The Antinomies of Russell and Burali-Forti and the Fundamental Problem of Set Theory

Dmitry Mirimanoff, translated by Paul Taylor

1917 and 2024

No mathematical theory has provided so many paradoxes and antinomies, at least apparent ones, as Cantor's famous set theory. The best known and most important of these were identified by Russell and earlier by Burali-Forti. But many other bizarre facts have been observed since the publication of Cantor's original work on set theory, by Borel, Peano, Richard and Cantor himself.

Cantor's antinomies, which almost always baffle at first sight, have been the subject of much despair amongst mathematical logicians. This is undoubtedly the cause of the exaggerated mistrust that the best-established Cantorian theories inspire in particularly hostile minds.

Do we really need to say that there is no fundamental justification for this mistrust or despair? Haven't we had comparable surprises in the theory of functions and in geometry?

Recall, for example, the discovery by Riemann and Weierstrass of continuous functions without derivatives and curves without tangents. These things could only be studied in depth with the help of set theory[1]. We could also mention the curious property of semi-convergent series, that Lejeune Dirichlet discovered, namely that their sums depend on the order of the terms.

In all of these cases, there is a clear conflict between the new facts and properties that we always believed to be true and which had seemed self-evident, but which were really only based on incomplete experience or intuition, only being valid under certain conditions.

Thus, in the case of semi-convergent series, the new fact pointed out by Dirichlet seems incompatible with the fundamental property of algebraic addition, namely commutativity. This property always holds in a finite domain, but Dirichlet's examples show that it can cease to be true when we permute infinitely many terms. The feeling of evidence rests here on an incomplete intuition.

Cantorian antinomies, particularly those of Russell and Burali-Forti, are similar to the examples we have just mentioned. It was believed, and seemed obvious, that the existence of individuals must necessarily entail that of their collection; but Burali-Forti and Russell have shown, by various examples, that a *set* of individuals need not exist, even though the individuals themselves do exist[2].

Since we cannot fail to accept this new fact, we are obliged to conclude that the proposition that seemed obvious to us and that we always believed to be true is inaccurate, or rather that it is only true under certain conditions [Kön14, chap II]. Therefore the following problem arises, which we may see as the fundamental one in set theory:

> What are the necessary and sufficient conditions for a *set* of individuals to exist?

Admittedly, the study of this problem is less advanced than that of semi-convergent series, but the first step has been taken, thanks above all to the research of Russell [Rus03], Poincaré [Poi08], König (*loc. cit.*, chaps II and IX), Hessenberg [Hes06], Schönflies [Sch08] and Zermelo[3] [Zer10].

[1] Mirimanoff cites two single-page abstracts with Mme Grace Chisholm Young, both on page 348 of L'Enseignement Mathématiques 17 (1915), but they don't contribute to set theory.

[2] Mirimanoff consistently uses the words "ensemble" and "exister", but it seems awkward in English (and presumably French too) to describe something and then say that it "doesn't exist", so I am tempted to say "(il)legitimate" for "(non-)existent" and "class" or "collection" for something that may not be legitimate.

[3] Mirimanoff seems not to be familiar with Zermelo's work.

In the final paragraphs of this work, we[4] will give a solution to this fundamental problem for the particular sets that we call *ordinary sets*. Our proofs are based on three postulates, which are already commonly applied in the study of set-theoretic problems.

On the other hand, the examples of Russell and Burali-Forti themselves ought to be examined more closely. We will show that it is easy to give a more precise form to Russell's example by ridding it of cosmetic difficulties that have nothing to do with Russell's antinomy itself. We will then transform Burali-Forti's example in the same way, allowing us to make a new connection between the two paradoxes.

In this work, we will not consider the new distinctions that J. König has introduced into the theory of arbitrary sets, and in particular well ordered sets (*op. cit.*). Two sets containing the same elements will never be considered as different (unless we consider order relations); and to every well ordered set, if it exists, will correspond, by definition, a particular order-type. I will explain elsewhere why König's theory has not been used here for this.

1. Russell's Paradox

We know that Russell distinguishes between two kinds of sets:

- A set E is of the *first kind* if it differs from each of its elements, *i.e.* it is not an element of itself; but
- it is of the *second kind* if does contain itself as an element.

It follows from this definition that any set of the second kind always always contains some set of the second kind among its elements. Hence,

Lemma. A set of sets that are all of the first kind is also a set of the first kind.

Now consider, following Russell, the set R of of *all* sets of the first kind.

It's easy to show that this set doesn't exist. Indeed, because of this lemma, if it did exist, it would have to be of the first kind, *i.e.* different from each of its elements. On the other hand, the set R must contain *all* of the sets of the first kind, and therefore, in particular, the set R itself, which is absurd.

Consequently, the conditions expressed by the words "first kind" and "all" are incompatible, and the set R does not exist.

This is the remarkable example given by Russell, and it shows, as observed in the introduction, that a *set* of individuals need not exist even though the individuals do exist.

2. Now we will give Russell's example in a slightly different form.

First, observe that the only property of the elements that are involved in this example is *how they are made up*. Is an element a set or not? If is, how is it made up? Are its elements in turn decomposable or not? . . . and so on. This is the only important thing to know.

To clarify matters, we introduce an idea that will be very useful in what follows.

Definition. Let E and E' be any two sets. We say that they are *isomorphic*[5] if the following conditions are satisfied:

- The sets E and E' are equivalent; in other words, there is a perfect correspondence (bijection) between the elements of E and those of E'.

[4] Mirimanoff mainly uses the first person singular (I), but at some point in the 20th century this became taboo in journals.

[5] "Isomorphe". Later in the text Mirimanoff also uses the word "semblable" (similar), without giving a definition, but maybe relying on Cantor for this. Only ordinals (S-sets) are called "semblable", whereas both ordinals and general sets are called "isomorphe" and it is not obvious whether or not he understands that these are the same. Decades later, this important concept arose in Process Algebra, where it is called "bisimilar" [San11].

- This correspondence can be made in such a way that to every indecomposable[6] element e of E corresponds an indecomposable element e' of E' and *vice versa*; and
- to every set-element F of E corresponds an equivalent set-element F' of E',
- the perfect correspondence between the elements of F and F' can be in turn made in such a way that
- to every indecomposable element of F corresponds an indecomposable element of F' and
- to every set-element of F an equivalent set-element of F'
- and so on.

If therefore two sets are isomorphic, the elements are also isomorphic, and *vice versa*.

For example, the two sets $\{e_1, e_2, \ldots, e_m, F\}$ and $\{e'_1, e'_2, \ldots, e'_m, F'\}$ are isomorphic, where the e_i and e'_i are indecomposable elements, and F and F' are sets that each contain the same number of indecomposables.

Here we write a set[7] whose elements are a, b, c, $\ldots$ as $\{a, b, c, \ldots\}$, whatever those elements are. For example, a bracketing of the form $\{E, F\}$, where E and F are sets, denotes the set whose *two* elements are the *sets* E and F, and not the *union* of the sets E and F.

We'll say that a set E set is of the *first kind* if it is not isomorphic to any of its elements and of the *second kind* if it is isomorphic to at least one its elements. For example, $E \equiv \{e, E'\}$ is of the second kind, if $E' \equiv \{e', E''\}$, $E'' \equiv \{e', E'''\}$ and in general $E^{(n)} = \left\{e^{(n)}, E^{(n+1)}\right\}$ for each n, where the e_i denote indecomposable elements[8].

The definition of "kind" that we have just given is not identical to the one in the previous paragraph, but the essential properties of the first and second kinds are the same and the lemma remains true.

Now, let's go back to Russell's paradox. Consider the set R' of all sets of the first kind, in the new sense. We showed in the previous paragraph that this does not exist. Indeed, if it did exist, by the lemma it would be of the first kind; whilst it would also be isomorphic to one of its elements, which is cannot be.

We will see that the new form in which we have just put Russell's paradox does not lend itself very well to comparison with that of Burali-Forti. For this we need to reformulate it again.

3. We start by introducing a concept that we will use frequently.

Definition. Let E be a set, E' one of its elements, E'' an element of that and so on. We call the passage from E to E' to E'' and so on a *descent*. Such a descent ends when we reach an indecomposable element. In this case the descent is *finite*, but it does not have to be: for example for any set E of the second kind has a descent from E to an element E' to which it is isomorphic, from E' to an isomorphic E'' and so on.

We say that a set is *ordinary* when it only has *finite* descents and *extraordinary* if at least one of its descents is infinite.

Any set of the second kind is therefore extraordinary. However, these two concepts (the second kind and extraordinary sets) are not equivalent, since an infinite descent can even occur in a set of the first kind.

For example, let $E \equiv \{e_1, E'\}$, where E' is a set of the form $\{e_2, e_2, E''\}$, $E'' \equiv \{e_1, e_2, e_3, E'''\}$ and in general[9] $E^{(n)} \equiv \left\{e_{n+1}, e_1, e_2, E^{(n+1)}\right\}$ for all n. The set E so defined is of the first kind, even though the descent E, E', E'', $\ldots$ is infinite.

[6] These were later called "ur-elements", but it is a pity that Mirimanoff was uncomfortable with the empty set.

[7] Mirimanoff actually used round brackets, $(a, b, c, \ldots)$, but it seems harmless to switch to the subsequently standard notation.

[8] So E is an infinite descending chain with single spurs at each step, making it isomorphic to E', E'' and so on.

[9] $E^{(n)}$ should probably be $\left\{e_1, e_2, \ldots, e_{n+1}, E^{(n+1)}\right\}$

Let us say that the *length* of a descent (in an ordinary set) is the number of steps it has. For any descent there is a particular integer n, but this is not bijective in general. Also, whilst the numbers n are finite, they may be unbounded for a particular set.

The properties of sets of the first kind that we have just seen also apply to ordinary sets. Our lemma is still valid and says:

Lemma. A set of ordinary sets is again an ordinary set.

Now consider the set V of *all* ordinary sets that exist. As in paragraphs 1 and 2, the set V cannot exist.

Now we consider a restricted version of V.

Let E be an ordinary set. By definition, all descents in E are finite and end with indecomposable elements. These, of course, are typically not elements of E itself. So to avoid confusion, we call them E-*nuclei*[10].

Now consider the ordinary sets whose nuclei e, f, g, ... belong to some set $N \equiv \{e, f, g, \ldots\}$ that exists. Let V' be the collection of *all* such sets. We see immediately that this collection, which is a sub-collection of V, cannot exist either.

In particular, the collection of all ordinary sets that have just a single nucleus e does not exist. In paragraph 7 we will consider sets whose nuclei are of a particular form.

From the sets that we have already introduced, we can define other illegitimate collections.

For example, let E be a set of the first kind in the new sense, and $\mathcal{E}$ the collection of all sets that are isomorphic to E. To each set E corresponds such a collection $\mathcal{E}$, and if a set E' is not isomorphic to E then the corresponding collection $\mathcal{E}'$ is different[11] from $\mathcal{E}$. From each collection $\mathcal{E}$ take some representative E_0. The collection of all such E_0, which is a sub-collection of R', does not exist as a set.

Similar sub-collections can be defined starting from the collections V and V'.

In the following paragraphs, we will have to rely on a property of existing sets which is far from obvious, but which we will treat as valid, at least for the sets that we are considering here.

Property 1: The existence of a set entails that of all of its sub-sets, *cf.* [Kön14, VI §16].

Because of this property, it is enough to show that V' does not exist in order to deduce that nor do V, R' or R.

4. The Burali-Forti paradox.

Burali-Forti [BF97] came to his paradox by considering the order-types of well ordered sets, *i.e.* Cantor's ordinal numbers. Recall that these numbers follow a definite law and form a kind of chain whose first links are the finite sequences of 1 (an improper sequence or set), 2, ..., n, ... elements, then the sequence ω, and the order-types $\omega + 1$, $\omega + 2$, ..., $\omega + n$, *etc.*

The properties of well ordered sets are very well known. We just recall two of them that will be particularly useful:

(a) Every well ordered infinite set is similar to the set of all of its segments. This property is still true for finite sets and, therefore, for all well ordered sets, if we include the fictitious segment whose order-type is defined to be 0, which we call by the letter e. The result is that any ordinal number π is the order-type of the set of ordinal numbers $\alpha < \pi$, including 0.

(b) A well ordered set is not similar to any of its segments.

Now consider, following Burali-Forti, the collection W of *all* of Cantor's ordinal numbers.

The collection W thus defined does not exist as a set. Indeed, if W existed, it would be well ordered and would have some order-type π (*cf.* the introduction); yet, any number π is an element

[10] "Noyaux".

[11] Mirimanoff only says "différent", but he was probably aware that they don't overlap at all, as is needed for the next remark.

of W, and the ordinal numbers $\alpha < \pi$ form a segment of W. By (a), the Burali-Forti set would therefore be similar to one of its segments, which is forbidden by (b).

This is the Burali-Forti paradox, the oldest and perhaps the most important of the known Cantorian antinomies.

It would appear at first glance that we are implicitly relying, in Burali-Forti's paradox, on the following assumption: All Cantor ordinal numbers exist. In fact, the Burali-Forti paradox is independent of this postulate. To see this, it is sufficient to consider the set of all ordinal numbers that *exist*. Indeed, if an ordinal number π exists, by property I it is the same as the set of all ordinals $\alpha < \pi$, and therefore every number π that exists is the order-type of the set of ordinal numbers that *exist* and are less than π.

Burali-Forti's reasoning applies without modification and we fall back on the same paradox as before. We write W for the set of all ordinal numbers *that exist*.

Notice that Burali-Forti's antinomy does not depend on the order relation on W. Yet the order-relations on a set are transferred to any equivalent set. From this we deduce the following property:

> *Property II*: if a collection is equivalent to the Burali-Forti set W then it does not exist as a set. More generally, nor does any collection that contains a subset equivalent to W.

5. We will now put Burali-Forti's example in a slightly different form.

Let E be any well ordered set and E' the set of all of its segments, including the empty one e, as in the previous paragraph. Because of property (a), E' is similar to E.

Now replace the segments forming E' with the set of segments of these segments, and apply the analogous transformation to the segments introduced in this way, and so on. Each well ordered set E thus corresponds to a set of a particular form that we call an *S-set*, because this is the initial letter of the word 'segment', as a reminder of the role played by segments in this transformation[12].

The fictitious empty segment e stays the same after this transformation. We see that the S-set corresponds to all of the well ordered sets of the same order-type α. We write α_S for this and call the ordinal number α the *rank* of the set α_S.

For example the S-sets that are derived from the well ordered sets of types 1, 2 and 3 are written

$$\{e\} \qquad \{e, \{e\}\} \qquad \text{and} \qquad \{\{e\}, \{e, \{e\}\}\}.$$

The element e will be considered as an indecomposable.

It follows from the previous definition that the *elements* of any S-set are also S-sets.

Now we see that every S-set is an ordinary set with a single nucleus e. Indeed, in the descents for an S-set, we go through a sequence of segments nested within one another, and we know that such sequences are always finite.

Any descent necessarily ends in the element e.

The S-sets are therefore ordinary sets. It follows, *inter alia*, that an S-set cannot be isomorphic to any of its elements.

Now let E and F be any two well ordered sets and E_S and F_S the corresponding S-sets. There are two possible cases: either the sets E and F are similar, or one of them (say E) is similar to some segment of the other (F).

We have seen that in the first case $E_S = F_S$ and in the second E_S is isomorphic and in fact equal to an element F_S.

Having established this, let's go back to the Burali-Forti paradox. Let W' be the set of all the α_S that exist. We claim that the set W' does not exist.

[12] This is apparently the representation of the ordinals later discussed by John von Neumann [vN23].

First hypothesis: To any existing number α there corresponds an existing set α_S. The set W' is then equivalent to W, but this does not exist, by property II.

Second hypothesis: The existence of a number α does not necessarily entail that of the set α_S. So let π be smallest number such that π_S does not exist. Nor does any α_S exist for $\alpha > \pi$, because the existence of α_S would entail that of π_S, which is an element of α_S. Then W' is the set of those α_S of rank less than π, so $W' = \pi_S$. But W' does not exist, so nor does π_S.

It is now easy to compare the Burali-Forti paradox with that of Russell. In fact, the set R relies on W as an intermediary between the sets R', V, V' and W'. Yet the set R' is a subset of R; the set V is a subset of R', every set V' is a subset of V; and finally the set W' is a subset of some V'. Hence the sets R, R', ..., W' form a sequence of sets that are nested one within another. It follows from this that it is enough to show that the Burali-Forti set does not exist, in order to conclude the same for each of the sets R, R', ..., W'. That is true, as we have seen, for the set W' and, by property I, it is also true of the sets R, R', V and V'. So there is no need to apply Russell's argument to these other sets: each of the partial results that we have obtained directly can be considered as a consequence of of the Burali-Forti paradox.

6. Next we can see that we may define S-sets without the intermediate step using well ordered sets. Let E be an S-set. We have seen that:

1. The set E is an ordinary set with a single nucleus e.
2. If x and y are any two distinct elements of E, one of them is an element of the other.
3. Moreover, if x is an element of E then every element of x is also an element of E.

These properties are characteristic of S-sets and can be used to define them. We show straight away that the sets E so defined are indeed the S-sets of the previous paragraph. To any set E corresponds a certain order-type, and the collection of all of the sets E is W'.

7. The solution of the fundamental problem in the case of ordinary sets.

The study of the various paradoxes that we have seen so far has identified the following facts: in each of our examples, it is possible to form bigger and bigger sets, but the set of *all* individuals[13] does not exist: No matter what set we imagine (so long as it exists), new individuals arise, and even larger sets necessarily appear. We are overwhelmed by indefinite extensions that never cease or have any bounds. In considering the fundamental problem, we will have to clarify this rather vague notion of "never ceasing or having bounds".

Recall, *à propos* of this, that the works that we have cited have deep logical and psychological analyses of the Cantorian paradoxes and the notion of set, but these are not needed for the point that we wish to make.

We will assume that the ordinary sets E that we intend to use in the study of the fundamental problem satisfy the following two conditions:

Condition (a) The elements of E are distinct, likewise the elements of each of these elements, and so on. This condition does not exclude sets E that have isomorphic elements nor those whose set-elements contain isomorphic elements, *etc.* Only identity is forbidden.

Condition (b) The nuclei e, f, g, ... of every set E belong to a set $N \equiv \{e, f, g, \ldots\}$ that we regard as given or known, *cf.* paragraph 3.

We therefore have to solve the following problem:

What are the necessary and sufficient conditions for a collection of distinct ordinary sets that satisfy these two conditions to exist as a set?

We start from the following three postulates:

[13] "Individus", a curious choice of word, since he surely means (nested) sets.

Postulate 1. If a collection of ordinary sets exists as a set, the same is true of the collection of its distinct subsets (the Power Set or Potenzmenge).

Postulate 2. If a set $\{E, F, ...\}$ exists, whose elements E, F, ... are ordinary sets, so does the union (sum) of the sets E, F, ... (the Union Set or Vereinigungsmenge), *cf.* [Kön14, VI §16].

Postulate 3. If a set $\{a, b, c, \ldots\}$ exists, so does every equivalent[14] set $\{E, F, G, \ldots\}$, where E, F, ... are ordinary sets that exist.

We begin the study of the fundamental problem with the particularly simple case of S-sets.

We have called the order-type α of a set α_S its *rank*. By postulate 3, to any ordinal number α that exists corresponds an existing set α_S; the second hypothesis in paragraph 5 should therefore be rejected. Indeed, any α_S is the set of all S-sets of rank less than α. Let π be the smallest existing number such that π_S does not exist; then all of the elements of π_S do exist, but on the other hand π_S is equivalent to an existing well ordered set; so this must exist by postulate 3, contrary to the supposition.

Therefore the collection W' of all of the α_S is equivalent to the Burali-Forti set W.

We say that the S-sets or their ranks *have a Cantorian bound* if there is an ordinal number greater than the rank of each of these sets. Otherwise, such S-sets *have no Cantorian bound.*

Hence we have the following criterion: in order that a set of non-isomorphic S-sets should exist, it is necessary and sufficient that they have a Cantorian bound.

Suppose, first of all, that the S-sets under consideration have no Cantorian bound. Then we say that the set $\mathcal{E}$ of these sets cannot exist. Suppose otherwise, and let π_S be such an S-set.

Write $A(\pi)$ for the set of elements of $\mathcal{E}$ whose ranks are less than π and $B(\pi)$ for the set of S-sets whose ranks are greater than all of those of the elements of $A(\pi)$, but no more than π. The set $B(\pi)$, which only contains a single element of $\mathcal{E}$, namely the set π_S, is a subset of $(\pi + 1)_S$, so it exists by property I.

To every element π_S of $\mathcal{E}$ corresponds a particular set $B(\pi)$.

By postulates 2 and 3, if the set $\mathcal{E}$ existed, it would be the union of the sets $B(\pi)$ extended to all of the elements π_S of $\mathcal{E}$, but the latter set is just the set W'. We know that W' does not exist, so the set $\mathcal{E}$ does not exist either.

The first part of our criterion is established.

Now suppose that the S-sets under consideration do have a Cantorian bound. We claim that the set $\mathcal{E}$ of all of these sets exists. Indeed, let π_S be an S-set with rank greater than the ranks of our sets. Then the set $\mathcal{E}$ is a subset of π_S, so it exists by property I.

Our criterion is established.

8. Before going on to study the general case, we make some remarks to clarify the problem.

Firstly, the criterion in the previous paragraph remains valid if, instead of the S-sets, we use Cantor's ordinal numbers.

Let $\mathcal{E} \equiv \{E_\alpha, E_\beta, \ldots\}$ be any set that is equivalent to a set of ordinal numbers α, β, ... If the numbers α, β, ... don't have a Cantorian bound, the set $\{\alpha, \beta, \ldots\}$ doesn't exist. Therefore, $\mathcal{E}$ cannot exist, by postulate 3, because its existence would entail that of $\{\alpha_S, \beta_S, \ldots\}$. Hence we have this

Lemma. If the numbers α, β, ... have no Cantorian bound then the set $\{E_\alpha, E_\beta, \ldots\}$ does not exist. Nor does any set that contains a subset of this nature, by property I.

We can now extend the notion of rank to the case of any ordinary set that exists.

Definition (r). The rank of an ordinary set is the smallest ordinal number greater than the ranks of its elements. The rank of a nucleus is zero.

[14] This seems to anticipate the Axiom-Scheme of Replacement and the following proof is that of the equivalence of the Cantor and von Neumann notions of ordinals.

This definition provides a definite rank for any ordinary set E. Indeed, suppose that the rank of each of the elements of E is known, according to (r). Since these have a Cantorian bound, the set E also has a rank, by the previous lemma.

If, on the other hand, E had no determined rank, there would be some element E' of E with the same property, likewise E' would have an element E'' without rank, and so on. But this is absurd, since any descent E, E', E'', ... reaches a nucleus, which has rank zero.

Therefore E has some definite rank, by (r).

9. We therefore have the following criterion:

To ensure that a set of distinct ordinary sets satisfying conditions (a) and (b) in paragraph 7 exists, it is necessary and sufficient that the ranks of these ensembles have a Cantorian bound.

We first prove

Lemma. For any ordinal number α, the set O_α of all distinct ordinary sets of rank α satisfies the conditions (a) and (b)[15].

To prove this lemma, we will use an argument that we have already used in the previous paragraph, simply adapted by the principle of complete induction.

Suppose that the lemma is true for all α less than some number π; we will show that it also holds for π.

Let Σ be the union of the sets O_α for all $\alpha < \pi$. This set exists, by postulates 3 and 2. Now, the set O_π is a set of subsets of Σ, so it exists, by postulate 1.

The lemma follows immediately. Indeed, if a set O_α were not to exist, nor would some sequence of sets O_α, $O_{\alpha''}$, ..., where $\alpha > \alpha' > \alpha' \ldots$. This is absurd, since any such sequence would end in the set O_0, *i.e.* at the set N that exists by hypothesis. Therefore O_α exists for any α. QED.

Now let's go back to our criterion. Let $\mathcal{E}$ be the set of all ordinary sets E.

First suppose that the ranks of the sets E have no Cantorian bound. In this case, the set $\mathcal{E}$ does not exist, by lemma in paragraph 8.

Suppose now that the ranks of the sets E have some Cantorian bound, and let π be some ordinal number that is greater than all of these ranks. Consider the set Σ that is the union of the O_α for all $\alpha < \pi$. This set exists, by the last lemma and postulates 3 and 2. But the set $\mathcal{E}$ is a subset of Σ, so it exists by property I.

Our criterion is therefore satisfied.

These are the main results that we aimed to establish in this work.

In summary, in the paragraphs about the paradoxes of Russell and Burali-Forti, we set out to describe the facts that were partly known and to organise them in a new way. Then we moved on to the fundamental problem, for which we have given a solution in the case of ordinary sets, making use of the Burali-Forti paradox and various postulates.

Even though these postulates are frequently used in studying set-theoretic problems, they are far from being obvious, and deserve careful examination and discussion.

We will have the opportunity to come back to these questions in another work devoted to the Cantorian antinomies and the theory of J. König. Unfortunately, I have been unable to get to know the publications since the beginning of the [First World] War.

Geneva, May–September 1916.

References

[BF97] Cesare Burali-Forti. Una questione sui numeri transfiniti. *Rendiconti del Circolo matematico di Palermo*, 11:154–164, 1897. English translation in [vH67], pages 104–111.

[15] O_α is apparently the hierarchy that was later named after John von Neumann and written V_α.

[Hes06] Gerhard Hessenberg. *Grundbegriffe der Mengenlehre*, volume 1. Abhandlungen der Friesschen Schule, 1906. openlibrary.org/books/OL20528826M.

[Kön14] Julius (Gyula) König. *Neue Grundlagen der Logik, Arithmetik und Mengenlehre.* De Gruyter, Leipzig, 1914. reprinted 2020.

[Mir17a] Dimitry Mirimanoff. Les antinomies de Russell et de Burali-Forti, et le problème fondemental de la théorie des ensembles. *Enseignement Mathématique*, 19:37–52, 1917.

[Mir17b] Dimitry Mirimanoff. Remarques sur la théorie des ensembles et les antinomies Cantoriennes, I. *Enseignement Mathématique*, 19:209–217, 1917.

[Mir19] Dimitry Mirimanoff. Remarques sur la théorie des ensembles et les antinomies Cantoriennes, II. *Enseignement Mathématique*, 21:29–52, 1919.

[Poi08] Henri Poincaré. *Science et Méthode. Dernières pensées.* Flammarion, Paris, 1908. gallica.bnf.fr/ark:/12148/bpt6k9691658b.

[Rus03] Bertrand Russell. *The Principles of Mathematics.* Cambridge University Press, 1903.

[San11] Davide Sangiorgi. Origins of bisimulation and coinduction. In Davide Sangiorgi and Jan Rutten, editors, *Advanced Topics in Bisimulation and Coinduction*, pages 111–151. Cambridge University Press, 2011. www.cs.unibo.it/∼sangio/DOC_public/history_bis_coind.pdf.

[Sch08] Arthur Schönflies. *Die Entiwicklung der Lehre von der Punktmanhigfaltigkeiten II.* Tuebner, Leipzig, 1908.

[Van45] Harry Schultz Vandiyer. Les travaux mathématiques de Dmitry Mirimanoff. *L'Enseignement Mathmatique*, 39:169–179, 1945.

[vH67] Jan van Heijenoort. *From Frege to Gödel: a Source Book in Mathematical Logic, 1879–1931.* Harvard University Press, 1967.

[vN23] John (Janos) von Neumann. Zur Einführung der transfiniten Zahlen. *Acta litterarum ac scientiarum Regiae Universitatis Hungaricae Franscisco-Josephinae, Section scientiarum mathematicarum*, 1:199–208, 1923. English translation in [vH67], pages 346–354.

[Zer10] Ernst Zermelo. *Collected Works, volume I.* Number 21 in Schriften der Mathematisch-naturwissenschaftlichen Klasse. Springer-Verlag, 2010. edited by Heinz-Dieter Ebbinghaus, Craig Fraser and Akihiro Kanamori, including parallel English translations.

The journal *L'Enseignement Mathématique* has an open-access archive at

www.e-periodica.ch/digbib/volumes?UID=ens-001

Dmitry Mirimanoff was born on 13 September 1861 about 150km north east of Moscow. He married a French woman but continued to live in Russia until 1900, when they moved to Geneva, where he worked for the rest of his life. According to Vandiyer's obituary [Van45], most of his work was in number theory, with some measure theory and special relativity. The present paper was the first of three on set theory. He died on 5 January 1945 in Geneva.