Ordinals as Coalgebras

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18 September 2023

This paper is still **work in progress**.

As you can see from the number of declared counterexamples, this is a particularly fiddly subject and likely to be riddled with more than the usual quota of errors.

1 Introduction

This paper is by no means a standard account of the ordinals with a few diagrams. It is an investigation led by *categorical* intuitions of some novel *categorical* ideas that just happen to have been suggested by elementary set theory. It is best viewed as as a worked example of *Well Founded Coalgebras and Recursion* [Tay23] with **Pos** instead of **Set**. This is intended as a model to be re-worked in other categories.

Category theory is the appropriate way to conduct such an investigation because it consists of *a small toolbox of very sharp tools* that have been forged by decades of experience across many mathematical disciplines. As a result, it usually advises the *one correct* definition for a concept, whereas symbolic formulations allow a free-for-all of unstructured ideas.

In fact, the set theoretic ideas will turn out not to be such straightforward instances of standard categorical ones as commonly happens in algebraic disciplines. The effort to give an account that matches them more accurately obliges us to explore some deeper (2-)categorical notions than were first envisaged.

The motivation is that set theory ($\in$-structures) provides *partial* initial algebras where the total one may not exist. So the intention is that future work (by me or others) will substitute other (perhaps much more complicated) categories and functors for the ones that we consider and thereby characterise the partial free algebras for them.

Georg Cantor originally defined *well-orderings* by saying that every non-empty subset has a least element. He showed how to "zip together" any two such structures, with the result that classical well-orderings are *linearly* ordered [Can97, §13 Thms N&E]. The same phenomenon re-appears in Zermelo set theory, but without the linear order, in the form of its bizarre notions of intersection and overlapping union.

The identification of sets with extensional well founded relations was made by Andrzej Mostowski [Mos49, Thm 3], relying on recursion and the axiom-scheme of replacement. The same argument can be used to *impose* extensionality on a well founded relation.

Since John von Neumann, set theorists have said that *ordinals* are *transitive sets of transitive sets*. Re-writing this definition without its ontology, an ordinal is a carrier with a *transitive* extensional well founded relation.

Remark 1.1 In the 1970s people began to develop mathematics without excluded middle and in particular Robin Grayson worked on intuitionistic set theory. He adopted the "transitive" definition of ordinal, along with the "one-point" successor,

$$\alpha^+ \equiv \alpha \cup \{\alpha\}.$$

He observed [Gra77, page 407] that, intuitionistically, these satisfy

$$\beta^+ \in \alpha^+ \iff (\beta^+ \in \alpha \ \vee\ \beta^+ = \alpha) \implies \beta^+ \subset \alpha \iff \beta \in \alpha$$

and
$$\beta^+ \subset \alpha^+ \iff \beta \in \alpha^+ \iff (\beta \in \alpha \ \vee\ \beta = \alpha) \implies \beta \subset \alpha,$$

but not the reverse of the remaining implications.

We will call these the *thin* ordinals. This formulation has continued to be the most popular one in constructive accounts, despite the alternative notions that were developed in the 1990s. For example, Michael Shulman used it in Homotopy Type Theory [Pro13, §10.3] and others have followed him.

Intuitionistic mathematics was developed in a *categorical* style at that time too, interpreted in its then recently introduced analogue of (Zermelo) set theory, namely an elementary topos [Law70]. In particular, Christian Mikkelsen [Mik22] gave a categorical proof of recursion for well founded relations. Gerhard Osius [Osi74] took the $\in$-structures of set theory seriously: he represented any binary relation as a *coalgebra* for the covariant powerset functor and characterised the subset relation as a coalgebra homomorphism, to re-construct Zermelo set theory within any elementary topos.

There was renewed interest in these ideas in the 1990s and in particular how to reverse the one-way implications above. It emerged that there are several (maybe many) kinds of ordinals and that they are best understood by treating the well founded relation ($\prec$) and the "inclusion" that it induces *independently*. This is how we proceed in this paper.

Remark 1.2 In one of these accounts, André Joyal and Ieke Moerdijk [JM95, Awo13] adapted the fibred category theory of open maps in topology to model the large–small distinction in set theory. They used this to present the systems of sets and ordinals as the (large) free algebras with all (small) joins and an operation s such that:
(a) with no extra condition: sets ($\in$-structures);
(b) with $x \leq sx$: thin ordinals;
(c) with $x \leq y \Rightarrow sx \leq sy$: plump ordinals; and
(d) with $s(x \vee y) = sx \vee sy$: directed ordinals.

The names *thin* and *plump* were introduced in my parallel investigation [Tay96] that was intended to be categorical but was in retrospect still too symbolic. The initial idea was to use the *fat* successor $\{\beta \mid \beta \subset \alpha\}$, but this was *too* fat and a non-trivial recursion was needed to obtain the correct definition. The characterisation will be much simpler in the present paper, because we will treat the two order relations in a *genuinely* independent and categorical way from the outset.

It was also observed there that Mostowski's construction amounts to a *quotient* of the carrier, which can be effected by an *equivalence relation* that it defined co-recursively [Tay96, Thm 2.11].

Both of these accounts had a goal of using their ordinals to prove that any monotone endofunction of a directed-complete poset with least element has a least fixed point. Joyal and Moerdijk did this by invoking an additional *axiom of collection*, that the image of a large object within a small one is small. I hoped to use Friedrich Hartogs' construction [Har15], but its application to the fixed point problem irretrievably uses excluded middle.

However, shortly afterwards and out of the blue, Dito Pataraia gave a far simpler proof that exploited *endofunctions* instead of *sub*sets or ordinals, and nothing beyond Zermelo set theory or an elementary topos. Unfortunately, he never published it before his death, but [Tay23, §2] gives an account of it and its context.

Osius's formulation using coalgebras was generalised to any endofunctor of **Set** that preserves inverse images [Tay99, §6.3], introducing the definition of well founded coalgebra.

After a further delay, this condition was weakened to require the functor just to preserve the monos themselves [Tay23]. Far from using a heavy technology of transfinite recursion and "large" objects to prove a simple fixed point theorem, for that work it was essential to *build on* (a development of) Pataraia's result to *prove* the recursion theorem.

That work examined exactly what properties are required of the underlying category to develop generalisations of the recursion theorem, set-theoretic intersection and union and the extensional quotient. In particular, it replaced the "plain" monos (1–1 functions) between sets with a factorisation system.

Against this background, the present paper may be seen as the *exercise book* for the previous one as the *textbook*. We give proofs both by exploiting that technology and directly, using both symbolic and categorical arguments. The purpose of this detail and repetition is to learn the techniques thoroughly before applying them to more complex situations in future work.

Specifically, instead of discrete sets we consider posets and in place of the powerset $\mathcal{P}$ we have the similar functor $\mathcal{D} : \mathbf{Pos} \to \mathbf{Set}$ that yields the (po)set of all *lower* subsets in the order. There are at least three factorisation systems instead of 1–1/onto functions.

Section 2 prepares for this by examining to what extent the category of posets enjoys the properties of sets that the original arguments exploited. We will need to work through rather a lot of elementary facts and fallacies, so this is a *toolbox* of such things and you may prefer to start with the main job in Section 3 and return as necessary. On the other hand, it is an *overview* of the structure of the underlying category and endofunctor that will be needed to develop a theory of ordinals for other situations.

Section 3 characterises coalgebras and their homomorphisms for the lower-sets functor, giving various terminology and illuminating examples that will be needed in the remainder of the paper. The larger category has a weaker characterisation of its morphisms than in the case of **Set**. The various categories of "ordinals" of headline interest are embedded in it, but in several cases with the same notion of morphism as in the case over **Set**.

Section 4 considers well-foundedness. The way in which it was defined in [Tay23] is in principle flexible enough to measure logical complexity or quantifier depth. However, the application in this paper essentially still uses the full *higher*-order logic of a topos, making the new versions equivalent to the old one. Nevertheless, adaptations of the present work that, for example, replace powersets with polynomial functors will need to take account of possibly different forms of induction and recursion.

Section 5 begins to show the power of our programme by considering the notion of *extensionality*. Recall that a binary relation $(\prec)$ is traditionally said to have this property if

$$\forall xy. \qquad \bigl(\forall z.\, z \prec x \Leftrightarrow z \prec y\bigr) \implies x = y,$$

which amounts to saying that the structure map of the coalgebra is a 1–1 function. Our principal innovation is to replace 1–1 functions with a factorisation system.

Remark 1.3 Well-foundedness and extensionality together provide the force of the mathematical structure that was behind Cantor's "zipping" property and then used for set theory. What we are generalising is what set theorists call *transitive* sets, rather than general ones, but beware that their use of this word is not the standard one. These are essentially *fragments* of a model of set theory.

Such things have some very strange properties, compared with other mathematical objects:

(a) Even though the way in which one set may be a subset of another is a *homomorphism* of $\in$-structures (or for us, of coalgebras), this can only happen in a *single* way;
(b) then there is a 1–1 matching of hereditary elements;
(c) hence sets have no automorphisms besides identities.
(d) Also, two *â priori* independent sets in general have a non-trivial intersection; and so
(e) their union is not a coproduct but a pushout over this intersection.

This behaviour remains a key feature of the general theory (Theorem 5.2) but beware that its proof takes three sections of the previous paper. It also assumes that the functor preserves the "monos", so it is valid for two of the factorisation systems that we consider here, but not the third.

Remark 1.4 In Section 6 we develop our analogue of the ordinal rank and Mostowski's extensional quotient. Categorically, these say that the subcategories of extensional coalgebras, according to (a different two of) the notions that we consider, are *reflective*. This was done in [Tay96] by means of a co-recursively defined equivalence relation, but now we take the co-recursion step by step. Each successive quotient results from factorising the structure map of the coalgebra according to the chosen factorisation system.

Whereas the *symbolic* development of the thin ("transitive") ordinals was easy but the plump ones required a difficult recursion, the latter are natural products of the *categorical* approach. Unfortunately, this does not straightforwardly give the thin ones, so in Section 7 we see a hint of the 2-categorical phenomena that seem to lie behind this, but did not appear in the earlier work. For the transitive closure to be a reflection into a subcategory requires explicit consideration of the poset order on the coalgebras. In fact, we see that three different such orders are needed to understand the situation.

Remark 1.5 Whilst *well founded* induction and recursion are based on the predecessor relation ($\prec$) or its abstraction as a coalgebra, *transfinite* recursion is defined in terms of "successors" and "limits" (joins, unions or colimits). In the classical theory, each ordinal can be classified as either zero, a successor or a limit, but we cannot expect this trichotomy to survive in a constructive situation. It does nevertheless still make sense to present instances of recursion in terms of successors and unions, but instead of a case-analysis, we must regard them as *simultaneous equations.* This is sufficient because every ordinal is the join of the successors of its elements.

Section 8 considers joins, in particular binary ones, these being the overlapping ones from set theory that result from zipping them together. For this we need to identify the axioms obeyed by pushouts in the *category* of sets and how well their properties transfer to posets. We find that only one of three systems of monos behaves well, but that thin ordinals actually behave quite similarly to plump ones for this part of the theory.

Section 9 shows how the thin and plump successors of the earlier accounts become two instances of a generic notion. Like transitivity (Section 7), this is derived from the unit of the monad structure.

Section 10 brings joins and successors together to prove transfinite recursion for thin and plump ordinals.

Remark 1.6 The systems of ordinals and indeed the categories that we consider are all "large". Joyal and Moerdijk can deal with such things using the technology that they develop. However, we prefer to consider them as *schemes*, so for example "**Pos**" is a shorthand for *what it is to be* a poset or monotone function, essentially just the axioms. (Our motivation for *not* introducing any kind of universes is the way in which we intend to approach the axiom-scheme of replacement in future work.)

The difficulty with the scheme approach is that transfinite recursion ostensibly yields a function from a large object to a small one.

In fact, these large preorders of ordinals are the (illegitimate) colimits of their slices or down-sets. These in turn have a simple characterisation as *small* objects, since the homomorphisms from other (thin or plump) ordinals into a given one are just lower subsets.

In general a map out of a colimit is equivalent to a *cocone* under the corresponding diagram. This brings the result of transfinite recursion back within our scheme point of view, so long as its partial approximants are compatible with the diagram.

A more complicated case is the *successor* operation, considered as an (illegitimate) function from the entire system of ordinals to itself. In the first instance we might hope that this will be a compatible scheme of endofunctions of the individual ordinals.

However, the successor endofunction for each particular ordinal is in general *partial*, but it can always be made total by allowing the values to be in a larger ordinal. In terms of colimit diagrams, this means that the "cocone" needs to be like a wonky ladder that links each level on one side to *a step up* on the other.

Overall, from the point of view of a structural mathematician, the plump ordinals (those whose successor operation preserves order) work the most neatly of the different systems. However, in Section 11 we see that they grow *very fast*, so that $\omega \cdot 2$ does not exist in the simplest non-classical topos. The method of Section 6 does not converge and so cannot construct the plump rank.

The set theorists will intervene here to say that the *axiom-scheme of replacement* in ZFC rescues these constructions. However, in future work we intend to turn this argument on its head: instead we will use the methods introduced in this paper to provide a *replacement for replacement*, in the *native language of category theory*, that is, to use *adjointness in foundations* as Bill Lawvere told us to do [Law69].

2 From sets to posets

Notation 2.1 As explained in the Introduction, we will compare

(a) the category **Set** of sets and functions (or indeed any elementary topos), equipped with the ***full powerset***, $\mathcal{P}$, considered as a covariant functor, *i.e.* acting as the direct image operation on functions:

$$\begin{aligned} \mathcal{P}(X) &\equiv \{U \mid U \subset X\} \\ \mathcal{P}fU &= \{f(x) \mid x \in U\} \qquad \text{for } f : X \to Y \text{ and } U \subset X \end{aligned}$$

(b) with the category **Pos** of posets (sets equipped with reflexive, transitive, antisymmetric relations) and monotone (order-preserving) functions, equipped with the *lower sets* functor $\mathcal{D}$:

$$\begin{aligned} \mathcal{D}(X, \leq) &\equiv \{U \mid U \subset_\downarrow X\} \equiv \{U \subset X \mid \forall x, y \in X .\, x \leq y \in U \Rightarrow x \in U\} \\ \mathcal{D}fU &\equiv \{y \in Y \mid \exists x \in U .\, y \leq_Y f(x)\} \qquad \text{for } f : (X, \leq) \to (Y, \leq) \text{ and } U \subset_\downarrow X. \end{aligned}$$

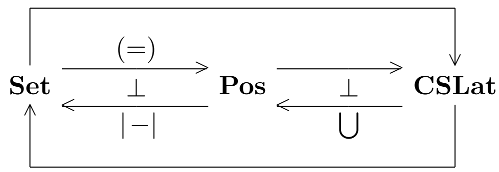
Here we use the symbol $U \subset_\downarrow X$ for a ***lower subset***, *i.e.* one for which

$$X \ni x \leq u \in U \implies x \in U.$$

Also note that $\mathcal{D}f$ preserves order even if the function f does not and that the antisymmetry axiom ($x \leq y \leq x \Rightarrow x = y$) will have an impact at several places in this work.

First we examine the relationship *between* the categories **Set** and **Pos** in more detail, to explain why we have chosen $\mathcal{D}$ as the analogue of $\mathcal{P}$.

Remark 2.2 There is a diagram of categories and adjoint functors



in which the leftward functors forget the relevant structure and $\mathbf{Set} \to \mathbf{Pos}$ assigns the *discrete* order $X \mapsto (X, =)$.

The composites *via* **CSLat** (the category of complete $\bigvee$-semilattices and join-preserving functions) provide the endofunctors $\mathcal{P} : \mathbf{Set} \to \mathbf{Set}$ and $\mathcal{D} : \mathbf{Pos} \to \mathbf{Pos}$. Since these functors arise from adjunctions, they have ***monad*** structures, for which we write η and μ as usual in both cases. However,

- for a *set* X, $\eta_X : X \to \mathcal{P}X$ gives the ***singleton***, $x \mapsto \{x\}$; whilst
- for a *poset* $(X, \leq)$, $\eta x \equiv \{y \mid y \leq x\} \equiv {\downarrow} x$ is the *lower* or *down-set*; and
- the *multiplication* $\mu : \mathcal{P}(\mathcal{P}X) \to \mathcal{P}X$ or $\mu : \mathcal{D}(\mathcal{D}X) \to \mathcal{D}X$ and the *structure maps* of $\mathcal{P}$- or $\mathcal{D}$-*algebras* are given by ***unions***.

Remark 2.3 Joyal and Moerdijk [JM95] showed that ***transfinite recursion*** over ordinals may be seen as a consequence of the theory of monads. (That ordinals form a proper class unfortunately obscures this relationship.) Briefly, the union (multiplication, μ) is of course connected with *limit* ordinals (Section 8), but less obviously, the unit η gives rise to *successors*, as we will see in Section 9. *Transfinite recursion* brings them together (Section 10).

Remark 2.4 Both $\mathcal{P}$ and $\mathcal{D}$ *add all joins*, yielding complete join-semilattices. However, whereas the power*set* then *forgets* this structure entirely, the down-sets endofunctor $\mathcal{D}$ on posets *respects the given order*.

For ordinary monads, any given *set* can in principle enjoy *many* different algebra structures. In contrast, *posets* can only have joins in *one* way, that is in fact *left adjoint* to the unit map (the inclusion of generators) with respect to the order *between* morphisms.

This is typical of the abstract 2-categorical situation, which was studied by Anders Kock [Koc95] and Volker Zöbelein [Zöb76], so it is called a ***KZ-monad***.

Other KZ-monads capture different *classes* of joins, possibly assuming some of them together with meets and other structure. They generalise to colimits and there are dual versions for meets and limits.

Here is simplest characterisation of a KZ-monad for joins:

Lemma 2.5 $f \leq g \Longrightarrow \mathcal{D}f \leq \mathcal{D}g$ and $\mathcal{D}\eta_X \;\leq\; \eta_{\mathcal{D}X}$.

Proof

$$\begin{array}{rrcl} & \mathcal{D}fU & \equiv & \{y \mid \exists x \in U.\, y \leq fx \leq gx\} \\ \subset & \mathcal{D}gU & \equiv & \{y \mid \exists x \in U.\, y \leq gx\} \\ & \mathcal{D}\eta_X U & \equiv & \{V \in \mathcal{D}X \mid \exists x \in U.\, V \subset \eta_X x\} \\ & & = & \{V \subset_{\downarrow} X \mid \exists x \in U.\, \forall y \in V.\, y \leq x\} \\ \subset & \eta_{\mathcal{D}X} U & \equiv & \{V \in \mathcal{D}X \mid V \subset U\} \\ & & = & \{V \subset_{\downarrow} X \mid \forall y \in V.\, \exists x \in U.\, y \leq x\}. \end{array}$$

□

We will see in Section 7 that KZ-monads have a novel relationship to the theory of ordinals, namely regarding *transitivity* of the well founded relation.

When we compare the notions of well founded $\mathcal{P}$- and $\mathcal{D}$-coalgebras in Section 4 we will need another specific technical consequence of the diagram of adjoints in Remark 2.2:

Notation 2.6 For any poset $(X, \le)$, write $\epsilon_{(X,\le)} : (X, =) \to (X, \le)$ for the function from the discrete poset $(X, =)$ to the given one. This is the counit of the adjunction between the discrete order and underlying set functors. It is monotone and surjective on points, but does not reflect the order.

$$\begin{array}{ccc} (X,=) & \xrightarrow{|f|} & (Y,=) \\ {\scriptstyle \epsilon_{(X,\le_X)}}\downarrow & & \downarrow{\scriptstyle \epsilon_{(Y,\le_Y)}} \\ (X,\le_X) & \xrightarrow{f} & (Y,\le_Y) \end{array} \qquad \begin{array}{ccc} \mathcal{P}X & \xrightarrow{\mathcal{P}f} & \mathcal{P}Y \\ {\scriptstyle \mathcal{D}\epsilon_{(X,\le_X)}}\downarrow & & \downarrow{\scriptstyle \mathcal{D}\epsilon_{(U,\le_Y)}} \\ \mathcal{D}(X,\le_X) & \xrightarrow{\mathcal{D}f} & \mathcal{D}(Y,\le_Y) \end{array}$$

Applying $\mathcal{D}$ to this yields yields the diagram on the right. Naturality of ϵ means that the square on the left commutes and functoriality of $\mathcal{D}$ makes the right one commute too.

Lemma 2.7 There is a right adjoint $\mathcal{D}\epsilon_X \dashv \epsilon_X^*$ as a monotone function, with

$$\epsilon_X^* \,;\, \mathcal{D}\epsilon_X = \mathsf{id}_{\mathcal{D}X} \qquad \text{and} \qquad \mathsf{id}_{\mathcal{P}X} \le \mathcal{D}\epsilon_X \,;\, \epsilon_X^*.$$

Proof For the equality, $V \in \mathcal{D}X$ is $V \subset_\downarrow X$, which ϵ_X^* takes to $V \in \mathcal{P}X$ and $\downarrow V = V \in \mathcal{D}X$. For the inequality, $U \in \mathcal{P}X$ is a general $U \subset X$, with goes to $U \subset \downarrow U$. □

So far we have largely described the *similarly* between the categories of sets and posets. The power of the generalisation to a new setting comes from replacing the *single* notion of 1–1 function between sets with the many possibilities that other categories offer.

Definition 2.8 We will consider the following classes of "generalised monomorphisms" between posets:

$\mathcal{I}$ ***plain monos***, injective functions: subsets with a possibly sparser order; these are the ***monomorphisms*** in **Pos** in the standard categorical sense.

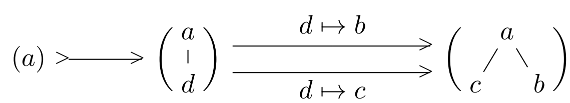
$\mathcal{R}$ ***full subsets***: subsets equipped with the restricted order relation; these are the ***regular monomorphisms*** in **Pos**.

$\mathcal{L}$ ***lower subsets***: if $x \le y$ in X with $y \in U \subset_\downarrow X$ and $U \in \mathcal{L}$ then $x \in U$, where U carries the restriction of the order relation on X.

In Section 6 we will see that that these classes of "monos" are accompanied by "epis" that form factorisation systems.

First we see how these classes arise.

Example 2.9 In **Pos**, this equaliser inclusion is in $\mathcal{R}$ but not $\mathcal{L}$:



Lemma 2.10 For any inclusion $i : U \rightarrowtail X$ in **Set**, $\mathcal{P}i : \mathcal{P}U \rightarrowtail \mathcal{P}X$ is lower (in $\mathcal{L}$).

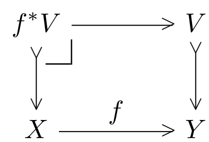
Proof Suppose $\mathcal{P}X \ni Y \subset V \in \mathcal{P}U \subset \mathcal{P}X$, so $Y \subset V \subset U \subset X$. Then $Y \subset U$, so $U \in \mathcal{P}U$. $\square$

Fundamental to their role in any adaptation of categorical logic is that these three classes are closed under pullback.

Proposition 2.11 The forgetful functor $|-| : \mathbf{Pos} \to \mathbf{Set}$ creates limits.

Proof This means that, to construct a pullback *etc.* of a diagram of posets, we first find it for the underlying sets and then equip the result with the unique order that is compatible with the diagram and the universal property. This is the densest order for which the maps in the limiting cone are monotone. $\square$

Lemma 2.12 The pullback of any $\mathcal{I}$-, $\mathcal{R}$- or $\mathcal{L}$-map $V \rightarrowtail Y$ along any monotone function $f : X \to Y$ belongs to the same class.



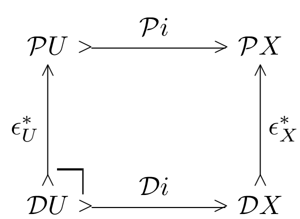
Proof For $\mathcal{I}$, this is just the result from **Set**, where the inverse image of a 1–1 function is again 1–1.

For $\mathcal{R}$, this just takes more careful reading of the Proposition, so f^*V inherits the order from V.

The $\mathcal{L}$ case is part of the proof of Lemma 2.19 below. $\square$

Pullbacks can also arise "spontaneously":

Lemma 2.13 For any $i : U \rightarrowtail X$ in $\mathcal{L}$ this square is a pullback:



Proof The square commutes because its sides are naïve inclusions, acting as the identity on $V \subset_\downarrow U \subset_\downarrow X$. It is a pullback or intersection because if $V_1 \in \mathcal{P}U$ and $V_2 \in \mathcal{D}X$ agree in $\mathcal{P}X$ then $V_1 = V_2 \in \mathcal{D}U$. $\square$

Example 2.14 For $i \in \mathcal{R}$, the square need not even commute.

Proof Let $V \equiv U \equiv \{\top\} \subset X \equiv \{\bot \leq \top\}$. This goes to $\{\top\}$ by the upper route and $\{\bot \leq \top\}$ by the lower one. $\square$

When we apply the downsets functor $\mathcal{D}$ to these classes we begin to see that some things work in the passage from one category to another, but others don't.

Lemma 2.15 The functor $\mathcal{D}$ preserves the class $\mathcal{L}$.

Proof Let $i : U \rightarrowtail X$ in $\mathcal{L}$, *i.e.* $U \subset_\downarrow X$. Then $\mathcal{D}X \ni W \subset V \in \mathcal{D}U$ is $W \subset V \subset_\downarrow U \subset_\downarrow X$ with $W \subset_\downarrow X$. Thus if $U \ni u \leq w \in W \subset_\downarrow X$ then $u \in W$, so $W \subset_\downarrow U$ and $W \in \mathcal{D}U$. $\square$

Lemma 2.16 The functor $\mathcal{D}$ preserves the class $\mathcal{R}$.

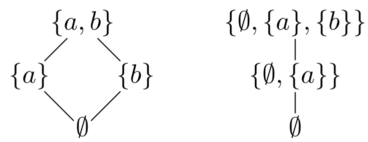
Proof Let $(i : X \to Y) \in \mathcal{R}$ and $U', U \subset_\downarrow X$ be lower subsets with $\mathcal{D}i(U') \subset \mathcal{D}i(U)$.

Recall that $\mathcal{D}i(U) \equiv \{y \mid \exists x \in U.\, y \leq ix\}$, so the inclusion says that

$$\forall x' \in U'.\, \exists x \in U.\, ix' \leq ix.$$

But we may delete i from this since it reflects the order ($i \in \mathcal{R}$), and then $U' \subset U$ since $U \subset_\downarrow X$. $\square$

Example 2.17 The functor $\mathcal{D}$ does not preserve the class $\mathcal{I}$. For example with $i : \{x, y\} \to \{x \leq y\}$, $\mathcal{D}i$ goes from a 4- to a 3-point lattice and $\mathcal{D}i\{y\} = \{\emptyset, \{a\}, \{y\}\} = \mathcal{D}i\{a, y\}$.



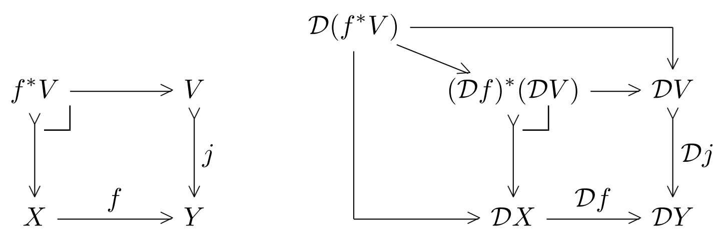
Indeed:

Lemma 2.18 For $f : X \to Y$ if $\mathcal{D}f : \mathcal{D}X \to \mathcal{D}Y$ is 1–1 on points ($\mathcal{D}f \in \mathcal{I}$) then f reflects order ($f \in \mathcal{R}$).

Proof Let $x', x \in X$ with $fx' \leq_Y fx$ in Y. Then $U \equiv \downarrow\{x', x\}$ and $V \equiv \downarrow\{x\}$ satisfy $\mathcal{D}fU = \mathcal{D}fV = \downarrow\{fx\}$. Since $\mathcal{D}f$ is 1–1 by hypothesis, we have $U = V$ and so $x' \leq x$. Note that X and Y could be discrete in this example. $\square$

The different behaviours of the three classes becomes more pronounced when we consider whether the functor $\mathcal{D}$ preserves inverse images (pullbacks):

Lemma 2.19 $\mathcal{D}$ preserves inverse images of the class $\mathcal{L}$.



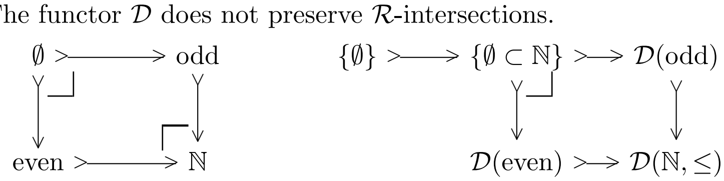
Proof If $V \subset_\downarrow Y$ and $X \ni x' \leq x \in f^*V \subset X$ then $fx' \leq fx \in V \subset_\downarrow Y$ and $fx' \in V$ too, whence $x' \in f^*V$, so $f^*V \subset_\downarrow X$. Hence pullbacks in **Pos** preserve $\mathcal{L}$.

The pullback $(\mathcal{D}f)^*(\mathcal{D}V)$ on the right consists of pairs (U, W) such that

$$U \subset_\downarrow X, \qquad W \subset_\downarrow V \subset_\downarrow Y, \qquad W = \mathcal{D}jW = \mathcal{D}fU = \{y \mid \exists x.\, y \leq fx \wedge x \in U\},$$

so W is redundant and we just require $\mathcal{D}fU \subset_\downarrow V$. But by the adjunction this is equivalent to $U \subset_\downarrow f^*V$ since $V \subset_\downarrow Y$. Hence the pullback is isomorphic to $\mathcal{D}(f^*V)$. $\square$

Example 2.20 The functor $\mathcal{D}$ does not preserve $\mathcal{R}$-intersections.



Proof Consider $\mathbb{N}$ as a poset with its usual arithmetical order. The lower closure of the subset of odd numbers is the whole of $\mathbb{N}$, and the same with the even numbers. Hence the subsets $\emptyset$ and $\mathbb{N}$ both belong to the pullback on the right, whereas $\mathbb{N}$ is not a (lower) subset of the pullback on the left. $\square$

Remark 2.21 Filtered colimits agree with unions of $\mathcal{I}$-, $\mathcal{R}$- and $\mathcal{L}$-maps, as required in [Tay23, Section 4].

3 Coalgebras and homomorphisms

Recall that a ***coalgebra*** for an endofunctor $T : \mathcal{C} \to \mathcal{C}$ of a category is an object $X \in \mathsf{ob}\mathcal{C}$ together with any $\mathcal{C}$-morphism $\alpha : X \to TX$.

Notation 3.1 A coalgebra for the powerset $\mathcal{P} : \mathbf{Set} \to \mathbf{Set}$ is a set X together with any binary relation $(\prec)$ whatever:

$$(y \prec x) \ \equiv\ \big(y \in \alpha(x)\big) \qquad \text{and} \qquad \alpha(x) \ \equiv\ \{y : X \mid y \prec x\}.$$

The relation $(\prec)$ has no particular meaning as yet, but following previous work our development will be inspired by aspects of set theory, where $(\prec)$ is called "membership" and we may derive a relation called "subset" $(\subseteq)$, *cf.* Notation 3.9.

On the other hand, $(\subset)$ means the standard notion of subobject, whilst $\mathcal{P}$ and $(\in)$ belong to the infrastructure, namely the logic of an elementary topos, which we call **Set**. Beware that the symbols $(\subset)$ and $(\subseteq)$ are not related, and our use of the latter most certainly *does not mean* that the former is *ir*reflexive.

As we have said, the purpose of this paper is to apply the same ideas to posets.

Proposition 3.2 A coalgebra for the lower sets functor $\mathcal{D} : \mathbf{Pos} \to \mathbf{Pos}$ is similarly given by a *po*set $(X, \leq)$ together with another binary relation, $(\prec)$, that corresponds to the coalgebra structure map as before. However, in order to define an order-preserving function

$$\alpha : (X, \leq) \longrightarrow \mathcal{D}(X, \leq),$$

these relations must be ***compatible*** in the sense that

$$z \leq y \prec x \Longrightarrow z \prec x \qquad \text{and} \qquad z \prec y \leq x \Longrightarrow z \prec x.$$

Proof The first condition says that $\alpha(x)$ is a lower subset (an element of $\mathcal{D}X$), *i.e.* $z \leq y \in \alpha(x) \Rightarrow z \in \alpha(x)$. The second says that α is monotone (a morphism of $\mathcal{C} \equiv \mathbf{Pos}$), $y \leq x \Longrightarrow \alpha(y) \subset \alpha(x)$. □

We will pass between these representations without further comment, but the most important thing to say at this point is that the two relations are *independent*, apart from the compatibility conditions: $(\leq)$ is not necessarily any particular poset relation that you may have in mind.

At one extreme we may essentially do without it:

Example 3.3 Any relation $(\prec)$ is compatible with equality, $(=)$. In this way, we will be able to import extensional well founded relations (the $\in$-structures for sets in the sense of Set Theory) into the same setting that we will use for ordinals. Indeed, for $X \equiv \mathcal{P}^3\emptyset$ or any further stage in the von Neumann hierarchy of sets, equality is the *only* reflexive relation that is compatible with $(\prec) \equiv (\in)$. □

The functor acts on coalgebras:

Lemma 3.4 The coalgebra $(\mathcal{D}X, \mathcal{D}\alpha)$ is $(\mathcal{D}X, \leq^{\mathcal{D}X}, \prec^{\mathcal{D}X})$, where

$$V \leq^{\mathcal{D}X} U \ \equiv\ V \subset U \qquad \text{and} \qquad V \prec^{\mathcal{D}X} U \ \equiv\ \exists u \in U.\, \forall v \in V.\, v \prec^X u.$$

Proof Apply Notation 2.1 to Proposition 3.2. □

Example 3.5 The subobject classifier (object of truth-values) $\Omega \equiv \mathcal{P}\mathbf{1} \equiv \mathcal{D}\mathbf{1}$ carries the coalgebra structure $(\Omega, \Rightarrow, \prec)$ where the only instance of $(\prec)$ is $\bot \prec \top$. Explicitly, for $\phi, \psi : \Omega$,

$$(\phi \prec \psi) \quad \equiv \quad (\phi \Leftrightarrow \bot) \wedge (\psi \Leftrightarrow \top) \quad \equiv \quad (\neg\phi \wedge \psi).$$

This is an application of the previous result to the empty $(\prec)$ relation on the singleton. The structure map is

$$\alpha(\bot) \;\equiv\; \emptyset, \qquad \alpha(\phi) \;\equiv\; \{\bot \mid \phi\} \qquad \text{and} \qquad \alpha(\top) \;\equiv\; \{\bot\}. \qquad \square$$

Ω is an example of how the two relations interact and of the role of the unit η of the KZ-monad:

Definition 3.6 (X, α) or $(X, \leq, \prec)$ is a ***transitive coalgebra*** if

$$\alpha \leq \eta_X \qquad \text{or} \qquad (\prec_X) \subset (\leq_X) \qquad \text{or} \qquad \forall xy.\quad y \prec x \Longrightarrow y \leq x,$$

where

$$\eta_X(x) \quad\equiv\quad \downarrow x \quad\equiv\quad \{y \mid y \leq x\}.$$

Lemma 3.7 If $(X, \leq, \prec)$ is a transitive coalgebra then $(\prec)$ is a transitive relation.
Proof By either of the compatibility conditions for $\mathcal{D}$-coalgebras. □

Immediately from this and Lemma 2.4 for a KZ-monad, we have:

Lemma 3.8 The functor $\mathcal{D}$ preserves transitivity, because $\mathcal{D}\alpha \leq \mathcal{D}\eta_X \leq \eta_{\mathcal{D}X}$. □

Notation 3.9 If $(\prec)$ is transitive, the relations

$$y \subseteq z \;\equiv\; (\forall x.\, x \prec y \Rightarrow x \prec z) \qquad \text{and} \qquad y \preceq z \;\equiv\; (y \prec z \;\vee\; y = z)$$

satisfy $y \preceq z \Longrightarrow y \subseteq z$ constructively and they are equivalent for classical ordinals. They are *candidates* for the poset relation $(\leq)$ that make the coalgebra transitive and we will study them in Sections 5 and 7 respectively.

Definition 3.10 A coalgebra ***homomorphism*** is a commutative square of the form

$$\begin{array}{ccc} TY & \xrightarrow{\;Tf\;} & TX \\ {\scriptstyle\beta}\uparrow & & \uparrow{\scriptstyle\alpha} \\ Y & \xrightarrow{\;f\;} & X \end{array}$$

In the case of $T \equiv \mathcal{P} : \mathbf{Set} \to \mathbf{Set}$, coalgebra homomorphisms are characterised as *bisimulations*, but this is modified for $\mathcal{D} : \mathbf{Pos} \to \mathbf{Pos}$:

Lemma 3.11 Let $(Y, \leq_Y, \prec_Y)$ and $(X, \leq_X, \prec_X)$ be $\mathcal{D}$-coalgebras and $f : Y \to X$ a function. Then f is a $\mathcal{D}$-coalgebra homomorphism iff

$$\begin{array}{lccl} \forall yy'\!:\!Y. & y' \leq_Y y & \Longrightarrow & fy' \leq_X fy \\ \forall yy'\!:\!Y. & y' \prec_Y y & \Longrightarrow & fy' \prec_X fy \\ \forall x\!:\!X.\, \forall y\!:\!Y. & x \prec_X fy & \Longrightarrow & \exists y'\!:\!Y.\, x \leq_X fy' \;\wedge\; y' \prec_Y y, \end{array}$$

where the last implication is reversible because of the other two and compatibility. If we do make the third implication reversible then the second is redundant.
Proof The first condition says that f is a **Pos**-morphism.

$$\begin{array}{ccc} Y & \xrightarrow{\;\beta\;} & \mathcal{D}Y \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \mathcal{D}f} \\ X & \xrightarrow{\;\alpha\;} & \mathcal{D}X \end{array} \qquad\qquad \begin{array}{ccccc} & & \exists y' & \overset{\prec_Y}{\cdots\cdots\to} & y \\ & & {\scriptstyle f}\,\vdots & & \downarrow{\scriptstyle f} \\ x & \xrightarrow{\;\leq_X\;} & fy' & \xrightarrow{\;\prec_X\;} & fy \end{array} \qquad \begin{array}{c} Y \\ \downarrow{\scriptstyle f} \\ X \end{array}$$

The square on the left commutes iff

$$\begin{array}{rrcl} & \mathcal{D}f(\beta y) & \equiv & \{x \mid \exists y'.\, x \le fy' \ \wedge\ y' \prec y\} \\ \text{and} & \alpha(fy) & \equiv & \{x \mid x \prec fy\} \end{array}$$

are equal in $\mathcal{D}X$. The inclusion $\mathcal{D}f(\beta y) \subset \alpha(fy)$ is

$$\forall xyy'.\, x \le fy' \ \wedge\ y' \prec y \Longrightarrow x \prec fy,$$

which, with $x \equiv fy'$, entails the second condition, but they are equivalent because $x \le fy' \prec fy \Rightarrow x \prec fy$ by compatibility.

The other inclusion is the third condition. It is reversible because of the second condition and since $(\le_X)$ and $(\prec_X)$ are compatible. □

Corollary 3.12 A subset inclusion $Y \subset X$ is a homomorphism and

(a) in $\mathcal{I}$ iff: we can't say anything useful;

(b) in $\mathcal{R}$ iff Y carries the restriction of $(\le_X)$ and $(\prec_X)$ and also satisfies

$$\forall x{:}X.\, \forall y{:}Y. \qquad x \prec y \Longrightarrow \exists y'{:}Y.\, x \le y' \prec y,$$

i.e. a weak form of lower-closure with respect to $(\prec)$.

(c) in $\mathcal{L}$ if it is lower-closed with respect to both $(\le)$ and $(\prec)$.

(d) Lemma 7.3 shows that homomorphisms between well founded $\mathcal{D}$-coalgebras that carry the $(\preceq)$ order are lower inclusions.

Proof Being in $\mathcal{R}$ means that $(\le)$ is the same for X and Y.

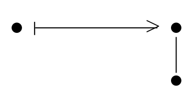
Putting $x \equiv fy''$ in the third condition in the Lemma gives

$$x \prec_X y \Longrightarrow (\exists y'.\, x \le_Y y' \prec_Y y) \Longrightarrow x \prec_Y y,$$

by compatibility of $(\le_Y)$ and $(\prec_Y)$, so the second condition is reversible and the $(\prec)$ relations agree. The displayed condition restates the third condition of the Lemma in the simplified notation. This condition simplifies if $f \in \mathcal{L}$. □

Notation 3.13 We write $x \equiv \emptyset$ for any element x of a coalgebra that has $\alpha(x) = \emptyset$, so there is no $y \prec x$. Classically, any inhabited well founded relation has such an element, which is unique by extensionality, but there need not be such a thing constructively[1]. In our examples we will also adopt some other set-theoretic notation in an informal way.

Example 3.14 Any homomorphism preserves $\emptyset$ (put $y \equiv \emptyset$ in the third condition in Lemma 3.11), but of course there are functions that preserve $(\prec)$ and $(\le)$ but not $\emptyset$, such as



where the vertical line indicates $(\prec)$.

Corollary 3.15 Example 3.3 defines a full and faithful embedding of $\mathcal{P}$-coalgebras in **Set** into $\mathcal{D}$-coalgebras in **Pos**:

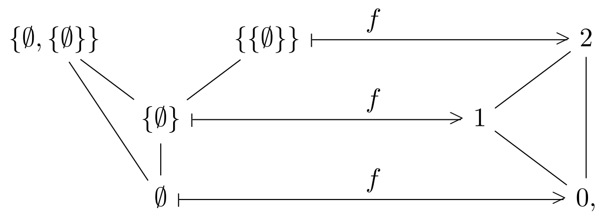
$$(X, \prec) \longmapsto (X, =, \prec).$$

[1] The object $X \equiv \{\emptyset \mid \xi\} \cup \{\{\emptyset \mid \xi\}\}$ is inhabited because $\{\emptyset \mid \xi\} \in X$ with full truth, but the truth value of "$\emptyset \in X$" is $\xi \vee \neg\xi$. This is easy if you know how to manipulate such objects, but otherwise no amount of explanation would help!

Proof When $(\leq_X)$ is equality, the conditions in the Lemma reduce to the characterisation of $\mathcal{P}$-coalgebra homomorphisms as bisimulations, where $y' \prec_Y y$ is a strict "lifting" of $x \prec_Y fy$, *i.e.* with $x = fy'$. □

The following key examples show why the $(\leq)$ order and the interpolant in Corollary 3.12(b) are so important to our account.

Example 3.16 Consider the function that assigns ordinal rank to sets ($\in$-structures), starting with



where the lines indicate the $(\prec)$ relation. Then $0 \prec 2 = f\{\{\emptyset\}\}$ does not lift because $\emptyset \notin \{\{\emptyset\}\}$.

In order to make it a $\mathcal{D}$-homomorphism, we need $0 \leq 1$ for the third condition in Lemma 3.11, which is what lies behind our Definition 3.6 of a transitive coalgebra. □

Notation 3.17 Since the various notions of ordinal coincide classically, in order to distinguish them we must introduce at least a *soupçon* of non-classical logic. For many purposes it will suffice to use the *three*-valued logic of the presheaf topos $\mathbf{Set}^{\rightarrow}$ (in which we mean that $\mathbf{Set}$ itself is Boolean). Then $\Omega = \{\bot, \xi, \top\}$ with a third proposition ξ that is neither provable nor refutable, although $(\neg\xi) \Leftrightarrow \bot$ and $(\neg\neg\xi) \Leftrightarrow \top$.

Example 3.18 In $\mathcal{D}\Omega$, recall that "$(\bot \in U)$" is a proposition, so it is an element of Ω, and this might belong to $V \subset \Omega$. Hence on $\mathcal{D}\Omega$ we may define the binary relation

$$U \prec V \qquad \equiv \qquad (\bot \in U) \in V,$$

which is Lemma 3.4 applied to Example 3.5.

Example 3.19 The map $f : \mathbf{3} \equiv \{0, 1, 2\} \to \mathcal{D}\Omega$ by

$$0 \mapsto \emptyset, \qquad 1 \mapsto \{\emptyset\}, \qquad 2 \mapsto \Omega$$

is a homomorphism of $\mathcal{D}$-coalgebras that is in $\mathcal{R}$ but not $\mathcal{L}$. It is not a $\mathcal{P}$-homomorphism.

Proof Consider $x \equiv \{\emptyset \mid \xi\}$ and $y \equiv 2$ in Lemma 3.11, so $x \subseteq \{\emptyset\} \equiv f(1) \subseteq \Omega \equiv f(2)$. In the third condition there we have $x \subseteq f(1)$ and $1 \prec 2$ but x is not the image of any element of $\mathbf{3}$. Hence the interpolant is needed. □

Corollary 3.20 The "underlying $\mathcal{P}$-coalgebra" operation, *i.e.* the assignment

$$(X, \leq, \prec) \mapsto (X, \prec) \qquad \text{or} \qquad (X, \alpha) \mapsto (X, \alpha\,;\epsilon_X^*),$$

where ϵ_X^* was defined in Lemma 2.6, is not a functor. □

This is why we write $(X, \leq, \prec)$ with the reflexive (poset) order *first*: it is a more intimate part of the structure. There is a functor $(X, \leq, \prec) \mapsto (X, \leq)$ from the coalgebra to its underlying poset, but none to $(X, \prec)$. $\mathcal{D}$-coalgebras are not $\mathcal{P}$-coalgebras with an additional order, but a *unifying framework* in which different structures are embedded.

Remark 3.21 These are not contrived examples.

We will exploit the difference between $\mathcal{P}$- and $\mathcal{D}$-coalgebra homomorphisms to embed both sets ($\in$-structures) and ordinals (of various kinds) in the same general setting of $\mathcal{D}$-coalgebras. In many of these cases, the morphisms *within* the subcategory are $\mathcal{P}$-homomorphisms.

In this setting, the transitive closure and ordinal rank become *reflections* into subcategories, *i.e.* left adjoints to their inclusions (Corollaries 6.14 and 6.15 and Section 7).

The universal properties make these examples the *principal* links between structures that would elsewhere be put into separate categories. So we might imagine the subcategories as *islands* in the *sea* of $\mathcal{D}$-coalgebras and these examples as *bridges* amongst them.

Notation 3.22 The categories that we have so far are related by the diagram

$$\begin{array}{ccccc} & & (X,\leq,\prec) & \longmapsto & (X,\leq) \\ (X,=,\prec) & & \mathcal{D}\textbf{-CoAlg} & \longrightarrow & \textbf{Pos} \\ \uparrow & & \uparrow & & \uparrow \dashv \downarrow \\ (X,\prec) & & \mathcal{P}\textbf{-CoAlg} & \longrightarrow & \textbf{Set} \end{array}$$

where

- the rightward functors forget the coalgebra structure or associated relation, $(\prec)$;
- the downward one forgets the reflexive order, $(\leq)$; and
- the upward ones assign equality $(=)$ for it and are full and faithful.

These horizontal functors create colimits, which means that to compute a colimit of coalgebras we do it for the carriers and then use the unique structure that makes the diagram commute. The same holds for inverse images of plain monos between $\mathcal{P}$-coalgebras and also (using [Tay23] and Lemma 2.19) for inverse images of $\mathcal{L}$-maps between $\mathcal{D}$-coalgebras. $\square$

4 Well-foundedness

A *well founded* relation is one over whih we may perform induction. This was abstracted in [Tay99, Tay23] to a coalgebra for an endofunctor. The predicates may be represented by a class of monos, so long as this is closed under the functor and inverse images. In this section we characterise well-foundedness using $\mathcal{R}$ and $\mathcal{L}$ for the predicates, but $\mathcal{I}$ is not suitable.

In fact these notions for **Pos** turn out to be the same as the ones for **Set**. The proof makes use of the map ϵ_X^* that we introduced in Notation 2.6.

Definition 4.1 A T-coalgebra $\alpha : X \to TX$ is ***well founded*** if, for every mono $i : U \to X$ in $\mathcal{C}$, whenever the pullback $K \rightarrowtail X$ of Ti against α factors through $U \rightarrowtail X$,

$$\begin{array}{ccccc} TU & & \xrightarrow{\quad Ti \quad} & & TX \\ \uparrow & \lrcorner & & & \uparrow \alpha \\ K & \rightarrowtail & U & \overset{i}{\rightarrowtail} & X \end{array}$$

we must have $i : U \cong X$. In fact, we ask this not for *all* (plain) monos, but for those belonging to the chosen class of monos, here $\mathcal{R}$ or $\mathcal{L}$.

Example 4.2 Well-foundedness for $\mathcal{P}$-coalgebras in **Set** is the familiar induction scheme:

$$\frac{\forall x.\, [\forall y.\, y \prec x \Rightarrow \psi y] \Longrightarrow \psi x}{\forall x.\, \psi x}$$

This is explained in [Tay96, Tay99, Tay23], but it is a special case of the ordered case:

Lemma 4.3 A $\mathcal{D}$-coalgebra $(X, \le, \prec)$ in **Pos** is well founded for an $\mathcal{R}$-predicate ϕ iff $\prec$ satisfies the induction scheme:

$$\frac{\forall x.\, [\forall z.\, z \prec x \Rightarrow \exists y.\, z \le y \prec x \wedge \phi y] \Longrightarrow \phi x}{\forall x.\, \phi x}$$

Proof The complication arises from the action of the functor $\mathcal{D}$ on the map $i : U \rightarrowtail X$ (Definition 2.1(b)). Saying that $i \in \mathcal{R}$ means that $U \subset X$ is an arbitrary subset equipped with the restriction of the order $(\le)$ on X. However, a lower subset V of U need not be lower in X, so it needs to be closed downwards. Therefore

$$\mathcal{D}iV \;\equiv\; \{y : X \mid \exists v.\, y \le v \in V\}, \qquad \text{whilst} \qquad V \;=\; U \cap \mathcal{D}iV$$

since V is lower in U. Now we may form the pullback K, where, for $x : X$ and $V \subset_\downarrow U \subset X$,

$$(x, V) \in K \quad \equiv \quad \alpha(x) = \mathcal{D}iV \quad \Longrightarrow \quad V = \alpha(x) \cap U.$$

If V is given by the formula on the right then $\mathcal{D}iV \subset \alpha(x)$ since $\alpha(x)$ is lower. Conversely

$$\alpha(x) \subset \mathcal{D}iV \quad \equiv \quad \forall z.\, z \prec x \Rightarrow \exists y \in U.\, z \le y \prec x,$$

or using the predicate ϕ instead of the subset U,

$$\forall z.\, z \prec x \Rightarrow \exists y.\, z \le y \prec x \;\wedge\; \phi(y).$$

This characterises K, or rather $\exists V.\, (x, V) \in K$, which is the induction *hypothesis*. The containment $K \rightarrowtail U$ is then the stated induction *premise*. □

Lemma 4.4 A $\mathcal{D}$-coalgebra $(X, \le, \prec)$ is well founded for an $\mathcal{L}$-predicate ϕ iff $(\prec)$ satisfies the familiar induction scheme,

$$\frac{\forall x.\, [\forall z.\, z \prec x \Rightarrow \phi z] \Longrightarrow \phi x}{\forall x.\, \phi x}$$

except that it only applies to *lower* subsets or predicates.

Proof If ϕ is an $\mathcal{L}$-predicate, the complication in the previous result is redundant. □

We temporarily call these schemes $\mathcal{P}$-, $\mathcal{R}$- and $\mathcal{L}$-induction respectively and now show that they are equivalent. We refer to the sub-formula in square brackets in each of them as the ***induction hypothesis***; it corresponds to the object K in the categorical definition. The upper line of each scheme is called the ***induction premise*** and corresponds to the property that K be contained in U.

Lemma 4.5 $\mathcal{P}$-induction entails $\mathcal{R}$-induction.

Proof The $\mathcal{P}$-induction *hypothesis* entails the one for $\mathcal{R}$ for the same predicate (with $y \equiv z$):

$$[\forall z \prec x.\, \phi z] \quad \Longrightarrow \quad [\forall z \prec x.\, \exists y.\, z \le y \prec x \wedge \phi y].$$

Since implication is contravariant on the left, it follows that the $\mathcal{R}$-*premise* entails the $\mathcal{P}$-*premise*. Similarly, $\mathcal{P}$-*induction* entails $\mathcal{R}$-*induction*. □

Lemma 4.6 $\mathcal{R}$-induction entails $\mathcal{L}$-induction.

Proof In the case of an $\mathcal{L}$- (lower) predicate, the two versions of the induction hypothesis are equivalent. Hence so are the premises and the induction schemes. □

Lemma 4.7 $\mathcal{P}$- and $\mathcal{L}$-induction are equivalent.

Proof They take the same form, except that $\mathcal{L}$-induction only applies to lower predicates, which means that $\mathcal{P}$-induction entails $\mathcal{L}$-induction.

For the converse, let ψ be a *general* predicate that satisfies the premise for $\mathcal{P}$-induction,

$$\forall x.\, [\forall y.\, y \prec x \Rightarrow \psi y] \Longrightarrow \psi x.$$

This says that $\forall x.\, \phi x \Rightarrow \psi x$, where

$$\phi x \quad \equiv \quad (\forall z.\, z \prec x \Longrightarrow \psi z).$$

Then ϕ an $\mathcal{L}$-predicate, since, by compatibility ($z \prec y \leq x \Rightarrow z \prec x$),

$$y \leq x \ \wedge\ \phi x \ \equiv\ y \leq x \ \wedge\ (\forall z.\, z \prec x \Rightarrow \psi z) \ \Longrightarrow\ (\forall z.\, z \prec y \Rightarrow \psi z) \ \equiv\ \phi y.$$

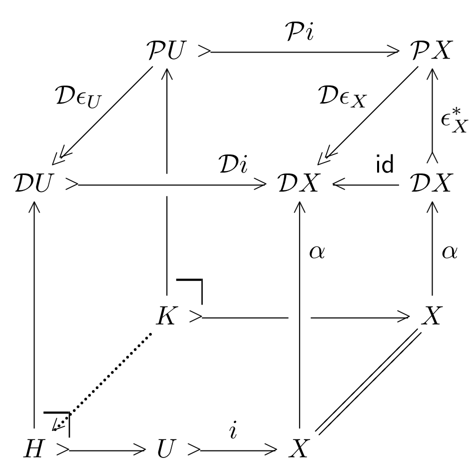
It satisfies the premise for $\mathcal{L}$-induction because

$$[\forall y.\, y \prec x \Rightarrow \phi y] \ \Longrightarrow\ [\forall y.\, y \prec x \Rightarrow \psi y] \ \equiv\ \phi x.$$

Hence $\forall x.\, \phi x$ by $\mathcal{L}$-induction, so $\forall x.\, \psi x$. □

Now we repeat these arguments in diagrammatic form, in order to help us see how they might be applied to other categories.

Lemma 4.8 $\mathcal{P}$-induction entails $\mathcal{R}$-induction: Let $\alpha : X \to \mathcal{D}X$ be a $\mathcal{D}$-coalgebra (in **Pos**) such that $(\alpha\,;\epsilon^*_X) : X \to \mathcal{D}X \to \mathcal{P}X$ is a well founded $\mathcal{P}$-coalgebra (in **Set**). Then α is well-founded with respect to $\mathcal{R}$-predicates.



Proof Lemma 2.6 defined ϵ^*_X. Suppose that $(i : U \rightarrowtail X) \in \mathcal{R}$ satisfies the induction premise for the $\mathcal{D}$-coalgebra α. That is, we have a broken pullback in **Pos**, including $H \rightarrowtail U$, as the front face of the cube.

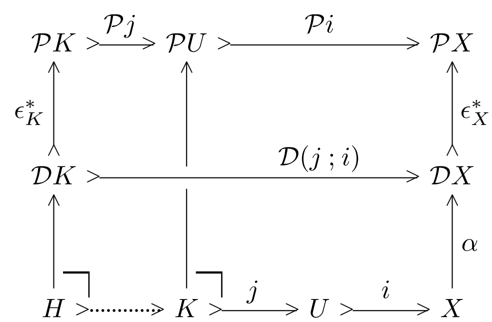
Form the corresponding $\mathcal{P}$-coalgebra $(\alpha\,;\epsilon^*_X)$ and the $\mathcal{P}$-induction hypothesis K, which is the pullback at the back. Whilst this diagram is *â priori* in **Set**, all the maps are monotone and **Pos** $\to$ **Set** creates pullbacks (Proposition 2.11), so K is also the pullback in **Pos**.

The top (rhomboidal) face of the cube and adjacent triangle commute by Notation 2.6 and Lemma 2.7.

Then $K \to H$ mediates to the pullback H, making K a broken pullback for the $\mathcal{P}$-coalgebra. By $\mathcal{P}$-induction, $i : U \cong X$ in **Set**. Since it was given to be in $\mathcal{R}$, this is also an isomorphism in **Pos**. $\square$

Lemma 4.9 $\mathcal{R}$-induction entails $\mathcal{L}$-induction (Lemma 4.6). $\square$

Lemma 4.10 $\mathcal{L}$-induction entails $\mathcal{P}$-induction: Let $\alpha : X \to \mathcal{D}X$ be $\mathcal{D}$-coalgebra in **Pos** that is well founded with respect to $\mathcal{L}$-predicates. Then $(\alpha\,;\epsilon^*_X) : X \to \mathcal{D}X \to \mathcal{P}X$ is a well founded $\mathcal{P}$-coalgebra in **Set**.



Proof The diagram is in **Set**. Let $i : U \rightarrowtail X$ be *any* sub*set* (corresponding to ψ) that satisfies the broken pullback (induction premise) on the right, from K to $\mathcal{P}X$.

K is now the $\prec$-lower closure of U, corresponding to ϕ in Lemma 4.7.

Then $(\mathcal{P}i : \mathcal{P}U \rightarrowtail \mathcal{P}X) \in \mathcal{L}$ by Lemma 2.10 and $(K \rightarrowtail X) \in \mathcal{L}$ by Lemma 2.19.

Form the $\mathcal{L}$-induction hypothesis H for the predicate $K \rightarrowtail X$ on the $\mathcal{D}$-coalgebra (X, α), which is the lower pullback rectangle H.

Since $(K \rightarrowtail X) \in \mathcal{L}$, the wide upper rectangle commutes by Lemma 2.13.

(In fact it is a pullback, but we don't need this. On the other hand, such a square need not even commute given just an $\mathcal{R}$-map, so we cannot insert $\mathcal{D}U$ in the middle of the diagram, *cf.* Example 2.14.)

It follows that $H \rightarrowtail X$ factors through $K \rightarrowtail X$. This is the $\mathcal{L}$-induction premise, so $\mathcal{K} \cong X$ by $\mathcal{L}$-induction and $U \cong X$ by cancellation. $\square$

Proposition 4.11 Let $\alpha : X \to \mathcal{D}X$ be a $\mathcal{D}$-coalgebra in **Pos** corresponding to binary relations $(X, \leq, \prec)$. Then the following are equivalent:

(a) (X, α) is well founded as a $\mathcal{D}$-coalgebra in **Pos** with respect to the class $\mathcal{R}$ as predicates;

(b) (X, α) is well founded as a $\mathcal{D}$-coalgebra in **Pos** with respect to the class $\mathcal{L}$ as predicates;

(c) $(X, \alpha\,; \epsilon^*_X)$ is well founded as a $\mathcal{P}$-coalgebra in **Set** using 1–1 functions as predicates; and

(d) $(X, \prec)$ is a well founded relation in the traditional sense. $\square$

At this point we defer to the "textbook" [Tay23] for the proof of the *recursion* theorems that may be deduced from well founded *induction*. The simpler of these is that there is a unique coalgebra-to-algebra homomorphism to any $\mathcal{P}$- or $\mathcal{D}$-*algebra*; from this we will deduce *transfinite* recursion (with successors and unions) in Section 10. A more complicated form of well founded recursion is the behaviour of *extensional* well founded coalgebras that we study in the next section.

Remark 4.12 The proof of these theorems relies on some other properties of the classes of monos that are used for predicates and initial segments. These are easily verified for our $\mathcal{R}$ and $\mathcal{L}$, but

for the record they are:
(a) all isomorphisms are “mono”;
(b) “monos” satisfy the cancellation property $\forall fg.\;\; f\,;i = g\,;i \Longrightarrow f = g$;
(c) the composite of two “monos” is “mono”;
(d) the pullback of any “mono” along any map exists and is “mono” (Lemmas 2.12 and 2.19);
(e) all maps $\emptyset \to X$ are “mono”;
(f) filtered colimits of monos are unions (Definition 2.21);
(g) “well powered”: the “monos” into any object $(X, \leq)$ form a *set*, in fact a subset of $\mathcal{P}(X)$; and
(h) the functor $\mathcal{D}$ preserves “monos” (Lemmas 2.15 and 2.16).

Notation 4.13 We now write $\mathcal{P}$- and $\mathcal{D}$-**WfCoAlg** for the categories of well founded $\mathcal{P}$- and $\mathcal{D}$-coalgebras, since the forms with $\mathcal{R}$ and $\mathcal{L}$ are the same. Adding them to the diagram in Notation 3.22, we have

$$\begin{array}{ccccc}
\mathcal{D}\text{-}\mathbf{WfCoAlg} & \underset{\twoheadleftarrow}{\overset{\rightarrowtail}{\bot}} & \mathcal{D}\text{-}\mathbf{CoAlg} & \longrightarrow & \mathbf{Pos} \\
\uparrow & & \uparrow & & \uparrow \dashv \downarrow \\
\mathcal{P}\text{-}\mathbf{WfCoAlg} & \underset{\twoheadleftarrow}{\overset{\rightarrowtail}{\bot}} & \mathcal{P}\text{-}\mathbf{CoAlg} & \longrightarrow & \mathbf{Set}
\end{array}$$

The rightward functors create colimits and also inverse images of 1–1 functions and $\mathcal{L}$-maps respectively [Tay23, Section 5]. Proposition 6.10 below gives the *right* adjoint, namely the largest well founded part. Therefore, since the remainder of this paper will be about constructing *left* adjoints, given well founded coalgebras, we will take them as our baseline from now on.

Although the flexibility that [Tay23] provided by using different classes of “mono” in the underlying category has had no real impact on the notion of well-foundedness, we will now see a very different story for the notion of extensionality.

5 Notions of extensionality

Recall that the axiom of extensionality is traditionally stated as

$$(\forall z.\quad z \prec x \iff z \prec y) \quad \Longrightarrow \quad x \;=\; y$$

or

$$\{z \mid z \prec x\} \;=\; \{z \mid z \prec y\} \quad \Longrightarrow \quad x \;=\; y,$$

so the structure map of the coalgebra $\mathcal{P}X \longleftarrow\!\!\prec X$ is 1–1. We transfer this idea from sets to posets.

Definition 5.1 A $\mathcal{D}$-coalgebra $\alpha : X \to \mathcal{D}X$ or $(X, \leq, \prec)$ is $\mathcal{R}$- or $\mathcal{L}$-***extensional*** if $\alpha \in \mathcal{R}$ or $\mathcal{L}$, respectively. As before, we will say $\mathcal{P}$-extensional for the corresponding idea in **Set**.

It will also be convenient to call a $\mathcal{D}$-coalgebra $\mathcal{I}$-***extensional*** if its structure map is in $\mathcal{I}$, which amounts to extensionality of the relation $(\prec)$ in the traditional sense. However, we must be very careful in doing so, because the functor $\mathcal{D}$ does not preserve this class.

The interest in $\mathcal{R}$- and $\mathcal{L}$-extensional well founded coalgebras is that they share the strange behaviour of set theory (Remark 1.3):

Theorem 5.2 The category of extensional well founded coalgebras and homomorphisms is a preorder [Tay23, Section 7]:
(a) there is at most one map between any two objects
(b) and this belongs to the chosen class of monos;
(c) there is a least member ($\emptyset$);
(d) there are binary meets; and
(e) there are directed unions. $\square$

We devote this section to characterising $\mathcal{R}$- and $\mathcal{L}$-extensional $\mathcal{D}$-coalgebras. The Theorem does not extend to $\mathcal{I}$-extensional ones: Example 3.16 is a $\mathcal{D}$-homomorphism between two of them that does not belong to $\mathcal{I}$. We will nevertheless need to consider $\mathcal{I}$-extensionality in Section 7.

Lemma 5.3 A coalgebra (X, α) corresponding to $(X, \leq, \prec)$ is $\mathcal{R}$-extensional iff

$$\forall xy. \quad \big(\forall z.\, z \prec x \Longrightarrow z \prec y\big) \iff (x \leq y).$$

Proof The left hand side is $\alpha(y) \subset \alpha(z)$, the reverse implication is monotonicity of α (part of compatibility, Proposition 3.2) and the forward one says that $\alpha \in \mathcal{R}$. $\square$

Notation 3.9 introduced the symbol $(\subseteq)$ for this relation, but it must satisfy another condition in order to define a $\mathcal{D}$-coalgebra:

Lemma 5.4 The relations $(\prec)$ and $(y \subseteq z)$ are compatible iff $(\prec)$ is extensional (in the usual sense) and ***meta-transitive***, *i.e.*

$$\forall w, x, y. \quad (\forall z.\, z \prec y \Rightarrow z \prec x) \;\wedge\; (x \prec w) \Longrightarrow (y \prec w).$$

Proof The relation $(\subseteq)$ is a preorder (reflexive and transitive) since $(\Rightarrow)$ is. It is antisymmetric by (traditional) extensionality and it satisfies the second compatibility condition for a coalgebra. Meta-transitivity re-states the first condition for this. $\square$

The reason for the name is the next result.

Lemma 5.5 If $(\prec)$ is a well founded meta-transitive relation then it is transitive in the usual sense and $(X, \subseteq, \prec)$ is a transitive $\mathcal{D}$-coalgebra (Definition 3.6).

Proof Consider the predicate ϕ on X that is defined by

$$\phi(x) \quad\equiv\quad (\forall y.\, y \prec x \Rightarrow y \subseteq x) \quad\equiv\quad (\forall yz.\, z \prec y \prec x \Rightarrow z \prec x).$$

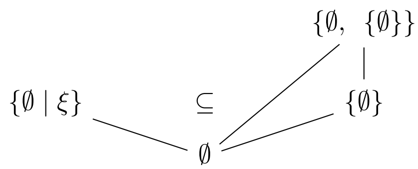
Then

$$\begin{aligned}\big(\forall y.\, y \prec x \Rightarrow \phi(y)\big) \quad&\equiv\quad \big(\forall zy.\, z \prec y \prec x \Rightarrow z \subseteq y \prec x\big)\\ &\Longrightarrow\quad \big(\forall zy.\, z \prec y \prec x \Rightarrow z \prec x\big) \quad\equiv\quad \phi(x),\end{aligned}$$

so $\forall x.\, \phi(x)$ by induction, but this states transitivity in both senses. $\square$

Remark 5.6 The predicate $\phi(x)$ says that x represents a *transitive set* in the restricted sense that is used in set theory, where the usual sense is called *hereditarily transitive*. The well founded induction that we have just used may be seen as an example of Pataraia induction, as explained in [Tay23, §2], namely that the *initial segments* of $(X, \prec)$ are all transitive in the weaker sense, whence so is the whole structure.

Example 5.7 Let $\xi \in \Omega$ be an undecidable truth value (Notation 3.17). Then



is a transitive relation but not meta-transitive. The lines indicate $(\prec)$, except that the statement $\emptyset \prec \{\emptyset \mid \xi\}$ is not "completely" true, but has truth-value ξ. Grayson observed that transitivity does not imply meta-transitivity [Gra77].

If we added $\{\emptyset \mid \xi\}$ as another member of the upper element, it would become meta-transitive. Example 8.11 is a more elaborate version of this. □

Definition 5.8 A ***slim ordinal*** is an $\mathcal{R}$-extensional well founded $\mathcal{D}$-coalgebra, or equivalently a carrier with a meta-transitive extensional well founded relation $(\prec)$, along with $(\subseteq)$ defined from it as above.

The Example suggests that meta-transitivity might be more natural than the usual notion. We have arrived at slim ordinals by putting together some commonplace categorical notions.

By Theorem 5.2, slim ordinals form a preorder, because there is at most one $\mathcal{D}$-coalgebra homomorphism between any two of them and then it is in the class $\mathcal{R}$. Example 3.19 shows that this need *not* be a $\mathcal{P}$-coalgebra homomorphism.

The characterisation of $\mathcal{R}$-extensionality using meta-transitivity suggests that the relation $(\leq)$ is redundant, but it is misleading to regard this as enough to model subsets:

Definition 5.9 A subset $U \subset X$ of a coalgebra is called ***representable*** if $\exists x.\, U = \alpha(x)$, where extensionality says that x is unique. By definition any representable subset is $(\prec)$-bounded above and by compatibility it must be $(\leq)$-lower:

$$\exists x{:}X.\, \forall u{:}X.\;\; u \in U \Longrightarrow u \prec x \qquad \text{and} \qquad \forall c, u{:}X.\;\; c \leq u \in U \Longrightarrow c \in U.$$

Lemma 5.10 A $\mathcal{D}$-algebra (X, α) or $(X, \leq, \prec)$ is $\mathcal{L}$-extensional iff every $(\prec)$-bounded $(\leq)$-lower subset $U \subset_{\downarrow} \alpha(x) \subset_{\downarrow} X$ is represented as $U = \alpha(y)$ for some unique $y \in X$, with

$$y \leq x \qquad \text{and} \qquad \forall u{:}X.\;\; u \in U \iff u \prec y.$$

Proof The conditions make $U \in \mathcal{D}(X)$, so the property re-states that $\alpha : X \to \mathcal{D}(X)$ is a lower inclusion. □

Definition 5.11 A ***plump ordinal*** is an $\mathcal{L}$-extensional well founded $\mathcal{D}$-coalgebra. All plump ordinals are slim, so they form a preorder, but now all $\mathcal{D}$-homomorphisms between plump ordinals are $\mathcal{P}$-homomorphisms and in the class $\mathcal{L}$. (We could deduce this directly from Theorem 5.2.)

Lemma 5.12 The functor $\mathcal{D}$ preserves slim and plump ordinals.

Proof $\mathcal{D}$ preserves $\mathcal{R}$ and $\mathcal{L}$ by Lemmas 2.15f and Lemma 3.4 gave the formulae for $(\prec^{\mathcal{D}X})$. From this,

$$V \subseteq U \quad \iff \quad \forall v \in V.\, \exists u \in U.\, v \subseteq u,$$

so $V \subset U \Rightarrow V \subseteq U$ easily. The converse also holds when the given ordinal is slim or plump, because then U is $(\subseteq)$-lower. □

Examples 5.13 The following are plump ordinals:
(a) $\emptyset$, $\mathbf{1} \equiv \mathcal{D}\emptyset$ and any subset of $\mathbf{1}$, with the trivial relations;
(b) $\Omega \equiv \mathcal{D}\mathbf{1}$, Example 3.5;
(c) any $(\Rightarrow)$-lower subset of Ω, with the same relations;
(d) In $\mathcal{D}\Omega$ (Example 3.18), $(\prec)$ is meta-transitive (since V is lower) and all subsets of $\{\emptyset\}$ are represented, by the corresponding element of Ω. Therefore $\mathcal{D}\Omega$ is $\mathcal{L}$-extensional.
(e) $\mathcal{D}\mathbf{2}$ is plump with, for $U, V \subset_\downarrow \{\bot, \top\}$,

$$V \prec U \qquad \equiv \qquad U = \{\bot, \top\} \quad \wedge \quad V \subset \{\bot\}.$$

Examples 5.14
(a) Let $\bot \prec \top$ and $\bot \leq \top$ in $2 \equiv \{\bot, \top\}$. This is slim but not plump, because the sub-singleton $\{\bot \mid \xi\} \subset \Omega$ is not representable.
(b) The two-element subset $\{\bot, \top\} \subset (\Omega, \Rightarrow)$ is a $(\prec)$-lower subset but is not $(\leq)$-lower, for the same reason. Then Ω is the plump rank of the thin ordinal 2.
(c) (Michael Shulman) In $X \equiv \{\xi, \top\} \subset \Omega$,

$$\{\phi : X \mid \phi \prec \xi\} \ = \ \emptyset \ = \ \{\phi : X \mid \phi \prec \top\},$$

because $\bot \notin X$, so X is not extensional, even in the traditional sense. □

Proposition 5.15 The slice preorder Plump/X over any plump ordinal X is equivalent to the CCD lattice $\mathcal{D}X$. We have no similarly simple result for Slim.

Proof By Theorem 5.2, every homomorphism $Y \to X$ of plump ordinals is a lower inclusion (in $\mathcal{L}$), with respect to both $(\subseteq)$ and $(\prec)$, by Corollary 3.12(c). Conversely, if $U \subset_\downarrow X$ then

$$\forall x, u{:}X. \quad (x \prec u \in U) \Longrightarrow (x \subseteq u \in U) \Longrightarrow (x \in U),$$

using Lemma 5.5. Hence the inclusion is a homomorphism by Corollary 3.12(c) and U is a plump ordinal by Lemma 5.10.

Moreover, any homomorphism $Y \to Z$, where Y and Z themselves have homomorphisms to X that necessarily form a commutative triangle, amounts to an ordinary inclusion of lower subsets $Y \subset_\downarrow Z \subset_\downarrow X$. That is, the (degenerate) categorical structure of Plump/X agrees with the poset order $(\subseteq)$ on $\mathcal{D}(X)$. □

Remark 5.16 Plump is a "large" or "class" preorder, *cf.* Remark 1.6. However, any single plump ordinal X is "small" — it is a coalgebra structure on an object of a given elementary topos. Then $\mathsf{Plump}/X \simeq \mathcal{D}X$ is also an object of that topos, namely a retract of the powerset of X, so it too is "small". Since X was arbitrary, this construction *covers* the whole *large* preorder by *small* posets.

For this to make sense, we must consider *change of base*: when X is replaced by Y along a homomorphism $i : X \to Y$, in which i must belong to $\mathcal{L}$, the posets are embedded by $\mathcal{D}(i) : \mathcal{D}(X) \rightarrowtail \mathcal{D}(Y)$, which is also in $\mathcal{L}$. Using these maps as the arrows in the diagram, we have an (illegitimate) colimit or union,

$$\mathsf{Plump} \ \simeq \ \operatorname*{colim}_{X \in \mathsf{Plump}} \mathsf{Plump}/X \ \simeq \ \bigcup_{X \in \mathsf{Plump}} \mathcal{D}(X) \ \simeq \ \bigcup_{X \in \mathsf{Plump}} X.$$

The additional equivalence on the right follows from the fact that if X is plump, so is $\mathcal{D}(X)$, whence the left hand side is contained in the right, and conversely any $\alpha : X \rightarrowtail \mathcal{D}(X)$ is a change of base. □

Notation 5.17 Here is a new summary of the categories, *cf.* Notations 3.22 and 4.13. In the literature, ordinals are often compared either

(a) as *initial segments* of each other, forming a *preorder*; or

(b) with *order-preserving functions* that may duplicate or omit values, forming a *category.*

Our formulation using $\mathcal{D}$-coalgebras makes categorical sense of these two *ad hoc* formulations: initial segments are coalgebra *homomorphisms* belonging to $\mathcal{D}$-**CoAlg**, whilst the order-preserving functions *forget* the coalgebra structure and just use the morphisms from the underlying category **Pos**.

We will use Sans Serif for the preorders and **Bold** for the categories:

$$\begin{array}{ccccccc}
\mathsf{Plump} & & \longrightarrow & & & & \mathbf{Plump} \\
\downarrow & & & & & & \downarrow \\
\mathsf{Slim} & & \longrightarrow & & & & \mathbf{Slim} \\
\downarrow & & & & & & \downarrow \\
\mathcal{D}\text{-}\mathbf{TrWfCoAlg} & \rightarrowtail & \mathcal{D}\text{-}\mathbf{WfCoAlg} & \rightarrowtail & \mathcal{D}\text{-}\mathbf{CoAlg} & \longrightarrow & \mathbf{Pos}
\end{array}$$

where $\rightarrowtail$ denotes a full and faithful functor and **TrWfCoAlg** consists of the transitive well founded coalgebras (Definition 3.6 and Section 7.).

The finite objects in **Plump** or **Slim** form the *simplex category* Δ, functors out of which are fundamental in homotopy theory, but we leave that subject to others.

Remark 5.18 We can similarly write Ens and **Ens** for the preorder and category of extensional well founded $\mathcal{P}$-coalgebras with homomorphisms or plain functions. Ens is the class preorder of "transitive sets" and set-theoretic inclusions (Remark 1.3). **Ens** is the full subcategory of a topos consisting of those objects that can carry such a structure, but it has fewer categorical properties than you might imagine, *cf.* Example 5.14(c), because Zermelo set theory does not work that way.

The next question is whether the inclusions of Slim and Plump have left adjoints. We consider yet another kind of ordinal in Section 7 and show how fast plump ordinals grow in Section 11.

6 Extensional reflection

We have used "monos" to define extensionality and generalised them to the classes $\mathcal{I}$, $\mathcal{R}$ and $\mathcal{L}$ in the previous section, so now we introduce the corresponding "epis" in order to give our version of the Mostowski extensional quotient and the ordinal rank (Remark 1.4).

Definition 6.1 Two maps $e : Y \twoheadrightarrow Q$ and $m : U \hookrightarrow X$ in any category are called ***orthogonal***, written $e \perp m$, if, for any two maps f and g such that the square commutes, there is a unique morphism $h : Q \to U$ making the two triangles commute:

$$\begin{array}{ccc}
Y & \xrightarrow{\;e\;} & Q \\
{\scriptstyle f}\downarrow & \overset{h}{\swarrow} & \downarrow{\scriptstyle g} \\
U & \xhookrightarrow{\;m\;} & X
\end{array}$$

Then a ***factorisation system*** is a pair $(\mathcal{E}, \mathcal{M})$ of classes of morphisms such that
(a) the classes $\mathcal{E}$ and $\mathcal{M}$ each contain all isomorphisms;
(b) they are each closed under composition;
(c) $e \perp m$ for every $e \in \mathcal{E}$ and $m \in \mathcal{M}$ and
(d) every morphism $k : Y \to X$ can be expressed as $k = e\,;m$ with $e \in \mathcal{E}$ and $m \in \mathcal{M}$.

The two classes determine one another *via* orthogonality and we will deal with several factorisation systems, so, in order to avoid introducing multiple new names, we will write ${}^{\perp}\mathcal{M}$ instead of $\mathcal{E}$ for the orthogonal class to $\mathcal{M}$. Since we are transferring intuitions, we will call $\mathcal{E}$- or ${}^{\perp}\mathcal{M}$-maps "epis", so we say "plain epi" for those that have the standard cancellation property $(e\,; f = e\,; g \Rightarrow f = g)$, just as we have done for "monos".

Proposition 6.2 Onto and 1–1 functions define a factorisation system in **Set**.

Proof The factorisation of the function $k : Y \to X$ is *via*

$$Q \;\equiv\; \{ky \mid y : Y\} \;\cong\; \{x : X \mid \exists y{:}Y.\, ky = x\} \;\equiv\; U.$$

For orthogonality in the diagram above, let $q \in Q$. Since e is onto there is some $y \in Y$ with $q = ey$ and we put $u \equiv fy$, so $mu = mfy = gey = gq$. If also $q = ey'$ then $mu = gq = mu'$, so $u = u'$ since m is 1–1. Hence if we put $hy \equiv u$ then the triangles commute and this is the only way of doing it. □

The next three results show that the classes that we introduced in Definition 2.8 are parts of factorisation systems:

Proposition 6.3 The class $\mathcal{I} \subset$ **Pos** of plain monos (functions that are 1–1 on points and preserve but don't necessarily reflect order) belongs to a factorisation system where ${}^{\perp}\mathcal{I}$ consists of the ***regular epis***, which are onto functions for which the order on the target is the transitive closure of the images of instances of the order on the source. The forgetful functor **Pos** $\to$ **Set** takes this system to the previous one.

Proof For the underlying sets, ${}^{\perp}\mathcal{I}$ and $\mathcal{I}$ agree with onto and 1–1 functions respectively, so on points the factorisation $Y \twoheadrightarrow Q \rightarrowtail X$ and orthogonality are the same as before. Each of the classes contains all isomorphisms and is closed under composition.

The order on the intermediate object Q is the transitive closure of instances like this:

$$q' \;=\; ey' \;\leq_Q\; ey \;=\; q \qquad \text{where} \qquad y' \;\leq_Y\; y.$$

Therefore, in order to show that $h : Q \to U$ is monotone it suffices to consider these situations, for which

$$hq' \;\equiv\; fy' \;\leq_U\; fy \;\equiv\; hq.$$ □

Remember that $\mathcal{D}$ does not preserve the class $\mathcal{I}$.

Proposition 6.4 The class $\mathcal{R} \subset$ **Pos** of regular monos (1–1 functions that preserve and reflect order) belongs to a factorisation system where ${}^{\perp}\mathcal{R}$ consists of ***plain epis*** (monotone functions that are onto on points). The forgetful functor also takes this system to the one for **Set**.

Proof Now the order on the intermediate object $U \subset X$ is the restriction of that on the target. For orthogonality, if

$$q' \;=\; ey' \;\leq_Q\; ey \;=\; q, \qquad u \;=\; hq \;=\; fy \qquad \text{and} \qquad u' \;=\; hq' = \; fy'$$

then $$mu' \;=\; mfy' \;=\; gey' \;=\; gq' \;\leq_X\; gq \;=\; gey \;=\; mfy \;=\; mu,$$

whence $u' \leq_U u$ since m reflects order. $\square$

Proposition 6.5 The class $\mathcal{L} \subset$ **Pos** of lower inclusions belongs to a factorisation system where ${}^{\perp}\mathcal{L}$ consists of ***cofinal functions***: $e : Y \to Q$ such that $\forall q{:}Q.\ \exists y{:}Y.\ q \leq ey$. Beware that these need not be onto or have the cancellation property for plain epis.

Proof Now the factorisation of $k : Y \to X$ is *via* the larger subobject

$$U \ \equiv\ \{x \in X \mid \exists y{:}Y.\ x \leq_X ky\} \ \subset_{\downarrow}\ X.$$

The values and monotonicity of h are as for $\mathcal{R}$, but we must still check that h is defined on all of Q. Given $q \in Q$, since e is cofinal there is some $y \in Y$ with $q \leq_Q ey$. Then $gq \leq_X gey = mfy \in U \subset_{\downarrow} X$, so $gq \in U$ and so $hq \equiv gq$ is well defined. $\square$

Lemma 6.6 Factorisation in **Set** or using $({}^{\perp}\mathcal{R}, \mathcal{R})$ or $({}^{\perp}\mathcal{L}, \mathcal{L})$ in **Pos** lifts to coalgebra homomorphisms.

$$\begin{array}{ccccc}
TY & \xrightarrow{\ Te\ } & TU & \xrightarrow{\ Ti\ } & TX \\
\beta\uparrow & & \gamma\uparrow & & \uparrow\alpha \\
Y & \xrightarrow{\ e\ } & U & \xrightarrow{\ i\ } & X
\end{array}$$

Proof Since the functor preserves monos, we may use the orthogonality $e \perp Ti$ to define the structure map $\gamma : U \to TU$ and make the squares commute. The other map, Te need not be epi. $\square$

Now we apply the general theory from [Tay23, Section 8] to factorising the structure maps of $\mathcal{P}$- and $\mathcal{D}$-coalgebras with respect to these systems. The first result says that epis preserve well-foundedness (*cf.* Corollary 8.4):

Lemma 6.7 Let $(\mathcal{E}, \mathcal{M})$ be a factorisation system such that $\mathcal{D}$ preserves $\mathcal{M}$. Let

$$(Y, \leq_Y, \prec_Y) \xrightarrow{\ e\ } (Q, \leq_Q, \prec_Q)$$

be a homomorphism in $\mathcal{E}$, where Y is well founded with respect to $\mathcal{M}$ as a class of predicates. (The map e need not be onto or plain epi.) Then Q is also well founded. $\square$

Corollaries 6.8 This applies directly to
(a) any onto homomorphism from a well founded $\mathcal{P}$-coalgebra in **Set**, *i.e.* a set with a well founded relation in the traditional sense;
(b) any ${}^{\perp}\mathcal{R}$-homomorphism (plain epi) out of a $\mathcal{D}$-coalgebra that is well founded with respect to predicates in $\mathcal{R}$; and
(c) any ${}^{\perp}\mathcal{L}$-homomorphism (cofinal) out of a $\mathcal{D}$-coalgebra that is well founded with respect to predicates in $\mathcal{L}$. $\square$

In fact, we saw in Section 4 that all of these notions of well-foundedness coincide. Therefore, although there is no such notion for the class $\mathcal{I}$ because $\mathcal{D}$ does not preserve it, there is still a version for $({}^{\perp}\mathcal{I}, \mathcal{I})$ as a special case of (b), because ${}^{\perp}\mathcal{I} \subset {}^{\perp}\mathcal{R}$:

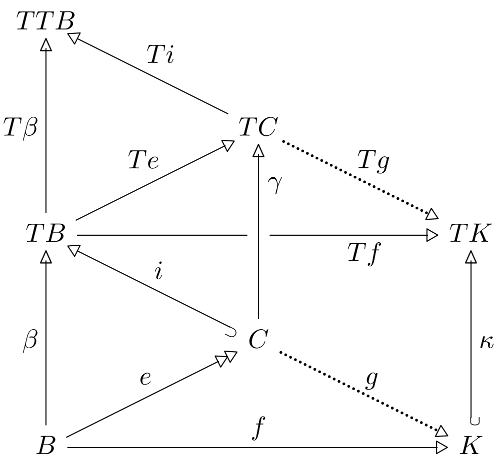
Corollary 6.9 The Lemma also applies to any homomorphism whose underlying map is in ${}^{\perp}\mathcal{I}$, out of a $\mathcal{D}$-coalgebra that is well founded in the traditional sense. $\square$

The first consequence of this in [Tay23, Section 8], for either $T \equiv \mathcal{P}$ or $\mathcal{D}$, was this:

Proposition 6.10 The inclusion T-**WfCoAlg** $\rightarrowtail$ T-**CoAlg** has a right adjoint, given by the largest well founded initial segment of any coalgebra. $\square$

However, the interesting application uses the "epi" that is given by factorising the structure map of the coalgebra:

Construction 6.11 The ***successor quotient*** (C, γ) of any coalgebra (B, β) is given by factorising its structure map β:



We define $\gamma \equiv i\,; Te$ but do not assume that $Te \in \mathcal{E}$ or $Ti \in \mathcal{M}$. Then
(a) both e and i are homomorphisms;
(b) if B is well founded then so is C;
(c) $e : B \cong C$ iff (B, β) is extensional; and
(d) any homomorphism $f : B \to K$ to an extensional coalgebra factors uniquely through C. □

We can *impose* extensionality (in any of our senses) on a coalgebra by iterating this construction, using Pataraia's fixed point theorem, in the form in [Tay23, Section 2]. For this we need that the co-slice category $(Y, \alpha)/{}^{\perp}\mathcal{M}$ of epis out of an object be equivalent to an ipo. The notion of being ***well co-powered*** is the analogue for outgoing epis of well powered for incoming monos.

Lemma 6.12 Onto functions in **Set** and ${}^{\perp}\mathcal{I}$ and ${}^{\perp}\mathcal{R}$ in **Pos** satisfy the cancellation property for epis and are well co-powered, whilst ${}^{\perp}\mathcal{L} \subset$ **Pos** fails both conditions.

Proof For sets, surjective functions out of Y are in bijection with equivalence relations on Y and therefore contained in $\mathcal{P}(Y \times Y)$. The result for ${}^{\perp}\mathcal{I} \subset$ **Pos** follows from this.

For ${}^{\perp}\mathcal{R}$, there may be additional instances of $(\leq)$ in the target, but we may represent these by their inverse images in Y, which form a preorder relation and so an element of $\mathcal{P}(Y \times Y)$.

On the other hand, there are at least ${}^{\perp}\mathcal{L}$-maps of the form $e : Y \to X + Y$ for *any* set X, however large, with $\forall yx.\, x \leq_X y$, so they cannot be encoded as elements of a single set. Nor can e control maps out of X. □

When the epis do form an ipo, [Tay23, Section 8] obtained:

Proposition 6.13 Amongst well founded $\mathcal{P}$- or $\mathcal{D}$-coalgebras, the $\mathcal{P}$-, $\mathcal{I}$- and $\mathcal{R}$-extensional ones form reflective subcategories and the unit is "epi" in the corresponding senses. □

Using Corollaries 6.8(a,b) we may interpret this to yield two familiar constructions in set theory:

Corollary 6.14 Applied to well founded relations or $\mathcal{P}$-coalgebras in **Set**, this is the extensional quotient described in Remark 1.4.

$$\mathcal{P}\text{-}\mathbf{WfCoAlg} \underset{\longleftarrow}{\overset{\text{Mostowski}}{\twoheadrightarrow}}\!\!\perp\ \mathsf{Ens}$$

The unit is an onto $\mathcal{P}$-coalgebra homomorphism. □

Since we now know about *meta*-transitive extensional well founded relations, we temporarily identify them with the classical ordinals and construct the ***slim rank***, as another example of the extensional reflection. It also illustrates how the phenomena of interest are subcategories that are like "islands the sea" of general $\mathcal{D}$-coalgebras (Remark 3.21).

Corollary 6.15 The (slim) ordinal rank imposes $\mathcal{R}$-extensionality on well founded $\mathcal{D}$-coalgebras:

$$\begin{array}{ccccccc}
& & (X, \prec_X) & \longmapsto & (X, =, \prec_X) & & \\
\mathsf{Ens} & \rightarrowtail & \mathcal{P}\text{-}\mathbf{WfCoAlg} & \rightarrowtail & \mathcal{D}\text{-}\mathbf{WfCoAlg} & \underset{\longleftarrow}{\overset{\text{slim rank}}{\twoheadrightarrow}}\!\!\perp & \mathsf{Slim} \\
& & & & (Y, \subseteq, \prec_Y) & \longleftarrow\!\!\!\dashv & (Y, \prec_Y)
\end{array}$$

The unit is a $\mathcal{D}$-coalgebra homomorphism in ${}^{\perp}\mathcal{R}$, *i.e.* surjective on points.

Proof The right-hand part of this diagram does the main work, for *general* well founded $\mathcal{D}$-coalgebras. Therefore, to construct the traditional rank of an (extensional) well founded relation ($\mathcal{P}$-coalgebra), we first embed that using the *discrete* order (Corollary 3.15) and then reflect into the subcategory of slim ordinals that are embedded using ($\subseteq$) (Definition 5.8). The $\mathcal{D}$-coalgebra structure, in particular the reflexive relation, is essential to formulating the rank as a left adjoint, because otherwise there is no forgetful functor to serve as the right one. □

Corollary 6.16 Amongst well founded $\mathcal{D}$-coalgebras the $\mathcal{I}$-extensional ones also form a reflective subcategory, but Example 3.16 shows that it is not a preorder and its maps need not be 1–1. □

Because of the failure of Lemma 6.12 for ${}^{\perp}\mathcal{L}$, the plump rank cannot be constructed in the logic of an elementary topos or Zermelo's original set theory, as we will see in Section 11. However, we may already have a bound:

Lemma 6.17 For any plump ordinal X,

$$\mathbf{WfCoAlg}/X \underset{\longleftarrow}{\overset{}{\twoheadrightarrow}}\!\!\perp\ \mathsf{Plump}/X \simeq \mathcal{D}(X)$$

Proof Factorise $Z \twoheadrightarrow Y \hookrightarrow X$. □

We will consider the usual notion of transitivity (and *thin* rank) in the next section.

7 Transitivity

We have now seen how certain *native* ideas of category theory reproduce several traditional notions from set theory, in particular sets, ordinals, the inclusion relation and the quotients that yield Mostowski's theorem and ordinal rank. In fact we have found *two* forms of "ordinal", simply by varying what we mean by a "mono" between posets. However, whilst both of these are classically equivalent to the traditional one, neither of them captures the most widely cited constructive definition (Remark 1.1), so we will now try to adapt the pattern to that.

Definition 7.1 A ***thin ordinal*** $(X, \prec)$ is a carrier with a *transitive* extensional well founded relation, where all of these words are understood in the standard senses.

As in Notation 5.17, we write $\mathsf{Thin} \subset \mathsf{Ens}$ for the full sub-*preorder* of thin ordinals and $\mathcal{P}$-coalgebra homomorphisms. When defined in this way, the thin ordinal $(X, \prec)$ becomes the *discrete* $\mathcal{D}$-coalgebra $(X, =, \prec)$ and by Corollary 3.15 this is a full embedding.

On the other hand, we would like thin ordinals to form a *reflective* subcategory of well founded structures (of some kind), as we had for slim ones in Corollary 6.15. However, Example 3.16 showed that we must use a non-trivial reflexive order in order to define the unit morphism for this reflection.

For these reasons we define a *second* embedding of thin ordinals, amongst *transitive* coalgebras instead of discrete ones. Recall from Definition 3.6 that this means that $(\prec) \subset (\leq)$.

Lemma 7.2 Let $(X, \prec)$ be a set with an acyclic binary relation, $(\lessdot\!\!\prec)$ its transitive closure and $(\preceq)$ its reflexive–transitive closure. Then $(X, \preceq, \lessdot\!\!\prec)$ is a transitive $\mathcal{D}$-coalgebra.

We call $(\preceq)$ the ***thin order***.

Proof Since $(\prec)$ is acyclic, $(\lessdot\!\!\prec)$ and $(\preceq)$ are antisymmetric, so $(X, \preceq)$ is a poset. The compatibility and transitivity conditions for coalgebras are easy.

Recall from its classical formulation using "infinite descent" that any well founded relation is acyclic; this remains so constructively (at least when the predicates form full higher order logic), whilst it also easily follows from extensionality. In the absence of those properties, the cycles define an equivalence relation, by which we could form the quotient to make this construction more general, but there is no need for that complication. □

This crude construction may look like classical recidivism, but its defence is the following result about coalgebra homomorphisms, which repairs Example 3.16:

Lemma 7.3 Let $f : (Y, \leq_Y, \prec_Y) \to (X, \preceq_X, \prec_X)$ be a $\mathcal{D}$-coalgebra homomorphism, where $(\prec_Y)$ is well founded, $(\prec_X)$ and $(\prec_Y)$ are transitive in the traditional sense and $(\preceq_X)$ is the reflexive closure of $(\prec_X)$. Then $f : (X, \prec_Y) \to (Y, \prec_X)$ is a $\mathcal{P}$-coalgebra homomorphism.

This does not assume extensionality.

Proof Recall from Lemma 3.11 that a $\mathcal{D}$-homomorphism f preserves both relations and

$$\begin{aligned} x \prec_X fy \quad &\iff \quad (\exists y''.\, x \preceq_X fy'' \wedge y'' \prec_Y y) \\ &\iff \quad (\exists y''.\, x = fy'' \;\wedge\; y'' \prec_Y y) \quad \vee \quad (\exists y'.\, x \prec_X fy' \;\wedge\; y' \prec_Y y). \end{aligned}$$

Consider the predicate ϕ_x on Y defined by

$$\phi_x y' \quad \equiv \quad (x \prec_X fy' \Longrightarrow \exists y''.\, x = fy'' \;\wedge\; y'' \prec_Y y').$$

In order to prove $\forall y.\, \phi_x y$ by induction, suppose that $\forall y'.\, y' \prec_Y y \Longrightarrow \phi_x y'$. In the disjunctive formula for $x \prec_X fy$ above, the first case is already

$$\exists y''. \quad x = fy'' \;\wedge\; y'' \prec_Y y.$$

The second, together with the induction hypothesis, give

$$\exists y'y''. \quad x = fy'' \;\wedge\; y'' \prec_Y y' \prec_Y y,$$

so $y'' \prec_Y y$ by transitivity. In both cases we have $\phi_x y$.

Hence $\forall xy.\, \phi_x y$, which is the third condition for a $\mathcal{P}$-coalgebra homomorphism. $\square$

Corollary 7.4 Even though thin ordinals are not $\mathcal{L}$-extensional and the functor $\mathcal{D}$ does not preserve them, their homomorphisms are $\mathcal{L}$-maps. Hence there is a full embedding

$$\mathsf{Thin} \rightarrowtail \mathcal{D}\text{-}\mathbf{TrWfCoAlg} \qquad \text{defined by} \qquad (X, \prec) \longmapsto (X, \preceq, \prec).$$

Proof Let $f : (Y, \preceq_Y, \prec_Y) \to (X, \preceq_X, \prec_X)$ be a $\mathcal{D}$-coalgebra homomorphism. By the previous lemma, $f : (Y, \prec_Y) \to (X, \prec_X)$ is a $\mathcal{P}$-coalgebra homomorphism.

Since thin ordinals are extensional $\mathcal{P}$-coalgebras, their homomorphisms are 1–1 and satisfy

$$x \prec_X fy \Longrightarrow \exists! y'.\ x = fy' \ \wedge\ y' \prec_Y y,$$

making them $(\prec)$-lower inclusions. Being $(\preceq)$-lower follows easily from this. $\square$

Corollary 7.5 The slice preorder Thin/X over any thin ordinal X is equivalent to the CCD lattice $\mathcal{D}X$.

Proof By a similar argument to Proposition 5.15. $\square$

Remark 7.6 As in Remark 5.16 we then have

$$\mathsf{Thin} \ \simeq \ \operatorname*{colim}_{X \in \mathsf{Thin}} \mathsf{Thin}/X \ \simeq \ \bigcup_{X \in \mathsf{Thin}} \mathcal{D}(X),$$

where $\mathcal{D}(X)$ carries its usual $(\subseteq)$ order, not $(\preceq)$. However, we do not have the further equivalence with $\bigcup X$, as we did in the plump case. This is because $\mathcal{D}$ does not preserve thin ordinals, so the colimits are over different systems of objects.

Examples 7.7
(a) $\mathbf{2} \equiv \{\bot, \top\}$ with $\bot \prec \top$ and $\bot \leq \top$ is both thin and slim, but not plump;
(b) $\mathcal{D}\mathbf{1} = (\Omega, \Rightarrow, \prec)$, also with $\bot \prec \top$, is slim and plump but not thin, because $(\Rightarrow)$ is not the thin order.
(c) The thin order on $\Omega \equiv \mathcal{D}\mathbf{1}$ is

$$\phi \preceq \psi \quad \equiv \quad (\phi \Leftrightarrow \psi) \ \vee\ (\neg\phi \wedge \psi).$$

(d) The thin order on $\Omega^{\to} \equiv \mathcal{D}\mathbf{2}$ is

$$U \preceq V \quad \equiv \quad (U = V) \ \vee\ (U \subset \{\bot\} \wedge V = 2).$$

Thin ordinals are also $\mathcal{I}$-extensional, but we will see that it is more appropriate to consider them to be well founded first, then transitive and *lastly* extensional. Forcing a coalgebra to be transitive ought to be a 2-categorical colimit of some kind, but I do not know what.

Lemma 7.8 Amongst acyclic $\mathcal{D}$-coalgebras, the transitive ones form a reflective subcategory.

$$\begin{array}{ccc}
(Y, \leq_Y, \prec_Y) & \xrightarrow{\quad f \quad} & (X, \leq_X, \prec_X) \\
& \searrow \qquad \nearrow & \\
& (Y, \prec\!\!\prec_Y \cup \leq_Y, \prec\!\!\prec_Y) &
\end{array}$$

Proof We add $(\leq_Y)$ to Lemma 7.2 by forming its union with the transitive closure $(\prec\!\!\!\prec_Y)$ (or equivalently with $\preceq_Y$). Using compatibility of $(\prec_Y)$ with $(\leq_Y)$, this union is transitive and compatible with $(\prec\!\!\!\prec_Y)$, whence it is also antisymmetric and we have a transitive $\mathcal{D}$-coalgebra.

Now let $f : (Y, \leq_Y, \prec_Y) \to (X, \leq_X, \prec_X)$ be a homomorphism to a transitive $\mathcal{D}$-coalgebra. Since it takes $(\prec_Y)$ to $(\prec_X)$ and the latter is transitive in the usual sense, it also takes $(\prec\!\!\!\prec_Y)$ to $(\prec_X)$. Since it takes $(\leq_Y)$ to $(\leq_X)$ and the latter includes $(\prec_X)$, it also takes $(\prec\!\!\!\prec_Y \cup \leq_Y)$ to $(\leq_X)$. For the third condition for $\mathcal{D}$-homomorphisms, if $x \prec_X fy$ then $x \leq_X fy' \wedge y' \prec_Y y$, so $y' \prec\!\!\!\prec_Y y$. ☐

Corollary 7.9 In the category of well founded $\mathcal{D}$-coalgebras, the full subcategory of transitive ones is reflective.

Proof We also need that transitive closure preserves well-foundedness, which follows from Lemma 6.7 since the carrier stays the same (e is a bijection). ☐

Beware of the distinction between transitive coalgebras and thin ordinals:

Example 7.10 The operation of replacing the poset order of a transitive coalgebra with the thin order $(\preceq)$ is not a functor. That is, there is no functor

$$\mathcal{D}\text{-}\mathbf{ExtTrWfCoAlg} \longrightarrow \mathsf{Thin}.$$

Proof Example 3.19 showed that

$$\mathbf{3} \equiv \{0,1,2\} \to \mathcal{D}\Omega \qquad \text{by} \qquad 0 \mapsto \emptyset, \quad 1 \mapsto \{0\} \quad \text{and} \quad 2 \mapsto \{0,1\}$$

is a $\mathcal{D}$-homomorphism between extensional transitive well founded coalgebras but it is not a $\mathcal{P}$-homomorphism. It therefore ceases to be a homomorphism of either kind if the $(\subseteq)$ order on $\mathcal{D}\Omega$ is replaced with $(\preceq)$. ☐

Besides this, Example 3.16 shows that transitive closure does not preserve extensionality, so we need to impose that first. Fortunately, the extensional reflection does preserve transitivity. To prove this, we add the following to Construction 6.11ff:

Lemma 7.11 Let $e : (Y, \leq_Y, \prec_Y) \twoheadrightarrow (X, \leq_X, \prec_X)$ be an ${}^{\perp}\mathcal{I}$-homomorphism from a transitive $\mathcal{D}$-coalgebra. Then X is also transitive.

Proof In fact we only need ${}^{\perp}\mathcal{R}$, that e be an onto $\mathcal{D}$-homomorphism. By Lemma 3.11,

$$\begin{aligned} x \prec_X ey &\implies \exists y' \!:\! Y.\; x' \leq_X ey' \;\wedge\; y' \prec_Y y \\ &\implies \exists y' \!:\! Y.\; x' \leq_X ey' \;\wedge\; y' \leq_Y y \\ &\implies \exists y' \!:\! Y.\; x' \leq_X ey' \leq_X ey \\ &\implies x \leq_X ey. \end{aligned}$$

☐

Lemma 7.12 Let $e : (Y, \prec_Y) \twoheadrightarrow (X, \prec_X)$ be an onto $\mathcal{P}$-homomorphism where $(\prec_Y)$ is a transitive relation. Then $(\prec_X)$ is also transitive, by the same argument with $(=)$ instead of $(\leq_X)$. ☐

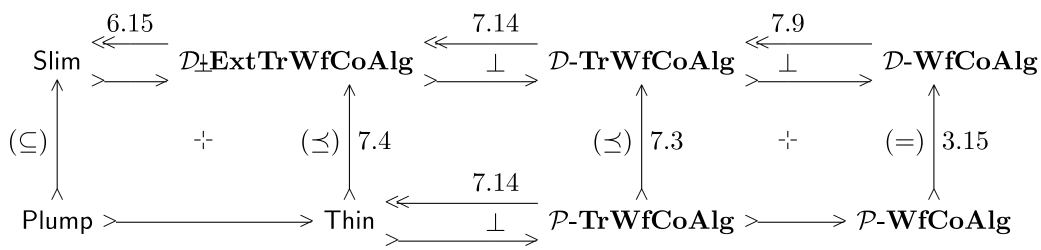
Lemma 7.13 Let $e : (Y, \preceq_Y, \prec_Y) \twoheadrightarrow (X, \leq_X, \prec_X)$ be an ${}^{\perp}\mathcal{I}$-homomorphism, where $(\preceq_Y)$ is the thin order. Then so is $(\leq_X)$.

Proof By Proposition 6.3, it suffices to consider the case where $x' \leq_X x$ is the image of $y' \preceq_Y y$, so either $y' = y$ and $x' = x$ or $y' \prec_Y y$ and $x' \prec x$ since any homomorphism preserves $(\prec)$. ☐

Corollary 7.14 The $\mathcal{I}$-extensional reflection (Corollary 6.16) preserves transitivity of relations and of coalgebras too, and also the thin $(\preceq)$ order. Moreover, amongst transitive well founded $\mathcal{P}$- or $\mathcal{D}$-coalgebras, the $\mathcal{I}$-extensional ones form a reflective subcategory.

Proof Use Pataraia induction [Tay23, Section 2] for Construction 6.11, since onto or ${}^{\perp}\mathcal{I}$ maps are epis and well co-powered (Lemma 6.12). □

Theorem 7.15 Any well founded relation has a thin ordinal rank.



Proof Peter Freyd's puncture symbol (-|-) [FS90] indicates that the outer squares do not commute, because we are using the *three different* order relations. On the left, plump ordinals are slim with $(\subseteq)$ but thin with no poset order. In the middle square, thin ordinals are given the thin $(\preceq)$ order, but in the final column the discrete $(=)$ one.

The transitive closure (Corollary 7.9) is defined as a functor, left adjoint to the inclusion, for $\mathcal{D}$- but not $\mathcal{P}$-coalgebras, *cf.* Example 3.16.

The $\mathcal{I}$-extensional reflection (Corollary 6.16) is defined for both $\mathcal{P}$- and $\mathcal{D}$-coalgebras and preserves transitivity (Corollary 7.14). However, only for $\mathcal{P}$-coalgebras does the category become a preorder.

The diagram as shown does not have a path from $\mathcal{P}$-**WfCoAlg** to Thin, but that is because we have not drawn a copy of the top row in which the reflexive order is required to be the thin one $(\preceq)$. Lemma 7.2 is the special case of Lemma 7.9 for $(=)$ and yields $(\preceq)$. Then this is preserved by Corollary 7.14, giving a thin ordinal.

We have another example of how the interesting subcategories are like "islands in the sea" of general $\mathcal{D}$-coalgebras (Remark 3.21). □

This section has unfortunately not met its promise to make the thin rank another instance of a *single* extensional reflection for some special class of monos, as we had for the slim rank. It is still a composite of two operations, the transitive closure and $\mathcal{I}$-extensional reflection, as in the traditional theory.

To do it as a single operation would require a factorisation system for $\mathcal{D}$-*homomorphisms*, whereas we have only defined them for the underlying functions. It is plausible that there is such a system, but we would need to check a lot of unenlightening details that would not be relevant to the rest of this paper.

It would be more appropriate to do this in the context of a more general investigation of ordinals for other KZ-monads (Definition 2.4). Then we might get a better understanding of the roles of factorisation of homomorphisms and of 2-categories.

8 Binary meets and joins

We now know that Ens, Thin, Slim and Plump are preorders and have directed joins inherited from **Set** or **Pos**. They have a least element $\emptyset$ too. We also know that Ens, Thin and Slim *must have* binary joins, because they are obtained by applying the reflection to coproducts in $\mathcal{P}$- or $\mathcal{D}$-**TrWfCoAlg**, which are inherited from **Set** or **Pos**.

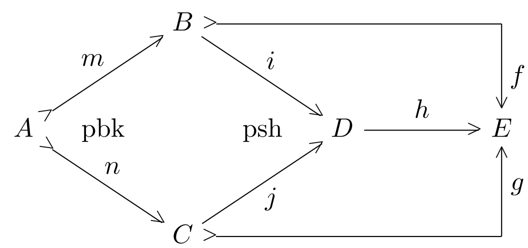
However, that is not very informative, so in this section we study binary joins more carefully. We show that Ens, Thin and Plump inherit them from pushouts in **Set** or **Pos** over their intrinsic binary intersections. This is the behaviour from set theory (Remark 1.3).

We therefore first need to understand the binary *meet* in these preorders.

Unfortunately, we find that all of this breaks down in the case of Slim. The binary meets and joins exist and the latter are quotients of the pushouts of their carriers, but there seems to be no easy way of understanding what order relations they carry.

First, we put coalgebras aside to understand pushouts in the underlying categories (**Set** and **Pos**).

Definition 8.1 A category with finite limits and colimits has the ***binary union property*** with respect to a given class of monos if



(a) the pushout D of a pair of monos $B \overset{m}{\longleftarrowtail} A \overset{n}{\rightarrowtail} C$ is another pair of monos and is also a pullback (this is known as the ***Amalgamation Lemma***); and

(b) if additionally A, B, C and E form a pullback with all these maps mono, then the mediator $h : D \to E$ is also mono.

Set or any pretopos satisfy this property.

Now we consider to what extent this behaviour transfers to **Pos**:

Lemma 8.2 **Pos** has the binary union property with respect to $\mathcal{L}$ as monos.

Proof We begin by taking advantage of the binary union property in **Set** and let D be pushout of sets, considering B and C as subsets of it that intersect exactly in A.

We define $x \le y$ in D if both x and y belong to B with $x \le_B y$, or both belong to C with $x \le_C y$.

For transitivity, suppose that $x \le y \le z$ with $x, y \in B$ and $y, z \in C$, so $y \in B \cap C \equiv A$. But since $A \subset_\downarrow B$, we have $x \in A \subset_\downarrow C$ too, so in fact $x, y, z \in C$ and already $x \le z$ by transitivity in C. Similarly with B and C reversed.

The order on D is also antisymmetric: if $x \le_D y$ because $x \le_B y$ whilst $y \le_D x$ because $y \le_C x$ then $x, y \in B \cap C \equiv A$ and $x \le_A y \le_A x$, so $x = y$ in all four preorders.

Hence D with this order is the pushout in **Pos**.

Now let $h : D \to E$ be the pushout mediator for some pair $f : B \to E$ and $g : C \to E$ of $\mathcal{L}$-maps. In particular, they are 1–1 functions, so h is too by the binary union property for **Set** and we may regard all of the objects as subsets of E, with $B \cup C = D$. The remaining question is whether $D \subset E$ is a *lower* subset. If $E \ni e \le_E y \in B$ then $e \in B$ too since $f : B \subset_\downarrow E$, so $e \in D$, and similarly with C in place of B. Hence $h : D \subset_\downarrow E$ is an $\mathcal{L}$-map. □

Example 8.3 The second part of the Definition fails for $\mathcal{R}$.

Proof Let $A \equiv \emptyset$, $B \equiv \{b\}$, $C \equiv \{c\}$, $D \equiv \{b, c\}$ and $E \equiv \{b \le c\}$ in the previous diagram. Then A is pullback with either root and D is the pushout, as before. However, whilst all the other maps are in $\mathcal{R}$, the pushout mediator $D \to E$ is not, since it does not reflect the order $b \le c$. □

Re-introducing the coalgebra structures, we need to transfer the well-foundedness property along a homomorphism. Lemma 6.7 did this forwards along an epi, but now we to do it backwards. This is the result that lies behind "induction on the length, height, depth, *etc.*" that is common in mathematics.

Lemma 8.4 If $f : A \to B$ is homomorphism and B is well founded then so is A.

Proof When the objects are $\mathcal{P}$-coalgebras, *i.e.* sets with binary relations, it suffices that f preserve the $(\prec)$ relation. This is easy to prove using either of the classical definitions of well-foundedness and it is shown intuitionistically using proof boxes in [Tay99, Prop. 2.6.2].

The results for **Pos** follow directly because the notions of well-foundedness in Section 4 coincide.

The general categorical proof is in [Tay23, Section 9], where it is also shown that it is necessary that the functor ($\mathcal{D}$ here) preserve inverse images of the chosen monos (in the role of predicates), so these must be $\mathcal{L}$ rather than $\mathcal{R}$ (Lemma 2.19ff). It is also necessary that the inverse image operations f^*, when applied to $\mathcal{L}$-inclusions, have right adjoints, as they do. □

Now we can calculate intersections of well founded coalgebras.

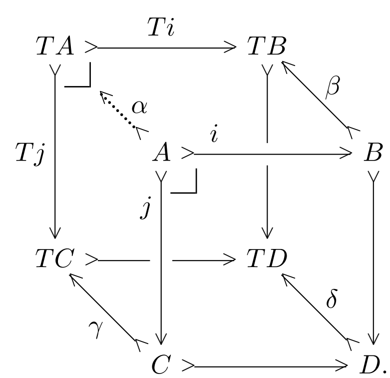
Lemma 8.5 The forgetful functors

$$\mathsf{Thin} \rightarrowtail \mathsf{Ens} \rightarrowtail \mathcal{P}\text{-}\mathbf{WfCoAlg} \rightarrowtail \mathcal{P}\text{-}\mathbf{CoAlg} \longrightarrow \mathbf{Set}$$

create binary intersections of 1–1 functions, whilst

$$\mathsf{Plump} \rightarrowtail \mathcal{D}\text{-}\mathbf{WfCoAlg} \rightarrowtail \mathcal{D}\text{-}\mathbf{CoAlg} \longrightarrow \mathbf{Pos} \longrightarrow \mathbf{Set}$$

create them for $\mathcal{L}$-maps.



Proof Working in **Set**, the functor $T \equiv \mathcal{P}$ preserves intersections. Hence if B, C and D carry coalgebra structures then so does their intersection A. By the previous Lemma, if B or C is well founded then so is A. If B or C is extensional (so β or γ is 1–1) then so is their subset A, by the cancellation property for 1–1 functions. If B or C is transitive then so is their subset A.

The same argument may be used in **Pos**, where $T \equiv \mathcal{D}$ preserves intersections of $\mathcal{L}$-maps (Lemma 2.19), to construct intersections of (well founded) $\mathcal{D}$-coalgebras and plump ordinals. □

In this construction, beware that, in general, in order to define an intersection as a pullback in a category, the diagram must first have a root D. If we change the root then the pullback A will also change.

In a preorder, the root is called a common *upper* bound. So far we know that these do *exist* for (well founded) $\mathcal{P}$- and $\mathcal{D}$-coalgebras and also in Ens and Thin, but we do not know much about them. We don't yet know that they exist in Plump and the construction above does not work for Slim because $\mathcal{D}$ does not preserve intersections of $\mathcal{R}$-maps.

However, the ideas from set theory provide these intersections without using an upper bound. Since it's a universal property, the two constructions for intersections of plump ordinals must agree. However, whilst the intersection also exists for slim ordinals, in Example 8.11 it is not the pullback of their carriers.

Lemma 8.6 Between any two extensional well founded coalgebras B and C, there is a greatest span $B \hookleftarrow A \hookrightarrow C$ of initial segments.

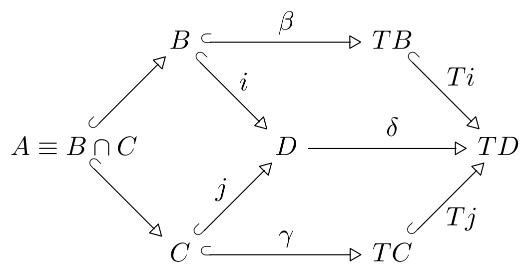
If they are embedded in some larger extensional well founded coalgebra D by $B \hookrightarrow D \hookleftarrow C$ then this forms a pullback square (in the category of extensional well founded coalgebras and homomorphisms), irrespectively of the choice of D.

Proof The greatest span is defined recursively, by "zipping" the two coalgebras together (Remark 1.3 and Theorem 5.2, proved in [Tay23, §7]). It just requires that the functor preserve monos, as $\mathcal{P}$ does in **Set** and $\mathcal{D}$ does for both $\mathcal{L}$ and $\mathcal{R}$ in **Pos**.

Any cone over this pullback diagram is also a span, so it is less than the greatest one, which is to say that it has a mediator to the pullback. □

We use this to show that binary joins behave as expected in Ens, Thin and Plump:

Theorem 8.7 The preorders Ens and Plump have binary joins, given by pushout over the greatest span.



Proof Let B and C be well founded coalgebras that are extensional with respect to 1–1 maps in **Set** or $\mathcal{L}$-maps in **Pos**, and let $B \hookleftarrow A \hookleftarrow C$ be the greatest span between them.

By the binary union property we may form their pushout D in **Set** or **Pos** and this recovers A as the pullback.

Being a pushout, D carries a well founded coalgebra structure [Tay23, §5] and all of the maps in diagram belong to the chosen class of monos, apart perhaps from $\delta : D \longrightarrow TD$.

As we have said, A is the pullback for *any* common bound: not just D but also TD. Then by the binary union property for the pushout D, the mediator $D \xrightarrow{\delta} TD$ also belongs to the chosen class of monos.

Hence this is the binary join in Ens or Plump. □

Corollary 8.8 The same construction also yields the binary join in Thin.

Proof We apply the Proposition in Ens, the monos being 1–1 functions in **Set**. This restricts to Thin because it is a full subcategory (Definition 7.1), and any sub-coalgebra A of a thin ordinal B or C is thin.

The remaining question is whether the order $(\prec)$ on the binary join D is transitive.

For this we switch to considering a thin ordinal $(X, \prec)$ as a transitive coalgebra $(X, \preceq, \prec)$, between which the *homomorphisms* are $\mathcal{L}$-maps (Corollary 7.4), even though the *structure* maps of the coalgebras are not.

By Lemma 8.2 the $(\leq)$ order on D is the union of the thin $(\preceq)$ orders on B and C, but this is the union of $(\prec_B)$, $(\prec_C)$ and $(=_D)$, which is $(\preceq_D)$. Hence D is a thin ordinal. □

We can now describe how to internalise colimits in the large preorders Thin and Plump (Remark 1.6) as joins in the CCD lattices $\mathcal{D}(X)$. Even for Thin we consider joins with respect to $(\subseteq)$, because Lemma 9.13 shows that they don't exist for $(\preceq)$.

Remark 8.9 All set-indexed colimits in Thin or Plump are bounded, *i.e.* they lie within some slice Thin$/X$ or Plump$/X$. This means that they are joins in $\mathcal{D}(X)$. In fact, we know this directly from the equivalences Thin$/X \simeq \mathcal{D}(X)$ and Plump$/X \simeq \mathcal{D}(X)$ that we proved earlier, whereas there is no such equivalence for Slim.

In the plump case, colimit diagrams in X are represented by elements of X (Definition 5.9), so there is a partial internal join operation $\bigvee : \mathcal{D}(X) \rightharpoonup X$.

These colimits or joins are preserved by any change of base, *i.e.* the embedding $\mathcal{D}i : \mathcal{D}(X) \rightarrowtail \mathcal{D}(Y)$ induced by a homomorphism $i : X \rightarrowtail Y$. Recall that for thin and plump ordinals any such homomorphism is a lower inclusion, an $\mathcal{L}$-map.

This is the sense in which the large preorders Thin and Plump are equipped with internal join operations of any set-indexed arity.

Lemma 9.13 shows why we must use joins with respect to $(\subseteq)$ for thin ordinals and not $(\preceq)$.

We conclude this section with some things that go wrong. Whereas it has usually been enough to consider the presheaf topos $\mathbf{Set}^{\rightarrow}$ to construct our counterexamples, we need slightly more complicated settings on this occasion.

Example 8.10 The functor $\mathcal{D}$ does not preserve binary joins of plump ordinals.

Proof Let $\xi_1, \xi_2 \in \Omega$ be independent non-decidable predicates (*cf.* Notation 3.17) and define

$$\begin{array}{lll} & B & \equiv & \{\phi : \Omega \mid \phi \Rightarrow \xi_1\} \subset \Omega \\ & C & \equiv & \{\phi : \Omega \mid \phi \Rightarrow \xi_2\} \subset \Omega \\ \text{so} & B \cup C & = & \{\phi : \Omega \mid (\phi \Rightarrow \xi_1) \vee (\phi \Rightarrow \xi_2)\} \\ & U \subset_{\downarrow} \Omega & \equiv & \forall \phi\psi.\, (\psi \Rightarrow \phi) \wedge (\phi \in U) \Rightarrow (\psi \in U) \\ & \mathcal{D}B & = & \{U \subset_{\downarrow} \Omega \mid (\forall \phi \in U.\, \phi \Rightarrow \xi_1)\} \\ & \mathcal{D}B \cup \mathcal{D}C & = & \{U \subset_{\downarrow} \Omega \mid \big((\forall \phi \in U.\, \phi \Rightarrow \xi_1) \vee \big(\forall \phi \in U.\, \phi \Rightarrow \xi_2)\big)\} \end{array}$$

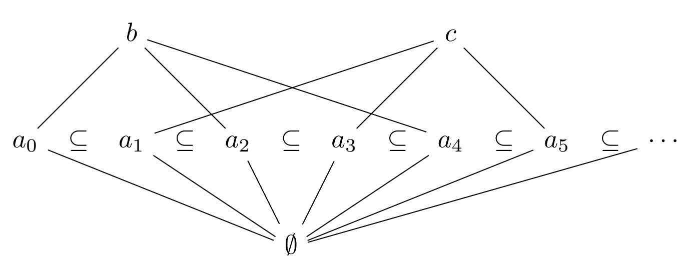
Then $(B \cup C) \in \mathcal{D}(B \cup C)$, but if $(B \cup C) \in \mathcal{D}B \cup \mathcal{D}C$ then $(\xi_2 \Rightarrow \xi_1) \vee (\xi_1 \Rightarrow \xi_2)$. $\square$

Example 8.11 Amalgamation fails for slim ordinals.

Proof Let $\xi_{(-)} : \mathbb{N} \to \Omega$ be a strictly increasing sequence of truth values, for example in the presheaf topos $\mathbf{Set}^{\omega}$, so

$$\forall n.\, \xi_n \Longrightarrow \xi_{n+1} \qquad \text{and} \qquad \forall n.\, \neg(\xi_{n+1} \Longrightarrow \xi_n).$$

Put $a_n \equiv \{\emptyset \mid \xi_n\}$, so $a_n \subset a_{n+1} \subset \{\emptyset\}$, $b \equiv \{\emptyset\} \cup \{a_n \mid n \text{ even}\}$ and $c \equiv \{\emptyset\} \cup \{a_n \mid n \text{ odd}\}$.



As in Example 5.7, the truth value of $\emptyset \prec a_n$ and of $\emptyset \subseteq a_n$ is ξ_n, but the upper lines indicate $(\prec)$ with full truth and we also understand that $\emptyset \prec b, c$. The inclusions $a_n \subseteq a_{n+1}$ are also fully true. Now let

$$B \;\equiv\; \{a_{2n}, b \mid n \in \mathbb{N}\} \qquad \text{and} \qquad C \;\equiv\; \{a_{2n+1}, c \mid n \in \mathbb{N}\},$$

for which membership is meta-transitive. So $(B, \preceq, \prec)$ and $(C, \preceq, \prec)$ are thin ordinals, whilst $(B, \subseteq, \prec)$ and $(C, \subseteq, \prec)$ are slim ones.

Then b and c are distinct in the *thin* binary union $B \cup C$ (the whole diagram), but are made equal in its meta-transitive closure D, *i.e.* as *slim* ordinals.

The example also shows that the whole development breaks down. The pullback $B \cap_D C$ in **Pos** consists of the top $(b = c)$ and bottom $(\emptyset)$ elements, but their common initial segment just contains the least element. This is a lower subset of B and of C, as in the thin case, where the pushout has distinct b and c, not equal as in D.

What about the "shift" functions $B \leftrightarrows C$ by $a_{2n} \leftrightarrow a_{2n+1}$ and $b \leftrightarrow c$? The rightward one preserves $(\prec)$ and $(\subseteq)$ but fails third condition in Corollary 3.12(b), whilst it is the opposite for the leftward one, because $\neg(\xi_{n+1} \Rightarrow \xi_n)$. $\square$

9 Successors

Like transitivity (Section 7), the successor operation is derived from the unit η of the monad structure on $\mathcal{P}$ or $\mathcal{D}$.

Definition 9.1 The ***internal successor operation*** on a coalgebra (X, α) is given by the pullback

$$\begin{array}{ccc} X & \xrightarrow{\;\alpha\;} & \mathcal{D}X \\ {\scriptstyle s_X}\uparrow & \lrcorner & \uparrow{\scriptstyle \eta_X} \\ P & \xrightarrow{\;p_X\;} & X \end{array}$$

so this is in the first instance a *binary relation* on X.

By antisymmetry of the poset $(X, \leq)$, the maps η_X and s_X are mono (indeed in $\mathcal{R}$), so this relation could be seen as a *partial function* $p : X \rightharpoonup X$, called *immediate predecessor*. However, this is always undefined for the element $\emptyset \in X$, if such exists, and also for ω in the classical ordinals.

On the other hand, when the coalgebra is extensional (in any of our senses), the maps α and p_X are also mono (in the corresponding sense). So there is another partial function $s : X \rightharpoonup X$ called *immediate successor* and there are plenty of situations in which this is total:

Example 9.2 For $\mathbb{N}$ with its usual orders $(\leq)$ and $(<)$, $sn = n + 1$, whilst $pm = m - 1$ for $m \neq 0$. $\square$

Lemma 9.3 For any $\mathcal{I}$-extensional $\mathcal{D}$-coalgebra $(X, \leq, \prec)$, if sx is defined for $x \in X$ then they satisfy

$$\forall y{:}X. \quad y \prec sx \iff y \leq x \qquad \text{so in particular} \qquad x \prec sx$$

and conversely this property characterises sx.

Proof It says $\alpha(sx)$ and $\eta_X(x)$ are the same member of $\mathcal{D}X$ or lower subset of X. $\square$

Corollary 9.4 The successor satisfies Remark 1.1, that

$$sy \prec sx \iff sy \leq x \implies sy \subseteq x \iff y \prec x$$

and $$sy \subseteq sx \iff y \prec sx \iff y \leq x \implies y \subseteq x.$$

Proof The first and fourth equivalences are just the Lemma. The forward directions of the second and third follow from $y \prec sy$ and the definition of $(\subseteq)$. For the reverse directions,

$$z \prec sy \ \wedge\ y \prec x \iff z \leq y \prec x \implies z \prec x$$

by the Lemma and one compatibility law. The other, $z \prec y \leq x \Rightarrow z \prec x$, gives the one-way implications. □

Remark 9.5 The successor therefore preserves all of the order relations iff the poset order $(\leq)$ is $(\subseteq)$. By Lemma 5.3, this is so iff $\alpha \in \mathcal{R}$. We may see this from the pullback definition: since $\eta_X, \alpha, s_X \in \mathcal{R}$, the object P inherits $(\subseteq)$ from $\mathcal{D}X$ *via* X and s takes it back to X.

The weaker property that $x \subseteq sx$ holds iff $(\prec) \subset (\leq)$, that is, iff the coalgebra is transitive (Definition 3.6 and Section 7).

All of these observations depend on the existence of the successor sx for all $x \in X$, so the following results show that this is "eventually" ensured.

Lemma 9.6 For any homomorphism $f : Y \to X$, if $s_Y y$ is defined then so is $f(s_Y y)$ and

$$f(s_Y y) \ = \ s_X(fy).$$

Proof By naturality of η with respect to f in a commutative cube: if $\beta y' = \eta_Y y$ then

$$\alpha(fy') \ = \ (\mathcal{D}f)(\beta y') \ = \ (\mathcal{D}f)(\eta_Y y) \ = \ \eta_X(fy),$$

so fy' obeys the defining property of $s_X(fy)$. Alternatively,

$$\begin{aligned} x \prec f(s_Y y) \quad &\iff \quad \exists y'.\, x \leq fy' \ \wedge\ y' \prec s_X y \\ &\iff \quad \exists y'.\, x \leq fy' \ \wedge\ y' \leq y \quad \iff \quad x \leq fy, \end{aligned}$$

so $f(s_Y y)$ satisfies the property that Lemma 9.3 required of $s_X(fy)$. □

Note that $s_X(fy)$ may be defined even when $s_Y y$ is not, indeed there is always a homomorphism that does this:

Lemma 9.7 For any $\mathcal{I}$-extensional $\mathcal{D}$-coalgebra $(X, \leq, \prec)$, for all $x \in X$

$$s_{\mathcal{D}X}(\alpha x) \ = \ \eta_X(x).$$

Proof Again by naturality of η with respect to α, or

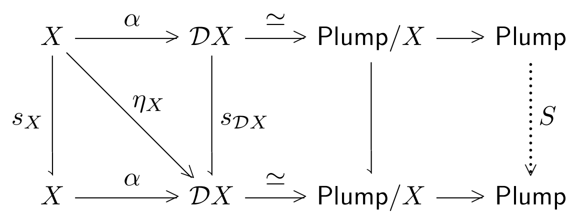
$$\begin{aligned} U \prec^{\mathcal{D}X} s_{\mathcal{D}X}(\alpha x) \quad &\equiv \quad \exists y \in s_{\mathcal{D}X}(\alpha x).\, \forall u \in U.\, u \prec^X y \\ &\equiv \quad \exists y \in \eta_X(x).\, \forall u \in U.\, u \prec^X y \\ &\equiv \quad \exists y.\, \forall u \in U.\, u \prec^X y \leq x \\ &\equiv \quad \forall u \in U.\, u \prec^X x \quad \equiv \quad U \subseteq^{\mathcal{D}X} \alpha x, \end{aligned}$$

which is the condition in Lemma 9.3 with αx for x and U for y. □

Examples 9.8 From this we recover three familiar constructions: $s_{\mathcal{D}X}(\alpha x) \equiv \eta_X(x)$ is as follows when the poset order is

(a)	discrete	$(=)$	$\{x\} \in \mathcal{D}(X) \equiv \mathcal{P}(X)$	no condition
(b)	thin	$(\preceq)$	$\{y \mid y \preceq x\} \equiv \alpha(x) \cup \{x\} \in \mathcal{D}(X)$	$x \subseteq sx$
(c)	plump	$(\subseteq)$	$\{y \mid y \subseteq x\} \in \mathcal{D}(X)$	s preserves $(\subseteq)$. □

Proposition 9.9 There is a monotone endofunction $S : \mathsf{Plump} \to \mathsf{Plump}$ that restricts to all of the partial successors $s_X : X \rightharpoonup X$ on plump ordinals.



Proof Recall that Plump is the illegitimate colimit of its slices, where $\mathsf{Plump}/X \simeq \mathcal{D}(X)$, and the colimit diagram consists of all of the homomorphisms (Remarks 1.6 and 5.16).

The simplest way of defining an endofunction of such a colimit is to give a system of endofunctions of its nodes that commute with the maps. In our case these would be the *partial* successors s_X or $s_{\mathcal{D}X}$. Another way is to "go up a step", using the *total* maps $\eta_X : X \to \mathcal{D}X$, or rather $\eta_{\mathcal{D}X} : \mathcal{D}X \to \mathcal{D}\mathcal{D}X$, that were given by Lemma 9.7. These also commute with the homomorphisms, by naturality of η. $\square$

In the thin case, the partial successors fail to preserve any of the order relations, so we only have a diagram in **Set**, not **Pos**. Besides this, $\mathcal{D}X$ is not thin, and it's no good just imposing the thin order on it (Example 7.10), but we have an alternative construction:

Lemma 9.10 Let $(X, \preceq, \prec)$ be a thin ordinal and define $(\prec)$ on the set $X \times \mathbf{2}$ by

$$y_0 \prec x_0 \;\equiv\; y \prec x \qquad \text{and} \qquad y_0 \prec x_1 \;\equiv\; y \preceq x \qquad \text{but} \qquad y_1 \prec x_0, x_1 \quad \text{never.}$$

Then $(\prec)$ is transitive, antisymmetric and well founded. The partial map s satisfies $x_0 \prec sx_0 \equiv x_1$ and $x_0 \subseteq sx_0$, but does not preserve any of the order relations.

The inclusion $f : X \to X \times \mathbf{2}$ by $x \mapsto x_0$ is a $\mathcal{P}$-homomorphism.

The coalgebra $X \times \mathbf{2}$ need not be extensional, since there may already be an element of X with the property of sx. We write X^s for the $\mathcal{I}$-extensional reflection of $X \times \mathbf{2}$, using Corollary 6.16.

Proof Simple exercises.

Proposition 9.11 There is an endofunction $S : \mathsf{Thin} \to \mathsf{Thin}$ that restricts to all the partial successors $s_X : X \rightharpoonup X$ on thin ordinals. It satisfies $\mathsf{id} \leq S$ but does not preserve the order relations.

Proof As in Proposition 9.9, Thin is the illegitimate colimit of its slices, $\mathsf{Thin}/X \simeq \mathcal{D}(X)$ over all *thin* ordinals. However, $\mathcal{D}$ does not preserve thinness, so for the morphism between the diagrams we use the total maps from $\mathcal{D}X$ to $\mathcal{D}(X^s)$ given by Lemma 9.10. Since these maps don't preserve order, the diagram is in **Set** and not **Pos**. $\square$

Having considered successors and joins for both kinds of ordinals, it remains to show that these operations generate all of them.

Proposition 9.12 In both the thin and plump cases, every ordinal is the join of the successors of its elements.

Proof Forming the joins in $\mathcal{D}X$,

$$\alpha x \;=\; \bigcup\{\eta_X y \mid y \prec x\} \;=\; \bigcup\{s_{\mathcal{D}X}(\alpha y) \mid y \prec x\}$$

because $$z \prec x \iff \exists y.\, z \leq y \prec x \iff \exists y.\, z \prec s_{\mathcal{D}X}(\alpha y) \;\wedge\; y \prec x.$$ $\square$

Now that we have recovered quite a lot of traditional set theory, we can show why we use (binary) joins with respect to $(\subseteq)$ and not $(\preceq)$. We may perhaps regard this as the intuitionistic result that underlies the trichotomous, linear or total order on the classical ordinals.

Lemma 9.13 If $x, y \in X$ in a thin ordinal admit the binary join $x \curlyvee y$ with respect to $(\preceq)$ then they are comparable: $x \prec y$, $x = y$ or $y \prec x$.

Proof By Lemma 9.10 we may assume that sx, $s(sx)$, sy and $s(sy)$ exist in X and then use Lemma 9.7 to work in $\mathcal{D}X$, where we have binary unions $(\cup)$ with respect to $(\subseteq)$. We repeatedly use Corollary 9.4 and the binary disjunction for $(\preceq)$.

If there is a cocone $x \preceq z \succeq y$ then either x and y are comparable or $x \prec z \succ y$ and so $sx \subseteq z \supseteq sy$.

If the join exists then any cocone $x \prec z \succ y$ must have either $x \curlyvee y = z$ or $x \curlyvee y \prec z$.

Now consider the binary unions with respect to $(\subseteq)$:

$$x \prec sx \subseteq (sx \cup sy) \succ y \qquad \text{and} \qquad x \prec sx \prec s(sx) \subseteq \big(s(sx) \cup s(sy)\big) \succ y,$$

so these are cocones. Since $(\cup)$ in $\mathcal{D}X$ is union of the underlying sets,

$$w \prec \big(s(sx) \cup s(sy)\big) \iff \big(w \prec s(sx)\big) \text{ or } \big(w \prec s(sy)\big) \iff (w \preceq sx) \text{ or } (w \preceq sy).$$

Putting $w \equiv x \curlyvee y$ in either of these cases,

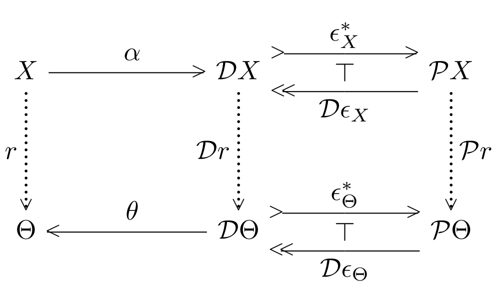
$$y \preceq (x \curlyvee y) \preceq sx \implies y \preceq sx \implies (x \prec sx = y) \text{ or } (y \prec sx) \implies (x \prec y) \text{ or } (y \preceq x),$$

making x and y comparable, and similarly when they are reversed. $\square$

10 Recursion

Now we pick up the story of *well founded* recursion from Section 4 and define *transfinite* recursion from it. Previously we justified the following result for any well founded coalgebra for either $\mathcal{P} : \mathbf{Set} \to \mathbf{Set}$ or $\mathcal{D} : \mathbf{Pos} \to \mathbf{Pos}$:

Theorem 10.1 Let (X, α) be a $\mathcal{D}$-coalgebra in **Pos** that is well founded in any of the senses in Section 4. Then it has ***well founded recursion***: for any $\mathcal{D}$-algebra $\theta : \mathcal{D}(\Theta) \to \Theta$ in **Pos**, there is a unique coalgebra-to-algebra homomorphism making the square on the left commute:



Proof The recursion theorem from [Tay23, §6] is directly applicable to $\mathcal{P}$-**WfCoAlg** and to $\mathcal{D}$-**WfCoAlg** with $\mathcal{R}$ or $\mathcal{L}$ as predicates and initial segments. This yields a unique coalgebra-to-algebra homomorphism $r : X \to \Theta$ that is monotone in the $\mathcal{D}$ case.

We can also deduce the result for $\mathcal{D}$ from that for $\mathcal{P}$. Recall that any object of the form $\mathcal{P}X$ or $\mathcal{D}X$ is a complete $\bigvee$-semilattice. In the square on the right, the rightward maps are inclusions and the leftward ones form the down-closure of any subset, which is a monotone and surjective operation. The action of $\mathcal{D}r$ is exactly the three-sided composite *via* $\mathcal{P}X$ and $\mathcal{P}\Theta$ (Definition 2.1(b), Lemma 2.7).

The coalgebra (X, α) corresponds to $(X, \leq, \prec)$ where $(\prec)$ is a well founded relation in the traditional sense, so defines a well founded $\mathcal{P}$-coalgebra, $(X, \alpha\, ;\, \epsilon^*_X)$. (This is not a functor, *cf.* Corollary 3.20, but that does not matter here.) By the recursion theorem for this, there is a unique $r : X \to \Theta$ in **Set** making the rectangle commute.

By the previous comments, the left-hand square commutes in **Set**, so the remaining question is monotonicity in the case of $\mathcal{D}$. But $\mathcal{P}r$ is monotone for *any* function r, whence so are $\mathcal{D}r$ and r because they are composites of monotone functions, since we have assumed that θ preserves order. □

Transfinite recursion is a universal property that compares Thin or Plump with any other complete $\bigvee$-semilattice Θ equipped with an endofunction $\sigma : \Theta \to \Theta$ that satisfies $\mathsf{id} \leq \sigma$ and monotonicity respectively. The latter refers to the order on Θ that is derived from its semilattice structure. Then

Lemma 10.2 The function $\theta : \mathcal{D}\Theta \to \Theta$ defined by

$$\theta \;\equiv\; \mathcal{D}\sigma\, ; \bigvee \qquad \text{or} \qquad \theta U \;\equiv\; \bigvee\nolimits_{\Theta} \{\sigma u \mid u \in U\}$$

preserves the joins and order with respect to Θ and so defines a $\mathcal{D}$-*algebra*. This does not require σ to preserve order.

Proof Just as for r_X in Theorem 10.1, $\mathcal{D}\sigma$ preserves order and joins for *any* function σ.

In symbols, if $U \subset V$ then the join defining θU is over a smaller subset than that defining θV. Also, since the join in $\mathcal{D}X$ is union,

$$\theta\big(\bigcup\nolimits_i U_i\big) \;=\; \bigvee \{\sigma u \mid \exists i.\, u \in U_i\} \;=\; \bigvee_i \bigvee \{\sigma u \mid u \in U_i\},$$

in which it doesn't matter how the values σu behave. □

Corollary 10.3 By Theorem 10.1, for any well founded $\mathcal{D}$-coalgebra $(X, \leq, \prec)$ there is unique monotone function $r_X : (X, \leq) \to \Theta$ such that

$$r_X \;=\; \alpha\, ; \mathcal{D}r_X\, ; \theta \;\equiv\; \alpha\, ; \mathcal{D}r_X\, ; \mathcal{D}\sigma\, ; \bigvee$$

or

$$r_X x \;=\; \theta\{r_X y \mid y \prec x\} \;\equiv\; \bigvee \{\sigma(r_X y) \mid y \prec x\}.$$

Monotonicity of r_X here means with respect to the given poset orders on X and Θ. When X is a thin or plump ordinal this order is $(\preceq)$ or $(\subseteq)$ respectively, so

$$y \prec_X x \implies y \preceq_X x \implies r_X y \leq_\Theta r_X x \qquad \text{or} \qquad y \subseteq_X x \implies r_X y \leq_\Theta r_X x.$$ □

Lemma 10.4 We have $r_X = \alpha\, ; R_X$, where the map

$$R_X \;\equiv\; \mathcal{D}r_X\, ; \mathcal{D}\sigma\, ; \bigvee : \mathcal{D}X \to \Theta \qquad \text{by} \qquad U \mapsto \bigvee \{\sigma(r_X u) \mid u \in U\}$$

preserves joins, for the same reason as in Lemma 10.2. □

Having dealt with joins, we consider successors.

Lemma 10.5 If $\sigma : \Theta \to \Theta$ is monotone and $(X, \leq, \prec)$ is an $\mathcal{I}$-extensional well founded coalgebra with elements x and sx that satisfy $\forall y.\, y \prec sx \Leftrightarrow y \leq x$ as in Lemma 9.3 then $r_X(sx) = \sigma(r_X x)$.

Proof $$r_X(sx) \;=\; \bigvee\{\sigma(r_X y) \mid y \prec sx\} \;=\; \bigvee\{\sigma(r_X y) \mid y \le x\} \;=\; \sigma(r_X x),$$

where the join is simply the top value of the set because r_X and σ preserve the poset order. □

That will provide transfinite recursion over *plump* ordinals. However, the successor for *thin* ordinals does not preserve order and *this did not matter* for Lemma 10.2. We therefore want to give their universal property without this assumption.

Lemma 10.6 If $\sigma : \Theta \to \Theta$ has $\mathsf{id} \le \sigma$ but is not necessarily monotone and $(X, \preceq, \prec)$ is a thin ordinal with elements x and sx that satisfy $\forall y.\, y \prec sx \Leftrightarrow y \preceq x$ then $r_X(sx) = \sigma(r_X x)$.

Proof We still have the same expansion of $r_X(sx)$, where $(\le)$ is now the thin order $(\preceq)$, but this is defined as a binary disjunction, so

$$y \prec sx \iff y \preceq x \;\equiv\; (y = x \;\vee\; y \prec x).$$

Therefore, evaluating the recursor at the successor gives

$$\begin{aligned} r_X(sx) &= \bigvee\{\sigma(r_X y) \mid y \prec sx\} \\ &= \sigma(r_X x) \;\vee\; \bigvee\{\sigma(r_X y) \mid y \prec x\} \\ &= \sigma(r_X x) \;\vee\; r_X x, \end{aligned}$$

which is just $\sigma(r_X x)$ since $\mathsf{id} \le \sigma$. □

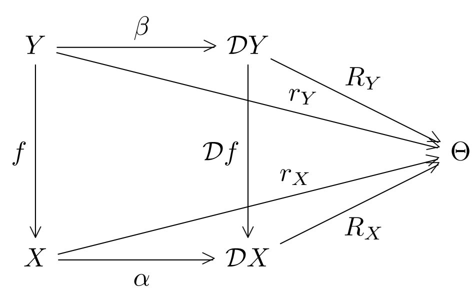
Lemma 10.7 Under the conditions of the previous two Lemmas and in the same sense, $R_X : \mathcal{D}(X) \to \Theta$ commutes with successors.

Proof

$$\begin{aligned} R_X(s_{\mathcal{D}X}(\alpha x)) &= R_X(\alpha(sx)) && \alpha \text{ homomorphism} \\ &\equiv r_X x && \text{definitions} \\ &= \sigma(r_X x) && \text{Lemma 10.5 or 10.6} \\ &\equiv \sigma(R_X(\alpha x)) && \text{definitions} \end{aligned}$$ □

The successor operation on any individual coalgebra or ordinal is typically partial, but becomes total as an endofunction S of the illegitimate colimits Plump or Thin, as in Propositions 9.9 or 9.11. We use these to obtain transfinite recursion.

Lemma 10.8 For any homomorphism $f : Y \to X$ of thin or plump ordinals, the diagram



commutes.

Proof The square says that f is a homomorphism. Then the bigger triangle commutes because $f\,; r_X$ obeys the recursion property for r_Y:

$$f\,; r_X \;=\; f\,; \alpha\,; \mathcal{D}r_X\,; \theta \;=\; \beta\,; \mathcal{D}f\,; \mathcal{D}r_X\,; \theta.$$

The little triangles relate r to R (by Corollary 10.3 and Lemma 10.4) and the equilateral one is

$$R_Y \;=\; \mathcal{D}r_Y\,;\theta \;=\; \mathcal{D}f\,;\mathcal{D}r_X\,;\theta \;=\; \mathcal{D}f\,;R_X. \qquad \square$$

Proposition 10.9 This therefore defines functions

$$\mathsf{Thin} \;\simeq\; \bigcup_{X\in\mathsf{Thin}} \mathcal{D}(X) \xrightarrow{R} \Theta \qquad \text{and} \qquad \mathsf{Plump} \;\simeq\; \bigcup_{X\in\mathsf{Plump}} \mathcal{D}(X) \xrightarrow{R} \Theta$$

that preserves joins and successors.

Proof The previous two lemmas define cocones $R_{(-)} : \mathcal{D}(-) \to \Theta$ (Remark 8.9). $\square$

Lemma 10.10 If there is any function $R : \mathsf{Thin} \to \Theta$ or $R : \mathsf{Plump} \to \Theta$ such that

$$R(Sx) \;=\; \sigma(Rx) \qquad \text{and} \qquad R(\bigvee x_i) \;=\; \bigvee\nolimits_i R(x_i)$$

then it must be given by $\qquad rx \;=\; \bigvee\nolimits_\Theta \{s(ry) \mid y \prec x\}.$

Proof Any ordinal is the join of the successors of its elements (Proposition 9.12)

$$\alpha x \;=\; \bigcup \{s_{\mathcal{D}X}(\alpha y) \mid y \prec x\},$$

so since R preserves joins and successors,

$$rx \;\equiv\; R(\alpha x) \;=\; \bigcup \{\sigma(R(\alpha y)) \mid y \prec x\} \;=\; \bigcup \{\sigma(ry) \mid y \prec x\},$$

which has a unique solution by Theorem 10.1. $\square$

This completes the proof of transfinite recursion:

Theorem 10.11 Let Θ be a complete $\bigvee$-join semilattice and $\sigma : \Theta \to \Theta$ an endofunction of it.
(a) If σ is monotone then there is a unique map $R : \mathsf{Plump} \to \Theta$, whilst
(b) if $\mathsf{id} \le \sigma$ then there is a unique map $R : \mathsf{Thin} \to \Theta$,
such that

$$R(Sx) \;=\; \sigma(Rx) \qquad \text{and} \qquad R(\bigvee X_i) \;=\; \bigvee\nolimits_i R(X_i). \qquad \square$$

11 Growth of plump ordinals

The class $\mathcal{L}$ fails Lemma 6.12 and so there is no reflection into the subcategory of plump ordinals. Indeed, we now show that they grow extremely fast, so that in general they cannot be constructed in the logic of an elementary topos but require the axiom-scheme of replacement in some form. Since we show this by a counting argument, our basic reasoning in this section is classical (using excluded middle). The object model is the one that we have been using for most of our counterexamples, namely the topos $\mathbf{Set}^{\to}$ of presheaves on a single arrow.

We start with some simple classical lattice-theoretic constructions in **Set** itself:

Definition 11.1 A subset $N \subset (X, \le)$ of a preorder is called an ***antichain*** if the restriction of the order to the subset is discrete.

Lemma 11.2 If $(X, \le)$ has an antichain of size 4 then $\mathcal{D}(X, \le)$ has one of size 6. [not needed]

Proof If $a, b, c, d \in X$ have no instance of $\leq$ between any two of them then there is no containment between any two of the following lower subsets of X:

$$\downarrow\{a,b\}, \quad \downarrow\{a,c\}, \quad \downarrow\{a,d\}, \quad \downarrow\{b,c\}, \quad \downarrow\{b,d\} \quad \text{and} \quad \downarrow\{c,d\}. \qquad \square$$

Lemma 11.3 If $\mathbf{2} \times N \subset (X, \leq)$ is an antichain then there is an antichain $\mathbf{2}^N \subset \mathcal{D}(X, \leq)$.

Proof For each decidable $U \subset N$, let

$$S_U \;\equiv\; \{x : X \mid \exists n \in N.\, \big(x \leq (1,n) \wedge n \in U\big) \vee \big(x \leq (0,n) \wedge n \notin U\big)\} \subset_{\downarrow} X.$$

Then $\quad S_U \subset S_V \Longrightarrow (U \subset V) \wedge (N \setminus U \subset N \setminus V) \Longrightarrow U = V.$

Hence the S_U for $U \in \mathbf{2}^N$ provide a big antichain in $\mathcal{D}(X, \leq)$. $\square$

Now we can start calculating the plump ordinals in our target model.

Definition 11.4 In the category $\mathcal{S} \equiv \mathbf{Set}^{\rightarrow}$

(a) the objects are functions $X_0 \to X_1$,

(b) the morphisms are commutative squares, and

(c) the products, monos and relations are computed componentwise.

Proposition 11.5 For any object $(X_0 \xrightarrow{f} X_1)$ of $\mathcal{S}$, the powerset $\Omega^{(X_0 \to X_1)}$ is the object

$$\left(\left\{ \begin{array}{c} U_1 \\ \uparrow \\ \emptyset \end{array} \;\middle|\; U_1 \subset X_1 \right\} \longrightarrow \left\{ \begin{array}{c} U_1 \\ \uparrow \\ U_0 \end{array} \;\middle|\; \begin{array}{ccc} U_1 & \subset & X_1 \\ \uparrow & & \uparrow f \\ U_0 & \subset & X_0 \end{array} \right\}\right),$$

or $$\left(\mathbf{2}^{X_1} \longrightarrow \{(U_0, U_1) : \mathbf{2}^{X_0} \times \mathbf{2}^{X_1} \mid \forall x{:}X_0.\; x \in U_0 \Rightarrow fx \in U_1\}\right).$$

Prove this: Similarly, for any internal preorder $(X, \leq)$ in $\mathcal{S}$, the lower set lattice $\mathcal{D}(X, \leq)$ is given in the same way, except that each $\subset$ becomes $\subset_{\downarrow}$. $\square$

Example 11.6 We know in general that the first three plump ordinals are $\mathbf{0}$, $\mathbf{1}$ and Ω, which in $\mathbf{Set}^{\rightarrow}$ are the objects

$$\mathbf{0} \;\equiv\; (\emptyset \to \emptyset), \qquad \mathbf{1} \;\equiv\; (\star \to \star) \qquad \text{and} \qquad \Omega \;\equiv\; (\{a,b\} \subset \{a,b,c\}),$$

where $\star$ is the singleton set consisting of the identity on $\emptyset$ and

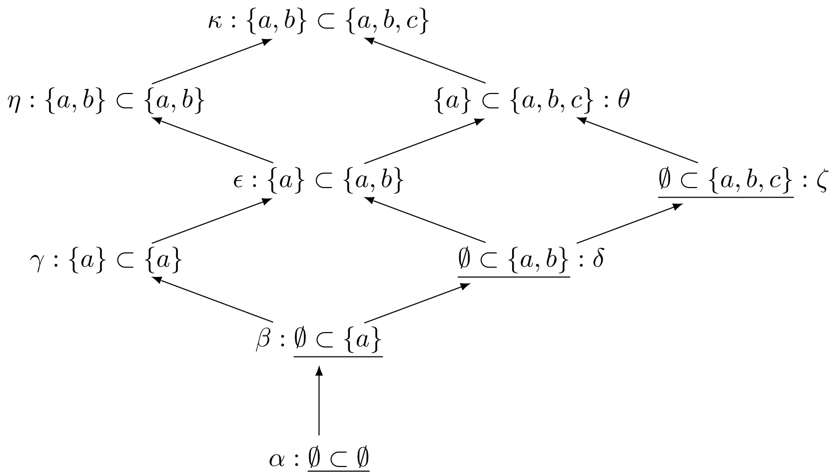
$$a \;\equiv\; (\emptyset \to \emptyset), \qquad b \;\equiv\; (\emptyset \to \star) \qquad \text{and} \qquad c \;\equiv\; (\star \to \star).$$

What is $(\prec)$ on Ω, for both total and partial elements?

Guess the coalgebra structures are:

$$\begin{array}{ccccccc}
\emptyset & \longrightarrow & \star & \xrightarrow{\;a\;} & \{a,b,c\} & \longrightarrow & \{\alpha, \ldots \kappa\} \\
\uparrow & & \uparrow & & \uparrow & \begin{array}{c} a \mapsto \alpha \\ b \mapsto \beta \\ c \mapsto \gamma \end{array} & \uparrow \\
\emptyset & \longrightarrow & \star & \xrightarrow{\;a\;} & \{a,b\} & \longrightarrow & \{\alpha, \beta, \delta, \zeta\} \\
 & & & & & & \\
\mathbf{0} & \longrightarrow & \mathbf{1} & \longrightarrow & \mathbf{2} \equiv \Omega & \longrightarrow & \mathbf{3} \equiv \mathcal{D}\Omega
\end{array}$$

Example 11.7 The next ordinal, $\mathcal{D}\Omega$, is more complicated:



The 1-part of the $\mathcal{S}$-object that we require is the whole of this lattice and the 0-part is the subset consisting of the four underlined elements.

What is $(\prec)$ for this? $\square$

Remark 11.8 What subobjects of $\mathcal{D}(\Omega)$ are slim ordinals, *i.e.* $\mathcal{R}$-sub-$\mathcal{D}$-coalgebras?

$\mathcal{R}$-inclusions that are $(\leq)$-cofinal.

Corollary 11.9 The 1-part of $\mathbf{4} \equiv \mathcal{D}^2(\Omega)$ has 56 elements, including an antichain of size 6:

$$\emptyset \subset {\downarrow}\kappa, \quad {\downarrow}\alpha \subset {\downarrow}\eta\theta, \quad {\downarrow}\beta \subset {\downarrow}\eta, \quad {\downarrow}\beta \subset {\downarrow}\theta, \quad {\downarrow}\delta \subset {\downarrow}\epsilon\zeta, \quad {\downarrow}\zeta \subset {\downarrow}\gamma\zeta,$$

where the Greek letters denote the elements of the previous lattice and we omit $\{\}$ for clarity. $\square$

Theorem 11.10 The 1-parts of the finite plump ordinals in $\mathbf{Set}^{\to}$ contain antichains of size at least

$$0, \quad 1, \quad 1, \quad 2, \quad 6, \quad 2^{6/2} = 8, \quad 2^{8/2} = 16, \quad 2^{16/2} = 256, \quad \ldots,$$

so plump ω, $\omega + 1$, $\omega + 2$, ... have them of size

$$\aleph_0, \quad 2^{\aleph_0}, \quad 2^{2^{\aleph_0}}, \quad \ldots.$$

Therefore plump $\omega \cdot 2$ cannot be defined in the language of an elementary topos with $\mathbb{N}$ or in Zermelo set theory, *i.e.* without the axiom-scheme of replacement.

Proof We are only applying Lemma 11.3 very weakly: it's enough to consider $X_0 \equiv \emptyset$, although doing so more carefully would yield faster growth for the finite ordinals. However, all that we need is that ω have an infinite antichain and that its successors grow by classical cardinal exponentiation as indicated. $\square$

Remark 11.11 By a similar technique, the same can probably be shown for just ω in the topos of presheaves on some infinite base category. $\square$

Corollary 11.12 Since $\mathcal{D}$ preserves infinitary intersections, the preorder Plump of $\mathcal{L}$-extensional well founded $\mathcal{D}$-coalgebras has them (and hence all limits) and the forgetful functor $\mathsf{Plump} \to \mathcal{D}$-$\mathbf{WfCoAlg}$ preserves them. However, this functor need not have a left adjoint unless we assume some form of the axiom-scheme of replacement. $\square$

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