Well Pointed Endofunctors and Recursive constructions in Category Theory

Paul Taylor
Honorary Senior Research Fellow

University of Birmingham
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www.Paul Taylor.EU/ordinals/

Not Category Theory 2025 in Brno

This was intended to be my talk at the international category theory annual conference in Brno (Czech Republic) in July.

Unfortunately, I completely missed the announcement with the "Call for Abstracts" in April.

This work is part of my proposal for completely expunging set theory from category theory, starting with the alleged need for transfinite constructions.

This talk will not consider the Axiom-Scheme of Replacement,

but

I have a definite proposal for a Replacement for Replacement that will be the subject of my talk at ItaCa on 18 November 2025.

Recursive constructions in Category Theory

This work was prompted by the 1980 paper

A unified treatment of transfinite constructions for free algebras, free monoids, colimits, associated sheaves, and so on

by some set theorist called Max Kelly.

It identified generating a monad by iterating a well pointed endofunctor as a simple case of this kind of problem, to which others could be reduced.

What would this paper have said if it had been written by the grandfather of Australian 2-category theory?

Reflective subcategories = idempotent monads

A subcategory is reflective if it is
- full (contains all morphisms),
- replete (closed under isomorphic copies) and
- the inclusion $U : \mathcal{A} \subset \mathcal{X}$ has a left adjoint $F : \mathcal{X} \to \mathcal{A}$.

The composite $M \equiv U \cdot F : \mathcal{X} \to \mathcal{X}$ is an idempotent monad with $\eta : \mathsf{id} \to M$ and $\mu : M \cdot M \to M$ satisfying

$$\eta M \,;\, \mu \;=\; \mathsf{id} \;=\; M\eta \,;\, \mu \qquad \text{and} \qquad M\mu \,;\, \mu \;=\; \mu M \,;\, \mu.$$

but also μ is invertible and therefore $\eta M \;=\; M\eta$.

The following are equivalent for an object $X \in \mathcal{X}$:
- X is a fixed point: $\eta X : X \cong MX$;
- X carries a (unique) (M, η, μ)-algebra structure;
- X is an image: $X \cong MY$ for some Y.

Intersection of reflective subcategories

Let's call objects of our reflective subcategory nice.

Suppose we have a second reflective subcategory, whose objects are pretty.

Do nice pretty objects form a reflective subcategory too?

Not obviously. (Idempotent) monads don't compose.

We have to make the given object nice, then pretty, then nice again, then pretty again, ...

and when we've done this infinitely often, we're still not finished, ...

You can see where this is going, but let's do some honest category theory instead!

Forget the multiplication μ

Since μ is redundant for an idempotent monad, forget it!

A pointed endofunctor $S : \mathcal{X} \to \mathcal{X}$ comes with a natural transformation $\sigma : \mathsf{id} \to S$.

A well pointed endofunctor has $S\sigma = \sigma S$.

Any idempotent monad is a well pointed endofunctor. (Characterised by invertibility of $S\sigma = \sigma S$.)

Unlike monads, well pointed endofunctors compose.

This is true but not *quite* obvious. Kelly doesn't state it.

When I asked about this subject on MathOverflow, a certain very smart categorist challenged me to prove it.

So here goes:

Composition of (well) pointed endofunctors

We compose (well) pointed endofunctors like this:

$$
\begin{tikzcd}
\mathsf{id} \arrow[r, "\sigma"] \arrow[d, "\rho"'] \arrow[dr, phantom, "\lambda"] & S \arrow[d, "S\rho"] \\
R \arrow[r, "\sigma R"'] & SR
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathsf{id} \arrow[r, "\sigma"] \arrow[d, "\rho"'] \arrow[dr, phantom, "\kappa"] & S \arrow[d, "\rho S"] \\
\mathsf{id} \arrow[r, "R\sigma"'] & RS
\end{tikzcd}
$$

using naturality of σ with respect to ρ and *vice versa*.

This is strictly associative with unit $\mathsf{id} \xrightarrow{\mathsf{id}} \mathsf{id}$ because that's true in the 2-category of categories.

If the functors are well pointed then

$$S\kappa \equiv S\sigma ; S\rho S = \sigma S ; S\rho S \equiv \lambda S.$$

and $$R\lambda \equiv R\rho ; R\sigma R = \rho R ; R\sigma R \equiv \kappa R.$$

Finally, $RS\kappa = R\lambda S = \kappa RS$ and $SR\lambda = S\kappa R = \lambda SR$.

Algebras for well pointed endofunctors

Just as for idempotent monads, these are just fixed points:

An object $X \in \mathcal{X}$ carries an algebra structure $\alpha : SX \to X$ for a well pointed endofunctor iff σX and α are mutually inverse.

Suppose that $X \xrightarrow{\sigma X} SX \xrightarrow{\alpha} X$ is id_X.

Since σ is natural with respect to α,

$$
\begin{tikzcd}
SX \arrow[r, "S\sigma X", "\sigma SX"'] \arrow[d, "\alpha"'] & SSX \arrow[d, "S\alpha"] \\
X \arrow[r, "\sigma X"] & SX
\end{tikzcd}
$$

we have
$$\alpha ; \sigma X = \sigma SX ; S\alpha = S\sigma X ; S\alpha = S\mathsf{id}_X$$

so α and σX are inverse and α is unique.

Generating an idempotent monad

We're looking for an idempotent monad (M, η, μ)
with the same fixed points as (S, σ).

We get it by "iterating" S.

(S, σ) is a small step, (M, η, μ) is the big step.

Whilst (M, η, μ) is the unique idempotent monad
(up to unique isomorphism, of course)
corresponding to its subcategory of algebras,

there are many well pointed endofunctors (S, σ)
fixing given objects, in particular
$(S \cdot S, \sigma \cdot \sigma)$, $(S \cdot S \cdot S, \sigma \cdot \sigma \cdot \sigma)$, ... do the same.

We will see that iterating well pointed endofunctors
is a mild categorical generalisation of the ordered case,
which we review briefly.

Fixed points in dcpos — again?

Let $\mathcal{X}$ be a directed-complete poset and
$\mathcal{S}$ a family of inflationary monotone endofunctions of $\mathcal{X}$:

$$\forall x.\, x \leq sx \quad \text{and} \quad \forall x, y.\ x \leq y \Longrightarrow sx \leq sy.$$

Then there is a closure operator $\mathsf{id} \leq m = mm : \mathcal{X} \to \mathcal{X}$
such that

$$\mathbf{Fix}\, m = \mathbf{Fix}\, \mathcal{S} \equiv \{x \in \mathcal{X} \mid \forall s \in \mathcal{S}.\, sx = x\}.$$

If $\mathcal{X}$ has a least element $\perp$ then
$m\perp$ is the least common fixed point of $\mathcal{S}$.
If further the $s \in \mathcal{S}$ satisfy my special condition,

$$(\forall s \in \mathcal{S}.\, x = sx) \wedge x \leq y \in \mathcal{X} \Longrightarrow x = y$$

then $m\perp$ is the top element $\top$ of $\mathcal{X}$, because then
$\mathbf{Fix}\, m \equiv \mathbf{Fix}\, \mathcal{S}$ has only one element, which must be $\top$.

Transfinite recursion in the ordered case

It is so seductive
to do something infinitely often ... and then some more.

How many times have you seen this?

$$\begin{aligned} f0 &= \text{some base data} \\ f(\alpha^+) &= \text{some operation applied to } f(\alpha) \\ f(\lambda) &= \bigcup \{f(\alpha) \mid \alpha < \lambda\} \end{aligned}$$

Then there is a fixed point, by appeal to your fairy godmother.
~~your fairy godmother~~
John von Neumann (1928) and Friedrich Hartogs (1917).

(Kelly doesn't claim fixed points exist for free.
but plenty of other authors do.)

Kazimierz Kuratowski (1922) showed this was unnecessary,
using an argument due to Ernst Zermelo (1908) that eventually
became known as the Bourbaki–Witt theorem (1949–51).

Prove it with a Galois connection

Just like in Galois theory (fields and groups), write

$$s \perp x \quad \text{for} \quad sx = x$$

and generalise this to subsets:

$$\mathcal{S}^{\perp} \equiv \{x \in \mathcal{X} \mid \forall s \in \mathcal{S}.\, s \perp x\} \equiv \mathbf{Fix}\, \mathcal{S}$$

$$^{\perp}\mathcal{A} \equiv \{\mathsf{id} \leq s : \mathcal{X} \to \mathcal{X} \mid \forall x \in \mathcal{A}.\, s \perp x\}$$

so that $\quad \mathcal{A} \subset \mathcal{S}^{\perp} \iff \mathcal{S} \perp \mathcal{A} \iff \mathcal{S} \subset {}^{\perp}\mathcal{A}.$

Then, since inflationary monotone endofunctions compose
(as domain theorists such as me should have noticed,
but Dito Pataraia had to point out to us),

${}^{\perp}\mathcal{A}$ is directed, so has a greatest element.

The greatest element of ${}^{\perp}(\mathcal{S}^{\perp})$ is the required closure operator m.

Categorical version

For $X \in \mathcal{X}$ and $\mathsf{id} \xrightarrow{\sigma} S : \mathcal{X} \to \mathcal{X}$ with $S\sigma = \sigma S$, write

$$(S, \sigma) \perp X \quad \equiv \quad \sigma X \text{ invertible}$$

and extend this to a Galois connection as before.

Since well pointed endofunctors compose, ${}^{\perp}\mathcal{A}$ is directed.

On the outrageous assumption that $\mathcal{X}$ is directed-complete,

- ${}^{\perp}\mathcal{A}$ has a terminal object (M, η);
- (M, η) is an idempotent monad; and
- if $\mathcal{A} \equiv \mathbf{Fix}\,\mathcal{S}$ then $\mathbf{Fix}\,M = \mathbf{Fix}\,\mathcal{S}$; so
- $\mathbf{Fix}\,\mathcal{S}$ is a reflective subcategory.

There is no other way

If $\mathcal{A} \subset \mathcal{X}$ is reflective then
its monad must be the terminal object of ${}^{\perp}\mathcal{A}$.

We deduce this from the equivalence amongst:
(a) there is a morphism $\phi : (S, \sigma) \to (M, \eta)$;
(b) $\sigma M : M \to SM$ is invertible;
(c) $M\sigma : M \to MS$ is invertible;
(d) for all $X \in \mathcal{X}$, MX is an S-algebra;
(e) $\mathbf{Fix}\,\mathsf{M} \subset \mathbf{Fix}\,(S, \sigma)$.
Moreover ϕ in part (a) is unique.

Proof: there is a bijection defined by

$$(\sigma M)^{-1} \;=\; \phi M \,;\, \mu \quad \text{and} \quad \phi \;=\; S\eta \,;\, (\sigma M)^{-1}.$$

(This result involves more interesting 2-category theory
in the case of *general* pointed endofunctors and monads.)

Calculating the colimits

All of the steps in the construction
of the category of well pointed endofunctors
lift colimits from the underlying category.

Except: co-slice only lifts connected colimits.

Kelly showed that colimits lift
from pointed to well pointed endofunctors.

But we're only interested in directed colimits,
so that's ok.

Why not filtered ones?

Because we're using the monoidal structure
of composition of pointed endofunctors
("proof-relevant directedness"),
whereas filteredness is related to finiteness.

That outrageous colimit

The colimit certainly doesn't always exist,
e.g. the covariant powerset functor has no fixed points.

The classical way of handling this restricts to functors
that preserve κ-filtered colimits.

The idea is that, assuming the Axiom of Choice,
the endofunctor $(-)^K : \mathbf{Set} \to \mathbf{Set}$
preserves colimits of diagrams $d : \mathcal{I} \to \mathbf{Set}$ for which,
for any function $f : K \to \mathcal{I}$,
there is already a bound $u \in \mathcal{I}$ with $\forall k \in K.\ \exists.\, fk \to u$,
plus a similar condition on arrows.

Then instead of just a set K, people talk about regular cardinals.
These are just isomorphism classes of sets in which we don't
care what the isomorphims are.

Can we do something with some algebraic meaning instead?

Before we discard the ordinals completely

André Joyal and Ieke Moerdijk in *Algebraic Set Theory* (CUP, 1995) characterised sets ($\in$-structures) and three kinds of ordinals in terms of the successor.

On the other hand, recall that ordinal addition

- is associative,
- is not commutative,
- preserves non-empty joins in the second argument.

Ordinal multiplication has similar properties.

Can we find similar natural algebraic structure in other recursive situations?

Number theorists don't just study 0, 1, 2, 3, . . . but algebraic field extensions (and lots of other things that I didn't understand as a student).

Could logicians, *e.g.* proof theorists, do something like that?

Recall the properties

Just like addition of ordinals,

composition of (well) pointed endofunctors

- is associative,
- is not commutative, and
- preserves connected joins in the second argument.

In fact *well* pointed endofunctors satisfy another property that does not seem appropriate for ordinal addition, although the classical tradition gives no clear idea what *categories* of ordinals should look like.

Maybe we need to repeat this for general pointed endofunctors.

Anyway, let's turn to an example where the colimit does exist.

Polynomial endofunctors

Consider endofunctions of **Set** or a presheaf topos of the form

$$SX \;\equiv\; \Sigma_{N\in C} A_N \times X^N$$

or more generally

$$SX \;\equiv\; \Sigma_{N\in C} A_N \times X^N / G_N$$

where G_N is a group acting on the object N.

These have been studied by several generations of categorists (including me) and variously called species, analytic functors, stable functors, containers.

Composition and directed colimits

Polynomial functors form a bicategory.

Composition, substitution or tensor product is clearly linear (preserves $\sum$) in the second argument, whilst the coefficients A_N can be incorporated into C, so we just need

$$\left(\Sigma_{N\in C} X^N\right)^M \;\cong\; \Sigma_{\Phi\in C^M} X^{\coprod\{\Phi m \mid m\in M\}}$$

By comparison, the directed colimits are much simpler, just acting on the indexing set C.

All of this can be done within a Π-pretopos and therefore the free algebras (W-types) exist, so long as the exponentials N are bounded.

Polynomial functors as spans

The polynomial functor $S : \mathbf{Set} \to \mathbf{Set}$

$$SX \;\equiv\; \sum_{a \in A} X^{B(a)}$$

can be encoded by the function

$$f : \sum_{a\in A} B(a) \longrightarrow A \qquad \text{where} \qquad B(a) \;\equiv\; f^{-1}(a)$$

and more generally $S : \mathbf{Set}^I \to \mathbf{Set}^J$ by a span

$$I \xleftarrow{\;s\;} B \xrightarrow{\;f\;} A \xrightarrow{\;t\;} J,$$

namely $\qquad S \;\cong\; \Delta_s \,;\, \Pi_f \,;\, \Sigma_t,$

where Δ is substitution and
$\Sigma \dashv \Delta \dashv \Pi$ are dependent sum and product.

We need to know how to compose spans...

Composition of polynomial functors

$$\begin{tikzcd}
&&& N \arrow[dl, "n"'] \arrow[rr, "p"] \arrow[dr, phantom, "\text{(iv)}"] && D' \arrow[r, "q"] \arrow[dl, "\epsilon"] & M \arrow[dd, "w"] \\
&& B' \arrow[dl, "m"'] \arrow[rr, "r"] \arrow[dr, phantom, "\text{(iii)}"] && A' \arrow[dl, "h"'] \arrow[dr, "k"] \arrow[rr, phantom, "\text{(ii)}"] && {} \\
& B \arrow[dl, "s"'] \arrow[rr, "f"] && A \arrow[dr, "t"'] \arrow[rr, phantom, "\text{(i)}"] && D \arrow[dl, "u"] \arrow[r, "g"] & C \arrow[dr, "v"] \\
I &&&& J &&& K
\end{tikzcd}$$

This glorious diagram is (I believe) due to Joachim Kock
in his unpublished draft book and subsequent joint paper
Polynomial functors and polynomial monads
with Nicola Gambino (2010). It is well explained there.

Cartesian (natural) transformations

(For a large number of reasons) the appropriate morphisms between polynomial functors are natural transformations for which the naturality square is a pullback:

$$\begin{tikzcd}
& S \arrow[r, "\phi"] & S' \\
X \arrow[d, "f"'] & SX \arrow[r, "\phi X"] \arrow[d, "Sf"'] \arrow[dr, phantom, "\lrcorner", very near start] & S'X \arrow[d, "S'f"] \\
Y & SY \arrow[r, "\phi Y"'] & S'Y
\end{tikzcd}$$

These are encoded as maps between spans like this:

$$\begin{tikzcd}
I \arrow[d, equal] & B \arrow[l, "s"'] \arrow[r, "f"] \arrow[d, "\beta"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[r, "t"] \arrow[d, "\alpha"] & J \arrow[d, equal] \\
I & B' \arrow[l, "s'"] \arrow[r, "f'"] & A' \arrow[r, "t'"] & J
\end{tikzcd}$$

(I've swapped the use of primes from Gambino–Kock.)

Well pointed polynomial endofunctors?

A pointed endofunctor $\sigma : \mathsf{id} \to S$ has this span:

$$\begin{tikzcd}
I \arrow[d, equal] & I \arrow[l, "\mathsf{id}"'] \arrow[r, "\mathsf{id}"] \arrow[d, "\beta"'] \arrow[dr, phantom, "\lrcorner", very near start] & I \arrow[r, "\mathsf{id}"] \arrow[d, "\alpha"] & I \arrow[d, equal] \\
I & B \arrow[l, "s"] \arrow[r, "f"] & A' \arrow[r, "t"] & J
\end{tikzcd}$$

which in the case $I \equiv J \equiv \mathbf{1}$ amounts to

$$A \;\equiv\; \{\alpha \equiv f\beta\} + A', \qquad B \;\equiv\; \{\beta\} + B'$$

so $\qquad \sigma X \;\equiv\; \nu_0 : X \longrightarrow X + S'X \;\equiv\; SX.$

When is this well pointed?

Sorry, it was only last week when I thought of including this example, so I haven't worked it out yet.

Directed colimits

$$
\begin{tikzcd}
I \arrow[d, equal] & B \arrow[l, "s"'] \arrow[r, "f"] \arrow[d, "\beta"'] \arrow[dr, phantom, very near start, "\lrcorner"] & A \arrow[r, "t"] \arrow[d, "\alpha"] & J \arrow[d, equal] \\
I \arrow[d, equal] & B' \arrow[l, "s'"'] \arrow[r, "f'"] \arrow[d, "\beta'"'] \arrow[dr, phantom, very near start, "\lrcorner"] & A' \arrow[r, "t'"] \arrow[d, "\alpha'"] & J \arrow[d, equal] \\
I \arrow[d, equal] & B'' \arrow[l, "s''"'] \arrow[r, "f''"] \arrow[d, "\beta''"'] \arrow[dr, phantom, very near start, "\lrcorner"] & A'' \arrow[r, "t''"] \arrow[d, "\alpha''"] & J \arrow[d, equal] \\
I & B' \arrow[l, "s'''"'] \arrow[r, "f'''"] & A''' \arrow[r, "t'''"] & J \\
\vdots & \vdots & \vdots & \vdots
\end{tikzcd}
$$

We need directed colimits to play nicely with pullbacks, which they do in a locally cartesian closed category.

Stratified models of type theories

Polynomial functors and W-types plainly model free algebraic theories, *i.e.* without equations.

The equations can be modelled as a dependent algebraic theory.

Exponentials $(-)^N$ *sort of* fit into this pattern.

The covariant powerset $\mathcal{P}$ is a polynomial functor analogous to e^X, where $n!$ becomes the symmetric group S_n:

$$\mathcal{P}(X) \;\cong\; \Sigma_n X^n / S_n.$$

Along with these (largely familiar) constructions comes an intrinsic generalised ordinal structure.

Maybe this could be used for categorical proof theory.

Generalised ordinal arithmetic

From the preceding remarks,
the bicategory composed of polynomial functors
admits ordinal-like addition.

Treating addition itself as a functor of this kind,
addition of such functors gives
ordinal-like multiplication,
essentially following the idea of
arithmetic for the Church numerals.

To do this we need a cartesian closed bicategory,
which we obtain by allowing the group quotients
and some other structure related to these groups.

The technology to do this already exists in the literature,
but this interpretation as generalised ordinals is new.

What about cardinals?

Apparently set theorists use these as a measure of logical strength rather than to classify sets up to isomorphism.

In the case of polynomial functors,
there are lots of parameters, including

- the "size" of the exponents, which amounts to
- how filtered the diagrams are whose colimits are preserved; and
- the complexity of the quotienting groups.

Categorically, these say what Π-pretopos we're using.

In other words, the strength of the logic, but
without the obscurantist language.

Several PhD projects

As you gather, there are lots of details here that I haven't worked out.

In particular, I worked on "polynomial functors" in the 1980s and don't want to return to it.

Much more structure of functors like this has been identified since then, and there is plenty of literature by some very clever people.

But my suggestion of generalised ordinal arithmetic is new structure and could make a good thesis topic.

For example, you could search for induction in
Well founded trees in categories
by Ieke Moerdijk and Erik Palmgren (1999)
and re-formulate the arguments
using my abstract notion of well founded object.

Food for thought

Set theory has had 150 years to put its case.

Not only did it fail itself to provide the simple intuitionistic fixed point theorem in order theory,

it inhibited other mathematicians from finding it.

I will be pleased to respond to questions about category theory but not set theory.

Thank you for your attention.

What this technique doesn't do

Returning to the iteration of general functors.

The "Galois connection" that we used to construct initial fixed points assumes that *some* fixed points exist.

In that case, this technique agrees with my work on well founded coalgebras.

When there are no fixed points, such as for the powerset, the techniques do not agree.

But we have not addressed the question of whether transfinite iterates exist.

In set theory this requires the Axiom-Scheme of Replacement.

We need a categorical replacement for that!

I will talk about that at ItaCa on 18 November.