Intuitionism

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Mathematicians and philosophers in French-speaking Switzerland will undoubtedly be delighted to read the wonderful lecture given in 1934 to his students by the eminent geometer who taught at the universities of Geneva, Lausanne and Fribourg. We thank his family for consenting to this publication. Rolin WAVRE.

Mirimanoff's lecture preceded the symposium on mathematical logic organised by the University of Geneva and the lectures given there by Mr Brouwer. It aimed to introduce students to these important subjects.

Introduction

For the past fifteen years or so, there has been much talk of a crisis in mathematics. I do not believe that this crisis, assuming it exists, is as serious as Brouwer's disciples claim, but it is certain that mathematicians do not agree.

On the one hand, there are the *idealists* or *formalists*, whose most illustrious representative is David Hilbert, that *princeps mathematicorum.* Hilbert firmly believes in the solidity of the mathematical edifice and the rigour of classical mathematical reasoning, and seeks to prove it (*cf. Grundlagen der Mathematik*) using a subtle method which Mr Bernays, one of his disciples, will no doubt discuss during his lectures.

On the other hand, there are *intuitionists* or *empiricists*, led by Brouwer, who not only attack the most recent achievements, but also overturn almost everything that has been done since the 17th century and seek to rebuild the mathematical edifice on a new basis, or rather in a new way.

What do they criticise the idealists for, what reasoning do they reject, and what is required for mathematical reasoning to be rigorous in the intuitionist sense? That is what I will try to explain to you today.

1 The origins of Intuitionism

Let us first examine the origins of the new theories. If we look carefully, we could find similar trends or criticisms among the Greeks. However, there is no direct link between Brouwer and Weyl and the philosophers or mathematicians of antiquity.

The origins of modern intuitionism can be found in the writings and teachings of the great German mathematician Leopold Kronecker. Kronecker was an arithmetician who believed that mathematics should be based on a solid foundation of axioms. He was also a strong advocate of the principle that mathematics should be based on logic. Kronecker's work laid the foundations for modern intuitionism. It is in the writings and teachings of the great German mathematician Leopold Kronecker that we must look for the origins of modern intuitionism. Kronecker was a genius in arithmetic, whose discoveries were, as Camille Jordan said, the envy and despair of geometers. Kronecker liked to say: 'Die ganze Zahl schuf der liebe Gott, alles Uebrige ist menschenwerk' (God created whole numbers, everything else is the work of man). For him, mathematical science had to be built on the concept of whole numbers.

It is often said that Weierstrass arithmetised analysis, but he did not arithmetise it in Kronecker's sense: he removed the geometric intuition from it. As Poincaré said, "Weierstrass reduced analysis to a kind of extension of arithmetic; one can read through all his books without finding a single figure." But this is not enough for Kronecker; mathematical reasoning must also be essentially constructive. A mathematical entity only exists if it can be constructed. For example, a proof of the fundamental theorem of algebra must, in order to be legitimate, provide a method for calculating the roots with the desired approximation. The existence of these roots then results from this very process. We therefore see in Kronecker's statements two theses that we will find later in Brouwer:

1) any mathematical concept must be reducible to the concept of an integer;

2) A mathematical entity only exists if it can be constructed.

For Weierstrass, an irrational number has an existence as real as any other object in the realm of thought, while Kronecker asserted that there are only equations between integers. When Lindemann established the transcendence of the number π, Kronecker said to him: "What use is your beautiful research on the number π? Why study these problems, since irrational numbers do not exist? Kronecker therefore questioned the results of which Weierstrass was most proud and criticised his most elegant methods. Weierstrass complained bitterly about this in his letters to his favourite student Sophie Kavalevskaïa. Quoting Kronecker's opinion of modern analysis, Weierstrass added, "Such a statement, coming from a man of such high calibre, whose work I admire... is not only humiliating for those whom he asks to recognise as an error and renounce what has been the subject of their thoughts and efforts... but it is also a direct invitation to the younger generation to abandon their current guides and gather around him as around the apostle of a new teaching that *must* be established."

But the success of Weierstrass's methods and Cantor's theory was so great, and the discoveries and results so brilliant, that after Kronecker's death in 1891, the victory of Weierstrass's and Cantor's methods seemed definitive.

In reality, it was only a lull. Towards the end of the 19th century, in 1897, Burali-Forti revealed the first Cantorian antinomy, which Cantor himself had already encountered two years earlier. I believe there is no need to discuss it here. This antinomy, which bears a certain resemblance to that of Sir Bertrand Russell, caused real concern among some mathematicians and logicians and cast doubt in their minds. On the other hand, in 1904, the German mathematician Zermelo published a famous proof of a fundamental theorem of set theory, which had been anticipated and asserted but not proven by Cantor. This proof sparked a new controversy involving several French mathematicians: Hadamard, Borel, Lebesque, Bair, with some (Hadamard, for example) accepting it and others criticising Zermelo's reasoning and refusing to accept the axiom. Borel, Lebesque, Bair, some (Hadamard, for example) accepting it, others criticising Zermelo's reasoning and refusing to accept the axiom (the axiom of choice) on which the entire proof is based.

But this was still, in Weyl's words, only 'conquests at the most extreme frontiers of mathematics'. Classical mathematics, that of Riemann and Weierstrass, was not threatened. As for Kronecker's criticisms, they were forgotten, and most mathematicians believed that in the analysis as constructed by Weierstrass, absolute rigour had been achieved.

But a few years later, around 1918, the empiricists and intuitionists, led by Brouwer, returned to the fray with renewed vigour.

2 Modern intuitionism

I have already said that Brouwer's criticism and assertions reflect Kronecker's theses, but Kronecker's criticism is becoming more precise and his dream of constructing analysis on a new basis is beginning to be realised.

Kronecker asserted, as you will recall, that:

1) every mathematical concept must be reducible to the concept of whole numbers;

2) a mathematical entity only exists if it can be *constructed* and he criticised Weierstrass ("those who come after me will also recognise the *inaccuracy* of *all* the conclusions on which *what we call analysis*").

However, as Poincaré said in 1900, "in Weierstrass's analysis (today's analysis), there are only *syllogisms* or appeals to the intuition of pure numbers (the only thing that cannot deceive us)".

But the intuition of whole numbers was also accepted by Kronecker (his first thesis).

You can see that the starting point is the same for Kronecker and Weierstrass. (What separates them is the way they define the notion of existence.)

In criticising Weierstrass, Kronecker was therefore implicitly criticising the way in which Weierstrass had built up analysis based on the concept of integers, *i.e.* certain forms of mathematical reasoning or the use of certain rules of logic. Only *constructive* proofs or definitions (in accordance with the second thesis) could be tolerated. A proof or definition that did not provide a method for *constructing* the mathematical entity had to be considered illegitimate.

Brouwer gave precise form to Kronecker's criticisms and desiderata.

It was probably by examining mathematical deductions more closely, in line with Kronecker's theses, that Brouwer was led to reject the use of one of the fundamental principles of logic — the *principle of the excluded middle* ('tertium non datur'), whenever it concerns an infinite mathematical domain.

It is fairly easy to understand why. Indeed, mathematical reasoning based on the principle of the excluded middle is necessarily indirect and therefore non-constructive. Take, for example, the reasoning so often used called *reductio ad absurdum.* To establish a theorem A, we assume that *not* A is true and we are led to an absurd result. We conclude that the hypothesis *not* A must be rejected. Therefore A. As you can see, this is the principle of the excluded middle that has been brought into play, and each time it is applied, it is by an indirect route that we are led to the final conclusion — the reasoning is not constructive.

When a mathematical domain is finite (and this is the least interesting case), we can replace indirect reasoning with direct (constructive) reasoning (we will give an example of this later). But this is not the case when dealing with an infinite mathematical domain. This is why we are led to abandon, at least partially, the use of the principle of the excluded middle when we accept Kronecker's second thesis.

To better understand Brouwer's point of view, I will begin by quoting a passage from a book by Pierre Boutroux that is in harmony with Brouwer's theses. The mathematical fact, says Boutroux, "is independent of the logical garb in which we seek to represent it; the idea we have of it is richer... than any combination of signs by which we can express it."

In other words, mathematics is independent of mathematical language. And it is also independent of *logic*, which merely symbolises a form of mathematical thought, one which, according to Brouwer, relates to *finite* mathematical domains. Mathematics therefore precedes logic, whose laws are deduced from it by abstraction. Mathematics first, logic second.

For Brouwer, classical logic is only valid for sciences onto which a finite mathematical system can be projected. Mathematical thought itself, which Brouwer considered to be the only true form of thought (and here we find one of Kronecker's theses), is essentially constructive. And if common mathematical reasoning, the kind found in the most well-known mathematical theories, does not always have this character, it is because it has been *corrupted* by classical logic, in particular by the application of the *principle of the excluded middle.* Therefore, as I have already said, Brouwer is led to reject this principle whenever it concerns an infinite mathematical domain. To restore the rigour that mathematics lacks, we must seek to reconstruct the mathematical edifice without relying on the principle of the excluded middle.

On what basis are we going to build the new analysis? We know that mathematical science (and in this Brouwer agrees with Weierstrass) must be constructed from the notion of whole numbers . Starting from this basis, according to Brouwer, we can arrive at a rigorous (*constructive*) definition of the concept of set and reconstruct analysis without relying on the principle of the excluded middle.

The notion of existence

For an idealist, a mathematical being exists if it is free of contradiction. To exist therefore means to be free of contradiction. The disadvantage of this definition is that it is often very difficult to show that this condition is satisfied, but when it is, it is considered sufficient.

Intuitionists are more demanding. For them, a mathematical being only exists if it can be constructed. Here again, we find, as we have said, a thesis of Kronecker.

Here is an example. I have already told you about a famous proof of a theorem by Cantor given by Zermelo in 1904. The aim was to show that a set, regardless of its power, can always be well-ordered. It does not matter to us at this point what exactly is meant by well-ordered. Let us simply say that, by virtue of this theorem, the elements of a set can always, at least theoretically, be arranged in a certain order: sequence ω, sequence ω followed by another sequence ω, sequence ω of sequences ω, etc. However, Zermelo's proof does not provide any means of ordering (*well ordering*) the elements of a given set, for example the points on a line (the real numbers). The proof is not constructive. Idealists are satisfied with this. Intuitionists reject it as insufficient.

The principle of excluded middle. — We have seen that it is for similar reasons that Brouwer rejects the excluded middle principle whenever it concerns an infinite mathematical domain.

I will give two examples: in the first, the principle of the excluded middle is applicable in the intuitionistic sense; in the second, intuitionists contest its validity, or rather criticise the way in which it is applied.

First example. Consider a finite sequence of integers, for example the first ten numbers of the form $n^{10} - 1$ ($n = 1, 2, 3, \ldots, 9, 10$). Is there a prime number in this sequence other than 11? What are the possible answers to this question, before examining the problem?

An idealist will say that there can only be two: either

1) there is at least one prime number in our sequence other than 11,

or 2) there are none.

1) is equivalent to not 2); 2) is equivalent to not 1).

No other case is possible; the principle of the excluded middle applies. *Tertium non datur:* either 1) or 2).

This also applies to intuitionists. Indeed, we can always go through our sequence and calculate the 10 numbers. If we come across a prime number to 11, we can say that there is at least one (in the intuitionist sense), since we have obtained (constructed, calculated) it and if we recognise that each of the 10 numbers is divisible by 11, we can say that none of them is prime to 11. In the particular case of our problem, all numbers are divisible by 11 : the answer is negative (Case 2).

But now let us suppose that it is an infinite sequence. Here is a classic example. Consider Fermat's equation

$$x^n + y^n = z^n$$

Let us ask ourselves the following question: is there a positive integer n greater than 2 such that this equation has an integer solution x, y, z?

Here again, an idealist sees only two possible answers to this question: 1) there is at least one n; 2) there is none.

There is no reason to consider any other possibility.

For an idealist, the principle of the excluded middle still applies: 1) is equivalent to not-2); 2) is equivalent to not-1).

But for an intuitionist, this classification is incomplete (inadequate).

Indeed, an intuitionist can only claim that we are dealing with the first case if we know of a process (a method) for calculating or finding a number n with the property in question, since we *can only prove the existence of an object by constructing it or giving it.*

On the other hand, one is only entitled to assert that we are dealing with the second case if one succeeds in proving by direct reasoning that no number x possesses the property in question.

But suppose that it is impossible to decide the question. Suppose, then, that it is impossible to show either that one of these n exists in the intuitionistic sense or that no n possesses the property in question.

There is no reason to believe that this third case could not arise.

We do not have the means to decide the question by examining all the integer values of separately, since the sequence we would have to go through is infinite.

Our two cases can therefore no longer be considered exclusive. The principle of the excluded middle no longer applies; or rather, by considering only two possible cases, we are applying it incorrectly.

Here is a rather curious consequence: suppose we have succeeded in showing that proposition 2 (Fermat's theorem) leads to a contradiction. We would reject it. But for an intuitionist, we cannot conclude that we are then faced with case 1, since 1) is not equivalent to not-2). Case 3) could arise.

You can see that it is indeed the notion of existence that underlies all these distinctions, notions that are narrower than the classical notion.

I believe it is unnecessary to point out other consequences of Brouwer's thesis, except for one that has been much discussed in recent years. You have seen that, alongside the answers yes or no.

You have seen that alongside the answers yes or no (cases 1 and 2), we are led to consider a third case — one where it is impossible to say *either yes nor no*, and where, consequently, the problem is *unsolvable* in the strict sense.

But the question of whether unsolvable problems exist is still a problem that, to date, seems unsolvable to us.

Some mathematicians believe that Fermat's problem is one of these problems. Moreover, it is not necessary to be an intuitionist to believe that unsolvable problems may exist (even in the *idealistic* sense), contrary to Hilbert's assertion ("every specific mathematical problem must necessarily be susceptible to a rigorous solution").

The notion of set following Brouwer. The continuum.

As I have already told you, Brouwer seeks to reconstruct the mathematical edifice without relying on the principle of the excluded middle, beginning with set theory, which he considers to be the foundation of mathematics. But his exposition is extremely complicated, especially when compared to Cantor's simple and clear theory himself, a theory which, in Brouwer's eyes, is merely a game of the mind and not a science.

I do not think it necessary to tell you how Brouwer seeks to reconstruct set theory. It suffices to give you an idea of his definition of the continuum.

We know how continuity is defined in Cantorian theory: arithmetic continuity is the set of all real numbers, rational and irrational. Since irrational numbers are defined by cutting (or by an analogous notion), the notion of cutting (or its equivalent) is implicitly contained in the notion of continuity.

But for intuitionists, this definition has no value.

We must therefore forge another definition of the notion of continuity (a constructive definition).

Let us form any *decimal expansion.* By arbitrary, I mean this: at each step, I freely choose the next digit or decimal place; for example, after the decimal place 2 that I have chosen, I write any decimal place (5, 7, ...).

We will say, with Brouwer, that this decimal expansion is a *free sequence.*

The free sequence defines, as the intuitionists say, a *medium of free becoming* ('Medium freien Werdens').

Well, for Brouwer, this *medium of free becoming* is the *continuum*; it is therefore a *sequence of arbitrary choices.*

A geometric interpretation then provides us with the definition of a point as a sequence in the making of nested intervals.

I will not dwell on this definition. I will only point out a consequence that can be deduced from it and which may seem strange to you.

In classical theory, two points on the reference line are either *confused* or *separated.* There is no tertium.

But for Brouwer (and for Weyl), there is nothing to say that two points could be neither conflated nor separate, again by virtue of the rejection of the tertium non datur, a consequence of the constructive definition of the notion of existence.

Unfortunately, the words "constructive", "construct" and "construction" lack precision.

You can also see that in the definition of the continuum (and this is in line with Kronecker's point of view), we have relied on the notion of a sequence of integers. We therefore seek to construct our mathematical beings from the set of integers, in the manner of Kronecker.

As for the very notion of integers on which our mathematical edifice is based, intuitionists, like Kronecker, consider it to be given by *intuition*.

This is even the primary or *fundamental intuition* of intuitionists (Urintuition).

In summary, here are the main theses of the intuitionists:

1) The concept of integers and the sequence of integers is given to us by intuition.

2) To exist means to be able to be constructed.

3) The principle of the excluded middle is applicable without restriction "only within a finite and determined mathematical domain".

I have sought to show that the third thesis is a consequence of the second. I believe, in fact, that by accepting Kronecker's second thesis, we are inevitably led to reject, as Brouwer did, the use of the principle of the excluded middle when dealing with an infinite mathematical domain.

Conclusion

What should we conclude?

Are mathematics, and in particular the analysis developed by the idealists, at risk of collapsing like a house of cards? I do not think so, and, deep down, the intuitionists do not believe so either, since they are seeking precisely to rebuild at least part of the mathematical edifice in a new way. If they succeed in replacing idealistic proofs with constructive proofs, we will have every reason to rejoice. In 1924, for example, three intuitionists, Skolem, Weyl and Brouwer, succeeded in giving a constructive proof of the fundamental theorem of algebra. This is an extremely interesting result.

A rather serious drawback of this intuitionist reconstruction is that the new proofs are mostly extremely complicated. But for all those who doubt the rigour of Weierstrassian reasoning, it is enough to know that these intuitionist proofs exist.

The fact remains that mathematicians do not agree. Should we say, with Poincaré, that people do not agree because they do not speak the same language and that there are languages that cannot be learned? Should we ask ourselves, along with Hadamard, whether brains are really as homogeneous and comparable to one another as we are tempted to believe and accustomed to thinking?

Note that alongside idealists such as Hilbert and intuitionists such as Brouwer, there are other types of mathematical minds.

On the one hand, there are the *logicians*, including Bertrand Russell, who devised a very curious system, criticised by Poincaré but adopted by some mathematicians. Fraenkel, who was very involved in the foundations of mathematics, prefers it.

And on the other hand, there are also the *realists*.

For realists, whose most illustrious representative was Hermite, mathematical beings have an existence outside of us. Poincaré said of Hermite: "Talk to Hermite: he will never evoke a tangible image, and yet you will soon realise that the most abstract entities are like living beings to him; he does not see them, but he feels that they are not an artificial assembly and that they have some kind of internal unity." "I believe," said Hermite himself, "that numbers and functions of analysis are not the arbitrary product of our minds; I think that they exist outside of us with the same character of necessity as things of objective reality, and that we encounter them, or discover and study them, like physicists, chemists and zoologists."

I will quote Juvet again, professor at the University of Lausanne, a staunch realist. "For us realists," he said in his inaugural speech, "rigour comes from mathematical beings, it is not a requirement of the mind; that is why we will reverse Poincaré's opinion by which is why we will reverse Poincaré's opinion by asking that we be forgiven for this great liberty: — the existence of a mathematical being comes from the fact that it does not imply contradiction; we would rather say: a mathematical being does not imply contradiction precisely because it is, because it exists."

In the face of these attitudes, which are so different and at first glance incompatible, is it necessary to make a choice? I do not think so. It is useful to examine them impartially.

Each of them brings something new, and the absolute does not exist. But I would be very embarrassed, I admit, if I were asked to specify the meaning of the word exists or rather does not exist that I have just used.