A Fixed Point Theorem for Categories

Paul Taylor

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Any monotone endofunction $s : X \to X$ of a poset with least element $\bot$ and directed joins $\bigvee^{\uparrow}$ has a least fixed point. Prior to 1997, the best proof of this was the so-called Bourbaki–Witt theorem (c.1950), although this was the core of Ernst Zermelo's second proof (1908) that any set could be well ordered. This required excluded middle and a tricky double induction.

Therefore Dito Pataraia's announcement of a simple constructive proof took everybody by surprise. Like a good computer scientist, he used *functions* instead of sets or relations: composition gives *directedness* of the poset of inflationary monotone endofunctions.

This lemma is not applied directly to the given poset, but to a certain subset of it. Like Zermelo, Bourbaki and Witt, Pataraia used the subset generated by $\bot$, s and $\bigvee^{\uparrow}$. But this particular use of second order logic (or impredicativity) can be eliminated by considering the subset

$$W \;\equiv\; \{w \in X \mid w \le sw \;\&\; \forall a \in X.\, sa \le a \Rightarrow w \le a\}.$$

The proof can now be presented in entirely finitary way, yielding a *specific* directed diagram shape such that the fixed point exists iff joins over this diagram do in the given poset.

This constructive proof is far simpler than the classical ones, but how does it work and relate to the old ones? We can understand this by moving from posets to categories.

The inequalities $w \le sw$ and $sa \le a$ become *coalgebras* and *algebras*. The condition in W captures *recursion* by *coalgebra-to-algebra homomorphisms*. There is at most one fixed point in W, which is its terminal object and the initial algebra in the original category.

Inflationary monotone endofunctions of W become *well pointed endofunctors*, that is, with a natural transformation from id_W that commutes with the functor. Composition of such functors defines a co-affine non-symmetric strict monoidal structure on the category.

Adapting Pataraia's observation, there is a *terminal* well pointed endofunctor. The values of this are algebras for any well pointed endofunctor. In his 1980 study of reflective subcategories, Max Kelly showed that the structure maps of such algebras are isomorphisms, *i.e.* fixed points.

Reflective subcategories need not be singletons. The special property of W that derives a terminal *object* from a terminal *endofunctor* is that there can only be one them.

What this proof for algebras for a functor teaches us is that we should identify the category of *partial sub-structures* of whatever system we want to study, such as Type and Proof Theories. The long-term hope is to identify *intrinsic* structure in the category W in place of the imposed arithmetic or ordinals in these subjects.

Then the relationship to the Zermelo–Bourbaki–Witt proof is that (the composition or monoidal structure of) the endomorphism category plays the role of the (addition of) ordinals.

Gregory Max Kelly, *A unified treatment of transfinite constructions for free algebras, free monoids, colimits, associated sheaves, and so on*, Bulletin of the Australian Mathematical Society **22** (1980) 1–83.

For my work, see `www.paultaylor.eu/ordinals/`