Free Algebras for Functors, with not an ordinal in sight!

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Friday, 26 April 2024

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Fixed points and free models

We will consider the construction and properties of

- the least fixed point of a monotone endofunction of a poset with least element and joins of a certain specific directed diagram, and
- the initial algebra for an endofunctor of a small category with initial object, coequalisers and colimits of a certain specific filtered diagram.

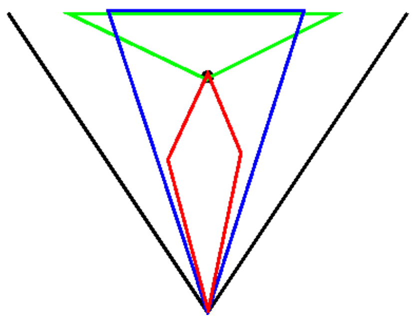
We will divide this problem into two parts:

- a specific finitary one and
- a general infinitary one.

We advocate that the finitary system of "partial solutions" be characterised in examples of these situations (whenever you think you need ordinals)

and generalised to more complex ones as a new algebraic approach to recursion for them.

The general scheme for fixed point problems



The ambient set/type/category $\mathcal{X}$ for the construction.
Those $x \in \mathcal{X}$ for which $x \le sx$ (coalgebras).
Those $x \in \mathcal{X}$ for which $sx \le x$ (algebras).
Partial solutions.
Everything outside these areas is useless to the problem.
How to define the red area is the subject of this seminar.

The theorem mis-attributed to Alfred Tarski

(It was well known at least 50 years before his paper.)

If $\mathcal{X}$ is a complete lattice,

just take the meet $\bigwedge \{x \mid sx \le x\}$ of the green area.

This theorem is packaged in many ways in mathematics, and is invoked whenever we say "the subset generated by..."

So it's impredicative or uses second order logic.

We also want to control the joins (and meets) that we use.

Finitary and infinitary parts of the problem

$\bot, s\bot, ss\bot, sss\bot, \ldots$ are some partial solutions.

We need some infinitary operation
to put all the partial solutions together
to get the total one (fixed point, initial algebra, *etc.*).

This could be a general purpose technique.

However, defining the class of partial solutions is

- specific to the problem, so
- may be rather complicated, but
- should be finitary, and
- could be a valuable "algebraic" structure in itself.

Transfinite recursion

Apparently a primordial reflex of mathematicians.

If $\bot, s\bot, ss\bot, sss\bot, \ldots$ don't work,

form $s^\omega\bot \equiv \bigvee^{\uparrow}_n s^n\bot$ and carry on,
$s^{\omega+1}\bot, s^{\omega+2}\bot, s^{\omega+2}\bot, \ldots$
$s^\lambda\bot, s^{\lambda+1}\bot, s^{\lambda+2}\bot, s^{\lambda+2}\bot, \ldots$

Until you get bored!
And then what?
All this does is label the iterates!
Why should we ever get a fixed point?
What properties can we deduce about it?

Purported proofs by transfinite recursion

Cardinality! — proof by intimidation.

Axiom of Collection! (the image of a class in a set is a set)
— proof by invoking a new axiom
when you don't know how to prove the theorem.

Hartogs' Lemma! — at least that is Mathematics!
(I have a draft translation of Hartogs' 5-page 1917 paper,
but need some help with the German.)

But all it proves is $\exists\lambda.\ \neg\forall\alpha,\beta\in\lambda.\ s^\alpha\bot = s^\beta\bot \Rightarrow \alpha=\beta$,
so to get a fixed point we need Excluded Middle.

It doesn't tell us about the properties of the fixed point.

This ancient "technology" is like
using a steam engine to power your smartphone.

Bourbaki–Witt theorem 1949/51

This is actually due to Ernst Zermelo, 1908.

Consider the subset $W_0 \subset \mathcal{X}$ generated by $\bot, s, \bigvee^{\uparrow}$.

It satisfies $\forall x,y\in W_0.\quad x\le y \quad\vee\quad sy\le x$,
whence W_0 is well ordered.

Therefore we can do induction and recursion over W_0.

But this never got into mainstream pure mathematics textbooks,
except in an *appendix* to a *reprinting* of Serge Lang's *Algebra*.

However, this form of "well ordering"
depends on Excluded Middle and the proof is quite tricky.

Nevertheless, we keep the idea that
W_0 is a set of partial solutions.

Dito Pataraia, 1996/7

Abandon Set Theory, ordinals and transfinite recursion!

Use functions instead (like a good Computer Scientist!)

The inflationary monotone endofunctions of any dcpo form a directed set F, with

$$x \quad \leq \quad fx, gx \quad \leq \quad f(gx), g(fx).$$

If ($\mathcal{X}$ and so) $\mathcal{F}$ are directed-complete then there is a greatest such function.

From this we may deduce the fixed point theorem.

It's constructive — no Axiom of (Choice or) Excluded Middle.

For once, the constructive proof is much easier than the classical one!

Two parts to Pataraia's Theorem

Any dcpo has a greatest inflationary monotone endofunction.

For example, consider the three-point dcpo like a V:
its greatest inflationary monotone endofunction is
the identity. (Not much help!)

So there must be something special about our dcpo W_0 so that the greatest inflationary monotone endofunction $t : W_0 \to W_0$ yields the greatest element of W_0.

The mysterious special condition that works is

$$\forall x, y \in W_0. \quad x \;=\; sx \;\leq\; y \Longrightarrow x = y.$$

Then, since $t\bot \;=\; s(t\bot) \;\leq\; s(tx) \;\geq\; x$,
$t\bot$ is the greatest element of W_0.

If you've got a fixed point, there's nothing more beyond it.

But where does this "special condition" come from?

Partial solutions for Pataraia

Dito Pataraia never wrote up his result and died in 2011.

There are two second hand accounts of it:

- an email from Mamouka Jibladze to Alex Simpson dated 20 January 1997 and
- notes taken by Peter Johnstone from Pataraia's lecture at a PSSL in Århus on 1 November 1997.

Alex Simpson made the proof much simpler and there are numerous fourth hand accounts based on that.

However, all of these took W_0 to be the subset generated by $\bot$, s and $\bigvee^{\uparrow}$, which uses second order or impredicative logic.

Can we avoid that?

Well founded elements

It is enough to use the subset

$$W \;\equiv\; \{x \in \mathcal{X} \mid x \leq sx \;\wedge\; \forall a.\, sa \leq a \Rightarrow x \leq a\},$$

instead of W_0 (although there are several variations on this).

This subset is closed under $\bot$, s and any joins that exist.

So it contains the subset W_0 generated by $\bot$, s and $\bigvee^{\uparrow}$.

But it's defined in a finitary, first order, or predicative way.

More importantly, it is defined
using the idioms of algebra, not logic.

And it satisfies the special condition,

$$\forall x, y \in W. \quad x \;=\; sx \;\leq\; y \Longrightarrow x = y,$$

so it's good enough to use in Pataraia's theorem.

Is the theorem predicative now?

I am not convinced by the motivations for predicativity, so that is for others (Type Theorists) to judge.

The subset W is defined in a finitary way, which is likely to be within any chosen foundational system.

We only need to take a single, specific directed join.

However, it may be objected that

- the diagram is defined from the problem, and
- the join is required in the function space F, not the set W itself.

Even though the fixed point is $\bigvee^{\uparrow}\{f\bot \mid f \in F\}$, we need to form $\lambda x.\ \bigvee^{\uparrow}\{fx \mid f \in F\}$ in order to deduce that we have a fixed point.

Using W in applications

Given a set Θ and a function $\theta : \mathcal{P}(\Theta) \to \Theta$, recursion over a (well founded) relation $(A, \prec)$ is

$$f(X) \;=\; \theta\big(\{f(Y) \mid Y \prec X\}\big).$$

John von Neumann (1928) showed how to solve this as a union of partial solutions.

So we could try to characterise the well founded partial functions.

But the recursion equation makes this system isomorphic to the system of well founded subsets.

Characterising W in applications

I advocate doing this in each specific example of an inductive or recursive situation.

Let $(A, \prec)$ be any set with a binary relation.

The full powerset $\mathcal{P}A$ has a least element $\emptyset$ and directed unions.

Consider the operation $s : \mathcal{P}A \to \mathcal{P}A$ by

$$sX \quad\equiv\quad \{a : A \mid \forall b{:}A.\ \ b \prec a \Longrightarrow b \in X\}.$$

Then any subset $X \subset A$ is

- a well founded element iff
- it is an initial segment on which $(\prec)$ is a well founded relation.

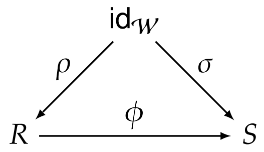
My well founded coalgebras are also an example.

Categorical Pataraia

Now let $S : \mathcal{W} \to \mathcal{W}$ be an endofunctor of a small category with an initial object I.

Consider the (small) category $\mathcal{F} \equiv \mathsf{id} \downarrow [\mathcal{W} \to \mathcal{W}]$ of pointed endofunctors (R, ρ) of $\mathcal{W}$, so $R : \mathcal{W} \to \mathcal{W}$ is a functor and $\rho : \mathsf{id}_{\mathcal{W}} \to R$ a natural transformation.

Morphisms $\phi : (R, \rho) \to (S, \sigma)$ of $\mathcal{F}$ are natural transformations $\phi : R \to S$ such that $\rho\,;\phi = \sigma$;



The identity $\mathsf{id} : (R, \rho) \to (R, \rho)$ is the identity natural transformation $\mathsf{id}_R : R \to R$ and composition is that of the natural transformations. The initial object of $\mathcal{F}$ is $(\mathsf{id}_{\mathcal{W}}, \mathsf{id}_{\mathsf{id}_{\mathcal{W}}})$, from which the unique morphism to (R, ρ) is ρ.

Categorical Pataraia

Pataraia's idea becomes the naturality square

$$
\begin{array}{ccc}
\mathsf{id}_{\mathcal{W}} & \xrightarrow{\ \rho\ } & R \\
\sigma\downarrow & \searrow^{\kappa} & \downarrow R\sigma \\
S & \xrightarrow{\ \rho_S\ } & Q \equiv R \cdot S
\end{array}
\quad \textit{i.e.} \quad
\begin{array}{ccc}
X & \xrightarrow{\ \rho_X\ } & RX \\
\sigma_X\downarrow & \searrow^{\kappa_X} & \downarrow R\sigma_X \\
SX & \xrightarrow{\ \rho_{SX}\ } & QX \equiv R(SX)
\end{array}
$$

whose common diagonal

$$\kappa \equiv \rho \,;\, R\sigma = \sigma \,;\, \rho_S \ :\ \mathsf{id}_{\mathcal{W}} \longrightarrow Q \equiv R \cdot S$$

defines another object of $\mathcal{F}$ and there are morphisms (natural transformations)

$$R \xrightarrow{\ R\sigma\ } Q \xleftarrow{\ \rho_S\ } S.$$

This property is directedness.

The categorical analogue has a further condition for parallel pairs of morphisms, for which we just assume that $\mathcal{W}$ has coequalisers. These are inherited by $\mathcal{F}$, which is then filtered.

Categorical Pataraia

If $\mathcal{W}$ and so $\mathcal{F}$ have colimits over this single, specific filtered diagram then $\mathcal{F}$ has a terminal object

$$T : \mathcal{W} \longrightarrow \mathcal{W}$$

Then any pointed endofunctor $S : \mathcal{W} \longrightarrow \mathcal{W}$
is an object of $\mathcal{F}$,
so it has a unique morphism $!_S : S \longrightarrow T$,
which is a natural transformation.
Similarly for the composite, $!_{S \cdot T} : S \cdot T \longrightarrow T$.

Now let $X \in \mathcal{W}$ be any object.
Applying the natural transformation gives a $\mathcal{W}$-morphism

$$S(TX) \xrightarrow{\ !_{S \cdot T} X\ } TX$$

which is an S-algebra in $\mathcal{W}$.

(This is just the categorical version of $s(tx) \leq tx$.)

We will come back to S-algebras in $\mathcal{W}$ shortly.

Small complete categories

A famous 1960s observation of Peter Freyd was that, classically, any small complete category is a poset (lattice).

However, we have not assumed all colimits (in particular coproducts) so our situation does not obviously trivialise.

Certainly there are endofunctors of large categories,
such as the covariant powerset on **Set**
that have no free algebras.
This technique is not going to change that.

For a functor that preserves monos and has a free algebra,
the set of subcoalgebras of the free algebra
provides the small category, albeit a poset.

Is there an example of a functor that has a free algebra
but does not preserve monos,
so might have a non-trivial category of partial solutions?

This is a problem I leave for another day or other people.

Categories of partial solutions

Here we concentrate instead on the finitary problem.

Consider the categorical form of the order-theoretic fixed point theorem:

We are given an endofunctor $S : \mathcal{X} \to \mathcal{X}$
of a category $\mathcal{X}$ with initial object I,
coequalisers of parallel pairs and
colimits for the diagram $\mathcal{F} \equiv \mathsf{id} \downarrow [\mathcal{X} \to \mathcal{X}]$.

How can we define "partial solutions"
and the small category $\mathcal{W}$ of them?

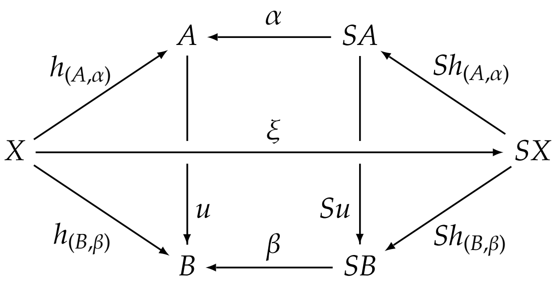
What is its "special property", so that the terminal pointed endofunctor gives the terminal object?

Recipes

A recipe consists of

- an object X of $\mathcal{X}$,
- a morphism (S-coalgebra structure) $\xi : X \to SX$ and
- a cone h under the forgetful functor from the category of S-algebras $\alpha : SA \to A$ and homomorphisms in $\mathcal{X}$.

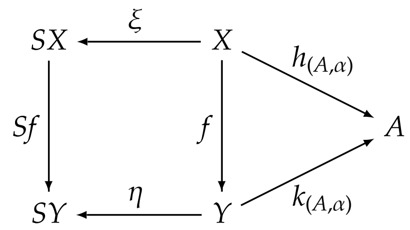
A cone h is a family of $\mathcal{X}$-morphisms $h_{(A,\alpha)} : X \to A$ indexed by the S-algebras $\alpha : SA \to A$, such that the diagram



commutes for each S-algebra homomorphism $u : (A, \alpha) \to (B, \beta)$.

Recipe homomorphisms

A recipe homomorphism $f : (X, \xi, h) \to (Y, \eta, k)$ is an $\mathcal{X}$-morphism $f : X \to Y$ that is a homomorphism with respect to the coalgebras and cones:



Recipes and homomorphisms form a category $\mathcal{W}$.
The initial object I of $\mathcal{X}$ has a unique recipe structure, which defines the initial object of $\mathcal{W}$.

Category of recipes

The endofunctor $S : \mathcal{X} \to \mathcal{X}$ lifts to $S : \mathcal{W} \to \mathcal{W}$, with

$$S(X, \xi, h) \equiv (SX, S\xi, H) \qquad \text{where} \qquad H_{(A,\alpha)} \equiv Sh_{(A,\alpha)} ; \alpha.$$

Then $\xi : (X, \xi, h) \to S(X, \xi, h)$ is a recipe morphism and $\sigma_{(X,\xi,h)} \equiv \xi$ is a natural transformation $\sigma : \mathsf{id} \to S$.

Colimits in $\mathcal{X}$ lift to $\mathcal{W}$.

Algebras over recipes

OK, it starts getting scary here.

Recall that the terminal pointed endofunctor gives

$$p : S(TX) \longrightarrow TX$$

for any object $X \in \mathcal{W}$ and pointed endofunctor $S : \mathcal{W} \to \mathcal{W}$.
This is an S-algebra.

What is an S-algebra $p : SP \to P$ in the category $\mathcal{W}$ of recipes?

It is a recipe (P, ξ, h) and a recipe morphism $p : SP \to P$.

Being a recipe, it has a cone $h_{(A,\alpha)} : P \to A$
over all S-algebras in the underlying category $\mathcal{X}$.

But $p : SP \to P$ itself is an S-algebra,
so the cone includes a map $h_P : P \to P$.

It turns out that h_P is an idempotent homomorphism of recipe-algebras.

Algebras and fixed points

For any recipe-algebra (P, ξ, p, h),
the map $h_p : P \to P$ is
an idempotent homomorphism of recipe-algebras.

Splitting this idempotent gives an object (Q, η, q, k)
that is a recipe-algebra with $SQ \cong Q$ and $k_Q = \mathsf{id}$.

Such a thing is equivalent to both

- the terminal object of $\mathcal{W}$ and
- the initial S-algebra in $\mathcal{X}$.

The new initial algebra theorem

Let $S : \mathcal{X} \to \mathcal{X}$ be an endofunctor of a small category with

- an initial object I,
- coequalisers of parallel pairs,
- colimits for a single specific filtered diagram $\mathcal{F}$.

Let $\mathcal{W}$ be the category of recipes
and $\mathcal{F} \equiv \mathsf{id} \downarrow [\mathcal{W} \to \mathcal{W}]$ its category of pointed endofunctors.

Then $\mathcal{F}$ has a terminal object

and S has an initial algebra,
which is a retract of TI.

What about the "special condition"?

We are looking for the analogue of

$$\forall x, y \in W_0. \quad x \;=\; sx \;\leq\; y \Longrightarrow x = y,$$

so $sx \leq x$ becomes an S-algebra in $\mathcal{W}$
and $x \leq y$ any morphism of $\mathcal{W}$.

Let $f : (X, \xi, p, h) \longrightarrow (Y, \eta, k)$
be a recipe morphism whose source is an algebra.

Then Y also carries an algebra structure
and splitting the idempotents h_X and k_Y
gives isomorphic results.

In fact the algebras TX
given by the terminal pointed endofunctor
already have this idempotent the identity,
ie they are already the initial S-algebra.

Where do we go from here?

The categorical Pataraia theorem doesn't depend on S:
it's a general purpose tool,
relying on whatever foundational system we are using.

The interesting thing is the construction of the category $\mathcal{W}$.
It is pure category theory,
with no foundational assumptions.

However, $\mathcal{W}$ is a system of recursion
that can be used to prove properties or make constructions
for the initial S-algebra.

Moreover, $\mathcal{W}$ is defined using algebraic ideas,
so there are homomorphisms of such structures.

Whose fixed point theorem is this now?

Pataraia's principal contribution was to tell us
to abandon Set Theory
and use domain theory, category theory and algebra instead.
His idea ends up playing a minor role in the construction,
and the categorical version has probably been done elsewhere.

Ideas of Joachim Lambek play a much more important part
in the construction of the category $\mathcal{W}$.
But really, this construction is part of a thread that runs
throughout the history of category theory and universal
algebra, at least back to start of the 20th century.

No-one is ever more than a baton-carrier for a mathematical
argument.