal{W} \to \mathcal{W}$
is a functor and $\rho : \mathsf{id}_{\mathcal{W}} \to R$ a natural transformation.

Morphisms $\phi : (R, \rho) \to (S, \sigma)$ of $\mathcal{F}$ are
natural transformations $\phi : R \to S$ such that $\rho\, ; \phi = \sigma$;

$$\begin{array}{ccccc} & & \mathsf{id}_{\mathcal{W}} & & \\ & \swarrow^{\rho} & & \searrow^{\sigma} & \\ R & & \xrightarrow{\quad\phi\quad} & & S \end{array}$$

The identity $\mathsf{id} : (R, \rho) \to (R, \rho)$ is the identity natural transformation $\mathsf{id}_R : R \to R$ and composition is that of the natural transformations. The initial object of $\mathcal{F}$ is $(\mathsf{id}_{\mathcal{W}}, \mathsf{id}_{\mathsf{id}_{\mathcal{W}}})$, from which the unique morphism to (R, ρ) is ρ.

Categorical Pataraia

Pataraia's idea becomes the naturality square

$$\begin{array}{ccc} \mathsf{id}_{\mathcal{W}} & \xrightarrow{\ \rho\ } & R \\ {\scriptstyle\sigma}\downarrow & \searrow^{\kappa} & \downarrow{\scriptstyle R\sigma} \\ S & \xrightarrow{\ \rho_S\ } & Q \equiv R \cdot S \end{array} \qquad \textit{i.e.} \qquad \begin{array}{ccc} X & \xrightarrow{\ \rho_X\ } & RX \\ {\scriptstyle\sigma_X}\downarrow & \searrow^{\kappa_X} & \downarrow{\scriptstyle R\sigma_X} \\ SX & \xrightarrow{\ \rho_{SX}\ } & QX \equiv R(SX) \end{array}$$

whose common diagonal

$$\kappa \equiv \rho\, ; R\sigma = \sigma\, ; \rho_S \ : \ \mathsf{id}_{\mathcal{W}} \longrightarrow Q \equiv R \cdot S$$

defines another object of $\mathcal{F}$
and there are morphisms (natural transformations)

$$R \xrightarrow{\ R\sigma\ } Q \xleftarrow{\ \rho_S\ } S.$$

This property is directedness.

(The usual categorical analogue of directedness is filteredness,
which has a further condition for parallel pairs of morphisms,
but there doesn't seem to be a natural way of getting this.)

Categorical Pataraia

If $\mathcal{W}$ and so $\mathcal{F}$ have colimits over this single, specific directed
diagram then $\mathcal{F}$ has a terminal object

$$T : \mathcal{W} \longrightarrow \mathcal{W}$$

This means that Pataraia's lemma that
every dcpo has a greatest inflationary monotone endofunction
does not require *all* directed joins
and generalises to categories.

This may not be a substantive generalisation, because

a famous 1960s observation of Peter Freyd was that,
classically, any small (co)complete category is a poset (lattice).

(His argument may not apply,
because we haven't asked for coproducts or pushouts.)

But that's not the point: we're trying to
understand how the dcpo argument works.

The more interesting question

Suppose now that we do have a terminal pointed endofunctor.
How does this help with the fixed point theorem?

More specifically,
what is the special condition on a category $\mathcal{W}$
such that the terminal pointed endofunctor
applied to the initial object
yields the terminal object of $\mathcal{W}$?

Recall that this doesn't happen if
$\mathcal{W}$ is the three-point poset V.

Given any endofunctor $S : \mathcal{X} \to \mathcal{X}$ of any category
(with an initial object I)
we need to construct the categorical analogue
of the sub-poset W of well founded or recursive elements.

Recursive coalgebras

Recall that, in the poset case,

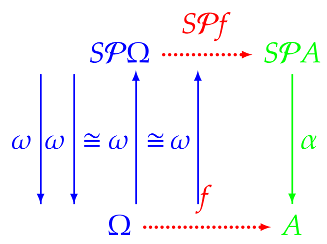
$$W \;\equiv\; \{x \in \mathcal{X} \mid x \le sx \wedge \forall a.\, sa \le a \Rightarrow x \le a\}.$$

We replace $x \le sx$ by an S-coalgebra $\xi : X \to SX$
and each $sa \le a$ by an S-algebra $\alpha : SX \to X$

But what is the analogue of $\forall a.\ \cdots \Rightarrow \cdots$?

(There are design choices in the proof here
and some are better than others.)

Categorical Set Theory



The initial algebra has invertible structure map.

So it's also a coalgebra satisfying

$$f \;=\; \omega \,;\, Sf \,;\, \alpha$$

and (even without invertibility) we say that this equation
defines a recursive coalgebra.

Gerhard Osius used recursive $\mathcal{P}$-coalgebras to encode Zermelo
set theory in any elementary topos.

The category of recursive coalgebras

Let $\mathcal{W}$ be the category of recursive S-coalgebras
and coalgebra homomorphisms.

Applying S to a recursive S-coalgebra gives another one.
So S is an endofunctor $\mathcal{W} \to \mathcal{W}$.

The structure maps of the coalgebras together define
a natural transformation $\sigma : \mathsf{id} \to S$:

$$\sigma_{(X,\xi)} \;\equiv\; \xi.$$

Then (S, σ) is called a pointed endofunctor.

But since we defined the structure map $SX \to SSX$ to be $S\xi$,

$$\sigma_{S(X,\xi)} \;\equiv\; \sigma_{(SX,S\xi)} \;\equiv\; S\xi \;\equiv\; S\sigma_{(X,\xi)}$$

so $\sigma_S = S\sigma$ — they commute.

Max Kelly called this situation a well pointed endofunctor.

Algebras for well pointed endofunctors

We have "our" well pointed endofunctor $\mathsf{id} \xrightarrow{\sigma} S$
and also the terminal one $\mathsf{id} \xrightarrow{\tau} T$ and the composite $S \cdot T$.

Since T is terminal, there are natural transformations

$$T \xrightarrow{\sigma_T} S \cdot T \xrightarrow{\alpha} T, \qquad \alpha \circ \sigma_T = \mathsf{id}$$

So for any object $W \in \mathcal{W}$ there are maps

$$TW \xrightarrow{\sigma_{TW}} S(TW) \xrightarrow{\alpha_W} TW, \qquad \alpha_W \circ \sigma_{TW} = \mathsf{id}_W$$

so that TW is an algebra for the pointed endofunctor.

But S is well pointed and Max Kelly showed (1980, Prop 5.1)
that any such algebra is a fixed point.

Reflective subcategories and the special condition

Kelly's 1980 study was about idempotent monads and
reflective subcategories (among many other things).

In general
well pointed endofunctors (such as idempotent monads)
may have many fixed points (members of a reflective
subcategory).

What is special about our situation?

In our category of recursive coalgebras,
only the terminal object can be a fixed point.

And that completes our fixed point theorem.

Where has the ordinal proof gone?

Instead of artificially forming the transfinite sequence

$$I, \quad SI, \quad SSI, \quad SSSI, \quad \cdots \quad \mathsf{colim}_{n\in\lambda} S^n I,$$

we have used natural category theory to collect

- all of the composites generated by id and S and
- whatever directed colimits exist in the underlying category.

Pataraia's trick (composition) is ordinal addition.

If we iterate the endofunctor category construction
$\mathsf{id} \downarrow [\mathcal{F} \to \mathcal{F}]$ *cf.* the Church Numerals, we can use λ-calculus
to define ordinal multiplication and higher operations.

Where do we go from here?

The categorical Pataraia theorem doesn't depend on S:
it's a general purpose tool,
relying on whatever foundational system we are using.

The interesting thing is the construction of the category $\mathcal{W}$.
It is pure category theory,
with no foundational assumptions.

However, $\mathcal{W}$ is a system of recursion
that can be used to prove properties or make constructions
for the initial S-algebra.

Moreover, $\mathcal{W}$ is defined using algebraic ideas,
so there are homomorphisms of such structures.

Whose fixed point theorem is this now?

Pataraia's principal contribution was to tell us
to abandon Set Theory
and use domain theory, category theory and algebra instead.

His idea ended up playing a minor role in the construction, and the categorical version has probably been done elsewhere.

Ideas of Joachim Lambek, Gerhard Osius and Max Kelly also play an important part in the construction of the category $\mathcal{W}$.

Besides, Pataraia's composition is a special case of the relationship between 2-categories and monoidal categories.

But really, this construction is part of a thread that runs throughout the history of category theory and universal algebra, at least back to start of the 20th century.

No-one is ever more than a baton-carrier for a mathematical argument.