

Replacement as an Adjoint Functor Theorem

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19 October 2026

This is a “discussion document” rather than a draft paper. I intend to share it with progressively wider but carefully chosen circles of colleagues. The text is likely to acquire more discussion about other understandings of Replacement, but even in those contexts this should provide a useful new Adjoint Functor Theorem.

This note was intended to be the write-up of my ItaCa lecture on 18 November 2025 called *A Categorical Replacement for Replacement*. It sketched the proof from IZF that any well founded coalgebra has an extensional reflection, in a very general situation in which, in particular, extensionality is defined using a general factorisation system.

However, I found a clean intellectual break with the material about well founded coalgebras. That has now been merged into (what is now) Section 9 of the 18 October 2026 version of [Tay21]. In addition to this, I have now written up the details of my CT26 lecture called *Transfinite iteration of functors as an extensional reflection* [Tay26a], which builds the “von Neumann hierarchy”.

It *almost* re-constructs IZF — apart from some issues that seem to me to be more about the first order formulation of set theory than about category theory (§8).

What is left (of the ItaCa lecture) is a Theorem about the existence of an adjoint that makes essential use of Unbounded Subset Selection and of Replacement.

Although this was originally intended for *reflections* to full replete subcategories (specifically extensional ones amongst all well founded coalgebras), it seems to work for general adjunctions.

I started writing some introductory material, intending the result to be understandable to mathematicians with only the most basic understanding of category theory and set theory. However, I realised that my prose was unlikely to survive debate with colleagues.

So this note just contains the key mathematical argument.

The introduction to a longer paper would argue that adjunctions are the most important feature of the “native language” of category theory and type theory. It would give some familiar examples to show how adjunctions or even reflections can be quotients, completions or generate bigger and more complex structures.

1 Octoberfest abstract

Octoberfest, 13:30 UTC, 31 October 2026.

Category Theory is a mature subject. We have rewritten most of 20th century mathematics in the language of an elementary topos and we have studied other foundational systems formulated using categorical properties. We do not need Set Theory, we should not formulate our subject in set-theoretic language and we should certainly not allow set theorists to tell us what our foundations are.

However, there are constructions, in particular infinite iteration of functors, that go outside topos logic but can supposedly be founded in ZF.

What, in our native language, is the principle underlying such constructions?

Our native language is written in adjunctions, so how can we derive a new way of justifying the existence of adjoints using IZF?

It is sometimes said that Replacement provides “big” colimits, but before invoking a colimit we must formulate its diagram. This may not be definable in topos logic.

However, it may be definable as the image under a putative left adjoint of a diagram in another category that is topos-definable.

Underlying this is an example of an unbounded predicate that is allowed in Zermelo set theory but not topos logic, namely that a particular morphism has a universal property.

Powerful adjunctions are related to recursive constructions but recursion need not terminate. To handle this, we introduce a notion of “well founded object”.

I need some guidance here on the correctness of my invocation of IZF, along with other suggestions of natively categorical properties that are unbounded predicates or make use of Replacement.

2 IZF *versus* topos logic

There are two significant differences between the logic of an elementary topos and (Intuitionistic) Zermelo–Fraenkel Set Theory as “officially” formulated.

Firstly, in topos logic we build up the collection of “sets” (objects) using products, powersets, natural numbers and a version of “subset selection” that admits those “elements” that satisfy predicates that are themselves built up using logical operations including quantification over objects that *have already been defined*. These are called ***bounded predicates***.

In Zermelo (–Fraenkel) Set Theory, “sets” are built up in a similar way, except that quantifiers may range over *the whole universe* of all sets, *i.e.* using ***unbounded predicates***.

(To be strictly accurate historically, Ernst Zermelo originally just said selection was for “definite properties” [Zer08]. Only with the influence of Thoralf Skolem in particular was this notion fixed using first order logic.)

Secondly, the axiom-scheme of Replacement says that, given any binary predicate $\Theta(x, y)$ that ranges over the universe of all sets and is formulated in first order logic, together with a set X , such that

$$\forall x \in X. \exists! y. \Theta(x, y),$$

there is a set Y with

$$y \in Y \iff \exists x \in X. \Theta(x, y).$$

On the face of it, $X \cong Y$, so any class that is bijective with a set is also a set.

But this is a vacuous statement to categorists.

Another way of saying this is that any function

$$\Theta : X \longrightarrow \mathbf{Set}$$

defines a discrete opfibration

$$\begin{array}{c} Y \equiv \sum_{x \in X} \Theta(x) \\ \downarrow p \\ X. \end{array}$$

But the “function” F is not defined in topos logic, so in fibred category theory we may just deal with this as the function $p : Y \rightarrow X$.

What we see is that Replacement just slips through our fingers if we’re not careful with it.

Two ways of dealing with this are to formulate category theory

- (a) in first order logic, just as set theory has been done since Skolem (ETCS, Section 9); or
- (b) as an internal category.

However, I consider that this *misses the point of* Replacement, which is somehow able *to build skyscrapers from plans on the ground*.

This should be expressed in *the native language* of category theory, namely as an *Adjoint Functor Theorem*.

3 The question of existence of an adjunction

Let $R : \mathcal{A} \rightarrow \mathcal{X}$ be a functor between categories, on which we will put some restrictions in the next section, but we suppose that the relevant constructions can be made within the logic of an elementary topos. We want to find a left adjoint $L : \mathcal{X} \rightarrow \mathcal{A}$ that may not be so definable.

We frame this question *one object at a time* in an older form [Sam48]:

Definition 3.1 A morphism $\eta_X : X \rightarrow RA$ of $\mathcal{X}$ is **universal** if for any $\mathcal{X}$ -map $f : X \rightarrow RB$ there is a unique $\mathcal{A}$ -map $h : A \rightarrow B$ that makes the triangle commute:

$$\begin{array}{ccc} X & \xrightarrow{\eta_X} & RA \\ & \searrow f & \downarrow Rh \\ & & RB \end{array} \qquad \begin{array}{c} A \\ \downarrow h \\ B \end{array}$$

We only need to find universal maps for each *object* $X \in \mathcal{X}$, not the morphisms, because of this result, which was apparently not noticed before [ML71, Thm. IV.2(ii)]:

Proposition 3.2 The adjoint $L \dashv R$ exists iff every object $X \in \mathcal{X}$ has a universal map $\eta_X : X \rightarrow RA$ and then $A = LX$.

Proof Use uniqueness to show, in this order: (a) $\eta_{(-)} : \text{id}_{\mathcal{X}} \rightarrow R \cdot L$ is natural, (b) L is defined on morphisms and is functorial, (c) there is a unique $\epsilon_A : L(RA) \rightarrow A$ with $\eta_{RA} ; R\epsilon_A = \text{id}_{RA}$, (d) $\epsilon_{(-)} : L \cdot R \rightarrow \text{id}_{\mathcal{A}}$ is natural and (e) $L\eta_X ; \epsilon_{LX} = \text{id}_{LX}$. $\square$

Remark 3.3 There is the usual annoying use of Choice here, which could be avoided by using profunctors or similar to throw in all of the isomorphic copies of A . However, every categorist knows that this is unimportant. Also, we’re only going to deal with universal maps from *single* objects or set-indexed families of them, not the whole of $\mathcal{X}$. $\square$

Remark 3.4 The relevance of the difference between IZF and the logic of an elementary topos is this:

- (a) The predicate in *three* variables (X , A and η_X) that $\eta_X : X \rightarrow RA$ is universal is typically definable in the logic of an elementary topos.
- (b) What may (for interesting cases) *not* be so definable is the *single*-argument predicate

$$\Phi(X) \equiv \exists A \in \mathcal{A}. \quad \exists \eta_X : X \rightarrow RA. \quad \eta_X \text{ is universal,}$$

because the quantifier $\exists A \in \mathcal{A}$ ranges over the universe (or at least the large category $\mathcal{A}$).

So Φ is a predicate that may be definable in Zermelo set theory but not in the logic of an elementary topos.

I think technically the three-variable universal property is called Δ_0 , whilst $\Phi(X)$ is Σ_1 and $L \dashv R$ is Π_2 .

Φ may be an example of the *usefulness* of “Replacement of sets”, contrary to Mike Shulman’s comment [Shu18, §7].

Remark 3.5 However, this observation is not specific to topos logic, set theory or replacement. In categorical logic, we *build up* the particular logic that we want to study by *decreeing that certain functors have adjoints*. (Give the familiar examples.) These correspond directly to the introduction and elimination rules from type theory (§9).

Remark 3.6 There is also a philosophical discussion that could be had about the meaning of the quantification over $f : X \rightarrow RB$ in Definition 3.1. Predicative type theorists cannot point their fingers at category theory about this, because the general indeterminate contexts in their rules, typically written Γ and Δ , play *precisely* the same role as universal properties. We skip this discussion by saying that we’re doing the usual thing in both subjects, but return to it in Section 9.

Remark 3.7 I encountered another unbounded predicate in category theory in [Tay26b]. Let $\rho : \text{id} \rightarrow R$ and $\sigma : \text{id} \rightarrow S$ be well pointed endofunctors of any category $\mathcal{X}$. Then

$$(R, \rho) \triangleleft (S, \sigma) \quad \equiv \quad \forall x \in \mathcal{X}. \quad \sigma_X : X \cong SX \implies \rho_X : X \cong RX$$

is an unbounded (Π_1) relation between these functors that is relevant to finding the idempotent monad generated by (S, σ) .

Question 3.8 What other examples are there of unbounded predicates that arise *naturally* in category theory?

4 The question of existence of colimits

Since the leading example of the use of Replacement is the von Neumann hierarchy, one may at first think that the issue is the *formation* of “big” colimits.

Remark 4.1 However,

- (a) before forming a colimit we must form the diagram;
- (b) the diagram may itself not be definable in topos logic;
- (c) it may instead be the image under an adjunction or universal property, expressed as the predicate Φ ;
- (d) where the diagram and colimit *are* so definable *in the other category*.

The role of the universal property and the predicate Φ is that the diagram of subobjects of X in $\mathcal{X}$ is definable in topos logic, whereas the image diagram in $\mathcal{A}$ may not be so definable.

This seems to be the essential point in this note. It is therefore the one most needs discussion with colleagues.

It's no good writing this with too much categorical abstraction: Since we need at some point to invoke a principle of Set Theory, the given categories $\mathcal{X}$ and $\mathcal{A}$ need to have some relationship to **Set**.

Assumption 4.2 Both categories are *locally small*, i.e. the homs $\mathcal{X}(X, Y)$, $\mathcal{A}(A, B)$ and $\mathcal{X}(X, RA)$ are sets. (Where do we actually use this?)

Assumption 4.3 Both categories have *initial objects*, $\emptyset \in \mathcal{X}$ and $Z \in \mathcal{A}$, so $\eta_\emptyset : \emptyset \rightarrow RZ$ is universal. Also $\emptyset \rightarrowtail X$ is a subobject and is the smallest one. Hence $\Phi(\emptyset)$ holds.

Assumption 4.4 The category $\mathcal{X}$ is *well powered*. It's enough for our purposes to say this for an individual object $X \in \mathcal{X}$:

- (a) there is a *set* $\mathbf{Sub}(X)$ of isomorphism classes of subobjects $Y \rightarrowtail X$;
- (b) $\mathbf{Sub}(X)$ has directed unions that are colimits.

Therefore the relevant infinitary colimit diagrams are *directed posets*. That these joins must agree with colimits may limit the applicability of the Theorem; for example $\mathbb{Z} \rightarrow R$ is not mono for general rings.

Being locally small and well powered seem to be the only way in which the category $\mathcal{X}$ needs to be related to **Set**.

The situation is different for $\mathcal{A}$. We know that, if the left adjoint L exists, then it must preserve colimits, so we state what we need regarding those.

Assumption 4.5 The category $\mathcal{A}$ has a “forgetful” functor $U : \mathcal{A} \rightarrow \mathbf{Set}$ that *creates colimits* of diagrams that are directed posets, as above.

These Assumptions are designed to prove this:

Lemma 4.6 The set of subobjects of $X \in \mathcal{X}$ that satisfy the predicate Φ is closed under unions indexed by any directed partially ordered set $\mathcal{I}$. (Also $\emptyset$.)

Proof

- (a) As we said in §3, this predicate Φ requires unbounded subset Selection and an annoying use of Choice.
- (b) Since $X \in \mathcal{X}$ is well powered, the argument is contained within a *set* $\mathbf{Sub}(X)$.
- (c) An $\mathcal{I}$ -indexed diagram defines subobjects $Y_i \rightarrowtail X$.
- (d) By hypothesis, they satisfy $\Phi(Y_i)$, i.e. they have universal maps $\eta_{Y_i} : Y_i \rightarrow RB_i$.
- (e) These define a diagram $B_{(-)} : \mathcal{I} \rightarrow \mathcal{A}$.
- (f) Composing this with the functor $U : \mathcal{A} \rightarrow \mathbf{Set}$ gives a diagram $UB_{(-)} : \mathcal{I} \rightarrow \mathbf{Set}$.
- (g) By Replacement, this has a disjoint union (Grothendieck construction) $\int_i UB_i$.
- (h) Using topos logic, the colimit $S \equiv \text{colim}_i UB_i$ exists in **Set**, as a quotient of $\int_i UB_i$.
- (i) Since $U : \mathcal{A} \rightarrow \mathbf{Set}$ creates colimits of this kind, the colimit $B = \text{colim}_i B_i$ exists in $\mathcal{A}$.
- (j) Then there is a universal map $\eta_Y : Y \rightarrow RB$, which means that $\Phi(Y)$ holds. □

Remark 4.7 I am not clear about the official definitions of Replacement and Collection. I understand one of them requires uniqueness, but uniqueness up to unique isomorphism should be the categorically relevant form. This is likely to be entwined with the Choice issue. Perhaps Mike Shulman's paper [Shu18] throws some light on this.

5 Well founded objects

“Ordinary” type constructors such as $+$, $\times$ and $\rightarrow$ don’t make “big” objects.

We need recursion for interesting results. But recursion may be non-terminating.

So we need some notion of “well-foundedness” [Mir17].

(Some very brief history of well-foundedness may be appropriate here, including Mirimanoff, von Neumann’s recursion theorem and my notion of well founded coalgebra. However, that already takes up quite a lot of space in [Tay21]. Theorem 7.1 later and so this definition of well-foundedness have roots in von Neumann’s 1923/8 derivation of recursion from induction [vN23, vN28].)

Definition 5.1 An object $X \in \mathcal{X}$ is called *well founded* if

- (a) it has a set (object of a topos) $\text{Sub}(X)$ of subobjects $Y \rightharpoonup X$ (Assumption 4.4);
- (b) there is an inflationary monotone function $s : \text{Sub}(X) \rightarrow \text{Sub}(X)$:

$$\text{if } Y \rightharpoonup X \text{ then } Y \rightharpoonup sY \rightharpoonup X$$

- (c) X (considered as its largest subobject $X \rightharpoonup X$) is the unique fixed point:

$$Y = sY \rightharpoonup X \iff Y = X.$$

Definition 5.2 A well founded object X has the *successor property for* Φ if

$$\Phi(Y) \implies \Phi(sY).$$

That is, given universal $\eta_Y : Y \rightarrow RB$, there is also universal $\eta_{sY} : sY \rightarrow RC$.

Well-foundedness and the successor property are typically provable within the logic of an elementary topos.

The new Section 9 of [Tay21] proves this for extensional coalgebras among well founded ones, but there’s a simpler argument in [Tay26a] (in fact one I had had earlier).

Remark 5.3 The single successor s may easily be replaced by a *family* S of them, in which case we require that

- (a) *each* $s \in S$ be monotone and inflationary;
- (b) X be the unique *common* fixed point of them: $(\forall s \in S. Y = sY) \iff Y = X$; and
- (c) *each* of the sY for $s \in S$ have a universal $sY \rightarrow B$.

Maybe these *unary* “successor” operations could be generalised to multiary constructors, even contravariant ones.

6 Patariaia induction

Lemma 6.1 The inflationary monotone endofunctions of any poset form a *directed* poset in the pointwise order, using composition. $\square$

Corollary 6.2 Any dcpo (directed complete partial order) has a greatest inflationary monotone endofunction. $\square$

Remark 6.3 That, really, is as much as Dito Patariaia did [Jib22]. What follows is mine, that I needed to develop for [Tay21].

Definition 6.4 A dcpo D with inflationary monotone endofunction $s : D \rightarrow D$ satisfies *maximality of fixed points* (I previously called this the “special condition”) if

$$\forall x, y \in D. \quad x = sx \leq y \implies x = y.$$

Proposition 6.5 Let D be a dcpo with

- (a) least element $\perp$;
- (b) an inflationary monotone endofunction $s : D \rightarrow D$
- (c) that satisfies maximality of fixed points.

Then D has a top element $\top$ and this is the unique fixed point of s . □

Corollary 6.6 Let Φ be a predicate on such a D satisfying

- (a) $\Phi(\perp)$ holds;
- (b) $\Phi(x) \Rightarrow \Phi(sx)$; and
- (c) for all directed $x_{(-)} : I \rightarrow D$, if $\forall i. \Phi(x_i)$ then $\Phi(\bigvee x_i)$.

Then $\Phi(\top)$ holds. □

7 The Theorem

Theorem 7.1 Every well founded object $X \in \mathcal{X}$ with the successor property has a universal map $\eta_X : X \rightarrow RA$.

Proof Definitions 5.1 and 5.2 were designed for parts (a–d):

- (a) Being well powered means that X has a small preorder $\text{Sub}(X)$ of isomorphism classes of subobjects.
- (b) Since $\mathcal{X}$ has an initial object with directed colimits, $\text{Sub}(X)$ is equivalent to a directed-complete poset (dcpo) with $\perp$.
- (c) That the object X is well founded means that it has a successor operation with maximality of fixed points.
- (d) The successor property for Φ is what is required for induction.
- (e) Likewise the colimit discussion in Section 4 shows that $\Phi(\emptyset)$ holds and Φ is preserved by directed unions.
- (f) Therefore $\Phi(X)$ holds, by Corollary 6.6. □

Corollary 7.2 If all objects of $\mathcal{X}$ are well founded with the successor property then there is an adjoint $L \dashv R$. □

8 Re-constructing intuitionistic Zermelo–Fraenkel

From the above, we have the von Neumann hierarchy. Is there a first order axiomatisation of this and the rank function that does not pre-suppose any form of Replacement?

I don’t see how to define unbounded subset selection in the hierarchy in a natural categorical way. Maybe [Shu18] can help with this.

I feel that this problem is more concerned with the official first order formulation of IZF than about category theory. It may be possible with the first order formulation (ETCS) of categorical

logic. However, I don't subscribe to ETCS as the foundation of categorical logic, so that is for someone else to do.

The following ideas are based on two blog posts by Joel Hamkins [Ham14, Ham18], although those are far longer than is needed.

For Replacement, we are given a functional binary relation $\Theta(x, y)$ (or just a function) between a set I and the universe V (von Neumann hierarchy).

We need to say at what stage in the hierarchy each of the sets in the image of this function occurs. Hamkins has an inordinately long discussion of this, but the answer is simply provided by the rank function. Any set x in the hierarchy satisfies

$$x \in V_{\text{rank}(x)+1},$$

as is easily provable by classical $\in$ -recursion, although I haven't yet seen the easy categorical trick.

Then we form the join

$$\alpha \equiv \bigvee \{\text{rank}(y) + 1 \mid \exists x \in I. \phi(x, y)\}$$

and finally the required Replacement set is

$$\{y \in V_\alpha \mid \exists x \in I. \phi(x, y)\} \in V_{\alpha+1}.$$

So the issue is about being able to form *the join of a set of ordinals*.

In categorical logic we standardly use fibrational methods to define operations on set-indexed families. We could, for example, regard this family as a disjoint union of ordinals, which is a set with a well founded relation, whose extensional reflection is the required join.

Both of these issues concern forming predicates on or functions into the universe, but I don't see how such things arise or could be definable categorically in such generality.

It may be that the Proof Theory of First Order Predicate Calculus shows that any such predicate or functional relation actually corresponds to some construction that could be specified with other tools.

On the other hand, it may be that all known mathematical applications of Replacement can actually be performed using the von Neumann hierarchy.

9 Disagreement with ETCS

If the Adjoint Functor Theorem in this note is indeed equivalent to IZF then this equivalence will have to be formulated using some version of Lawvere's ETCS [Law64], simply for compatibility with the "official" formulation of set theory.

However, if this paper is to become a collaboration, I am keen that it should describe ETCS as *one possible* foundation for categorical logic, not the *definitive* one.

The way that Ernst Zermelo conceived his axiomatisation of set theory [Zer08] was similar to modern type theories. He didn't approve of the re-formulation in first order predicate calculus by Thoralf Skolem in particular

Such a first order formulation pre-supposes a universe of *â priori* anonymous individuals that are then given their roles by the axioms.

Where does this universe come from? This is a "turtles all the way down" problem.

ETCS is similarly formulated in a first order way, except with separate universes of objects and morphisms. The axioms axiomatise the category and then, following the lead of set theory, the type connectives (product, powerset, *etc.*).

It has exactly the same foundational problem.

In contrast, type theories and predicate calculi (lower case, because I don't mean a specific calculus) do not have this problem. They are axiomatised in the same way as primary school arithmetic and Euclidean geometry:

We have some symbols on the page. Whichever system we are using states “rules of the game” for adding new symbols to the page. The motivation is just the same as for Turing machines [Tur35], except that we have free choice of the next step, instead of having just one, dictated by Turing's transition table.

In category theory, *all* possible compositions are supposed to exist in advance: they are not introduced in any particular well founded order. The solution to this problem is to say that the category is generated by the elementary sketch that is defined directly by a type theory [Tay99, Ch. VIII].

So I would say that a topos, for example, is generated by the sketch that is defined by a certain type theory, such as [LS86].

Of course this applies to categories whose structure is equivalent to a type theory, such as pretoposes and cartesian closed categories. The category of groups, on this other hand, is usually understood as super-structure on the category of sets.

Type theories and categorical logic are *built up* by adding connectives that are defined by systems of introduction and elimination rules, which are exactly equivalent to asserting the existence of various adjoints.

I envisage that the Adjoint Functor Theorem in this note will be used in the same way, to provide a menu of adjunctions from which one could build new categorical logics and their equivalent type theories.

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