Replacement as an Adjoint Functor Theorem

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Exorcising Ghosts on Halloween

Set Theory is not so much a foundation for Mathematics
as a dungeon equipped with instruments of torture
that are brought out to scare ordinary mathematicians
who ask awkward questions.

The spookiest ghost is Replacement.

Professional mathematicians claim to believe in this liturgy,
without being able to say what it means.

In 100 years, the set theorists have failed to give any
tutorial account to explain to others how to use Replacement.

Undoubtedly we do need it,
for transfinite iteration of functors,
such as the von Neumann hierarchy.

Let's do category theory instead!

There are lots of ways of missing the point
about what Replacement is for.

How can we make a useful general theorem out of it,
in the native language of category theory?

We first have to look at the principal example.

Then to understand the two differences between
topos logic and Zermelo–Fraenkel set theory.

Do we actually already use these differences
when doing category theory?

What new categorical axioms can we add
to the topos ones to handle the examples?

Categorical axioms are formulated as adjunctions.

Canonical example from set theory

The von Neumann hierarchy is the transfinite iteration
of the covariant powerset functor. In particular,

$$V_{\omega\cdot 2} \quad \cong \quad \bigcup_{n\in\mathbb{N}} \mathcal{P}^n(\mathbb{N})$$

is (enough to construct) an elementary topos with $\mathbb{N}$,
but cannot be constructed in elementary topos theory.

So far as I can gather, other applications of Replacement,
such as that "instrument of torture" Borel Determinacy,
factor through the von Neumann hierarchy
or other transfinite iterations of functors.

Apparently, the infinite iterations in homological algebra
are of "small enough" functors
that they don't need Replacement.

Missing the point, semantically

On the face of it, Replacement is about big unions.

There are two familiar ways we could construct these:

(1) It could be an internal category,
such as in HoTT or inside a topos.

Then colimits are just like joins in a complete lattice.

(2) We could employ small and large objects,
such as in Algebraic Set Theory,
with an axiom to say that the image of a small object is small.

This is more like the statement of Replacement in Set Theory.

Both of these miss the point of why Replacement is interesting:
it builds skyscrapers from plans on the ground.

The two ideas above make it look trivial.
They are like dropping building materials from a satellite.

Missing the point, syntactically

Set Theory (since Thoralf Skolem)
has been formulated as a first order theory.

(Ernst Zermelo did not approve of that.)

Categorical logic has also been formulated
as a first order theory, Bill Lawvere's
Elementary Theory of the Category of Sets.

(Paul Taylor does does not approve of that.)

This seems to me to be mere trans*liter*ation.

If we're going to prove a formal equivalence,
it would have to be done like that,
cf. Comparing Material and Structural Set Theories
by Mike Shulman.

But I am not claiming equivalence and
first order logic doesn't help us understand the skyscrapers.

Definite, unbounded and bounded predicates

Ernst Zermelo (1908):
"Axiom III. Whenever the propositional function $\mathfrak{E}(x)$
is definite for all elements of a set M,
M possesses a subset $M_{\mathfrak{E}}$ containing as elements
precisely those elements x of M for which $\mathfrak{E}(x)$ is true."

Thoralf Skolem (1919) gave a meaning to "definite"
as a first order formula, possibly involving
quantification over the whole (class) universe.
(Zermelo did actually approve of that much.)

These are called unbounded predicates.

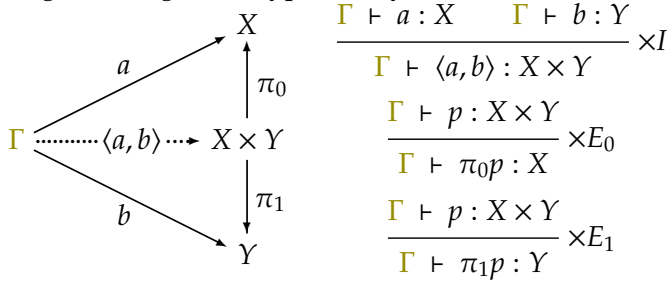
Bill Lawvere (1969) showed how quantifiers are adjoints,
ranging over individual already defined objects (sets).

These are called bounded predicates.

Do we actually use unbounded predicates in category theory?

The generic object in a universal property

Here are the rules for binary product
in categorical logic and type theory:



$$\frac{\Gamma \vdash a : X \qquad \Gamma \vdash b : Y}{\Gamma \vdash \langle a, b\rangle : X \times Y}\times I$$

$$\frac{\Gamma \vdash p : X \times Y}{\Gamma \vdash \pi_0 p : X}\times E_0$$

$$\frac{\Gamma \vdash p : X \times Y}{\Gamma \vdash \pi_1 p : Y}\times E_1$$

Is Γ unboundedly quantified? ("Show of hands?")

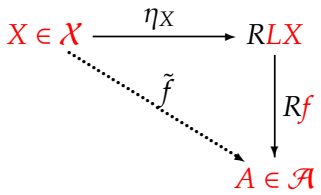
I say no: Γ is not unboundedly quantified.

These rules are absolutely standard
and have nothing to do with what Replacement does.

I argue that Γ is instead legislative or schematic
(but we can debate that elsewhere).

Universal properties

Instead of products (a right adjoint),
we will discuss the putative left adjoint $L? \dashv R$
to a given functor $R : \mathcal{A} \to \mathcal{X}$,
one object $X \in \mathcal{X}$ at a time (Pierre Samuel 1948):



We have agreed that "$\forall A.\ \forall f.\ \exists!\ \tilde{f}$"
is standard and not unboundedly quantified.

But $\exists LX$ is unboundedly quantified!
This is the (first) key point.

Back to the big colimit

We cannot form the colimit

$$\bigcup_{n=0}^{\infty} \mathcal{P}^n(\emptyset)$$

because we cannot form the colimit diagram.

We want to form $\mathbb{N} \longrightarrow \mathbf{Set}$ by $n \longmapsto \mathcal{P}^n(\emptyset)$.
But that doesn't mean anything in topos logic.

We still can't do it using a discrete fibration

$$\sum_{n=0}^{\infty} \mathcal{P}^n(\emptyset) \downarrow \mathbb{N}$$

because that's essentially the same as the colimit!

The second key point is that forming this diagram
is more or less the statement of Replacement.

Left adjoints and colimits

The left adjoint $L : \mathcal{X} \to \mathcal{A}$ (if it exists) must preserve colimits.

As we have just said,
we cannot define the $\mathsf{colim}\, A_i \in \mathcal{A}$ that we need.

But we can define the $\mathsf{colim}\, X_i \in \mathcal{X}$ that we have.

Then $\mathsf{colim}\,(LX_i) \equiv L(\mathsf{colim}\, X_i)$

The difference uses the unbounded predicate
that we have been discussing, namely
the existence of a universal object.

This point is the key subtlety!

But what are these colimits that "we have"?

What are these colimits that "we have"?

A well powered object $X \in \mathcal{X}$ is one with
a "set" $\mathsf{Sub}\,(X)$ of subobjects.

This can be defined in topos logic, using fibrations
(although I don't know of a clear textbook account).

We can manipulate $\mathsf{Sub}\,(X)$ like any other object in a topos.

We can use it to define unions of subobjects.

We can apply the order-theoretic fixed point theorem to it.

All in topos logic, *i.e.* with bounded quantifiers.

It is the putative left adjoint L
that introduces the unbounded predicate
and provides the definition of the big colimit diagram.

Summary of the requirements so far

Both categories must be locally small.

Both categories must have initial objects,
$\emptyset \in \mathcal{X}$ and $Z \in \mathcal{A}$,
so $\eta_\emptyset : \emptyset \to RZ$ is universal.

We will also need $\emptyset \rightarrowtail X$ to be a subobject,
indeed the smallest one.

The category $\mathcal{X}$ must be well powered and
$\mathsf{Sub}(X)$ must have directed unions that are colimits.

The category $\mathcal{A}$ must have
a "forgetful" functor $U : \mathcal{A} \to \mathbf{Set}$
that creates directed colimits.

What has this achieved?

If $\mathcal{X}$ is well powered, with
unions of subobjects that agree with colimits and
$\mathcal{A}$ has a "forgetful" functor $U : \mathcal{A} \to \mathbf{Set}$
that creates colimits.

Then the colimit diagrams that we use

- start off as unions in $\mathsf{Sub}(X)$,
- which are colimits in $\mathcal{X}$,
- get transferred to colimits in $\mathcal{A}$, using
  the unbounded predicate in the universal property,
- evaluated in **Set** using Replacement and
- created back in $\mathcal{A}$ by $U : \mathcal{A} \to \mathbf{Set}$.

It remains to explain which joins we're going to use,
though the unbounded predicate will be needed again!

Other adjoint functor theorems

We pause to make a comparison with them.

If A and X are complete $\bigwedge$-semilattices
and $R : A \to X$ preserves $\bigwedge$ then

$$Lx \quad = \quad \bigwedge \{a \in A \mid x \leq Ra\}.$$

Categorical adjoint functor theorems
are based on the same idea
(but with set-theoretic fudges).

Those pull limits "back" along the right adjoint.

Our method pushes colimits forward along the left adjoint.

But we haven't finished describing it.

Adjoints and recursion

Chemistry is static.

Biology is dynamic.

Static operations don't give us anything very interesting.

Dynamics (recursion) is the powerful thing.

But recursive definitions may fail to terminate.

They must be well founded (Dimitry Mirimanoff 1917).

Well founded objects

A well founded object X has an operation

$$(Y \subset X) \quad \longmapsto \quad (sY \subset X)$$

such that $s : \mathsf{Sub}(X) \to \mathsf{Sub}(X)$ is

- monotone and inflationary and
- $(X \subset X)$ is the unique fixed point.

Are we now going
to do something infinitely often,
and then some more?

Surely this idée fixe of Georg Cantor
is the set-theoretic idea
that has held back mathematics the most.

This induction scheme, without ordinals was already in

- Zermelo 1904 and 1908, with arbitrary joins; and
- Kazimierz Kuratowski 1922, with joins of chains.

But, to our (my) utter shame it took until 1997 to think
of composing the function being iterated.

Induction for our adjoint functor theorem

For any well founded object X,

consider the (unbounded) predicate $\Phi(Y \subset X)$
that $Y \subset X$ has a universal map $\eta_Y : Y \longrightarrow R(LY)$.

- $\Phi(\emptyset)$ holds;
- $\Phi(Y) \Longrightarrow \Phi(sY)$;
- Φ is closed under directed unions; and
- $(X \subset X)$ is the unique fixed point.

Therefore $\Phi(X)$ holds:
there is a universal map $\eta_X : X \longrightarrow R(LX)$.

Altogether, there is a left adjoint $L \dashv R$.

The induction step

Nothing so far has depended on the given functor R !

Suppose that $Y \subset X$ has a universal map $\eta_Y : Y \longrightarrow R(LY)$.

Then we need to construct from that a universal map

$$\eta_{sY} : sY \longrightarrow R(LsY)$$

(within topos logic).

Of course this is done in some way that is peculiar to R.

I devised this method for my study
of well founded and extensional coalgebras,

proposing that
every well founded coalgebra have an extensional reflection.

Transfinite iteration of functors

From here on it is more conventional
(but not trivial) category theory, using

my theory of well founded coalgebras;

factorisation systems, where

- monos must be closed under rich structure, whilst
- epis can be extremely sparse, generating that structure;

fibrations of categories (preorders are enough).

I have written this up (since my CT26 lecture)
as a draft paper, on my website.

Does this tie categorical logic to ZF?

Absolutely the opposite!

ZF is only interesting because
- pure mathematicians claim to use it, but
- in over 100 years, they have not found it inconsistent.

So we now have a new tool that
- is native category theory;
- can probably prove whatever theorems were proved in ZF;
- is as consistent as ZF is (or we could hope for); but
- could be a new weapon to find its inconsistency.

Alternatively, now that we have
re-formulated our Adjoint Functor Theorem without ZF,
it could be added to other foundational systems.

Somebody's PhD thesis!

(These are suggestions from an AI conversation.)

There are existing accounts of Borel Determinacy
written by non-set theorists,
including a formalisation in Lean.

Ulm's classification theorem for countable Abelian p-groups.

Other classical results in measure theory and point–set topology.

These are apparently based on
the von Neumann hierarchy up to ω_1.

Re-formulating them
in category theory using the new adjoint functor theorem
would make an excellent PhD thesis.

Writing this up

I usually work on my own.

In this case it would be better to collaborate
and gather other points of view
on category theory and Replacement.

So I have written a "discussion document"
rather than a draft paper about this.

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