Ordinals as Coalgebras

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Why should categorists study ordinals?

Friends, Categorists, Colleagues, lend me your ears!
I come to bury the Ordinals, not to praise them!

I would like to encourage my fellow categorists,
not to transcribe set theory into diagrams,
but to re-think the natural algebraic structure
of the recursive situations in which ordinals have been used.

Why should categorists bother with ideas from set theory?
Because they suggest notions of partial structure
whose behaviour could be adapted to complex situations
such as type theories and proof theory.

Today I will tell you about some highlights of the theory.
More details are in the papers and other slides on the webpage.

Free structures that may or may not exist

When they do exist

- they are fixed points of some operation(s);
- we can do induction and recursion over them;
- recursion is the universal property of the left adjoint.

Classically, such operations are said to have rank.

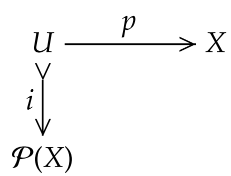
When they don't exist (or are very complicated), the iteration may go on forever, but

- they may have partial sub-structures
- that form a recursive system,
- maybe with other algebraic structure (like arithmetic for ordinals).

Sets ($\in$-structures) as partial algebras

Georg Cantor: there is no set X with $\mathcal{P}(X) \cong X$.
Hence there is no initial algebra $\mathcal{P}(X) \longrightarrow X$ for the covariant powerset functor (Joachim Lambek).

What about algebras whose structure maps are partial functions?



If we concentrate on $p \equiv \mathsf{id}$, this is a coalgebra, $X \longrightarrow \mathcal{P}(X)$,
which is the same as a binary relation $(\prec) \subset X \times X$.

When $X \rightarrowtail \mathcal{P}(X)$ is mono the relation is extensional.

Most of the theory works with functors that preserve monos.

Fixed point theorems

When the free structure exists,
it is the greatest or terminal partial free structure
and is typically a least fixed point.

When all joins and meets exist,
any monotone endofunction has a least fixed point.
This easy result is mis-attributed to Alfred Tarski (1955),
but was well known in the early 1900s.

In fact, joins of chains are enough,
for which the difficult classical proof called the Bourbaki–Witt Theorem (1949) was actually due to Ernst Zermelo (1908).

When the least element and directed joins exist,
there is an easy constructive due to Dito Pataraia (1997),
with improvements by others.

Pataraia's fixed point theorem

Consider inflationary monotone endofunctions of any dcpo.

Composition gives directness,
but the system is also directed complete,
so there is a greatest such endofunction.

When does a poset with a greatest inflationary monotone endofunction have a greatest element?

This works for (*e.g.*)

$$W \equiv \{x \in X \mid x \le sx \;\&\; \forall a.\, sa \le a \Rightarrow x \le a\}$$

the subset of coalgebras with "recursion" for all algebras.

All of this can be done finitarily,
yielding a specific directed diagram
that may or may not have a join.

Pataraia for categories

More complicated but more informative than the poset case.

The construction splits into two (almost) separate parts:

A finitary construction of the category $\mathcal{W}$ of partial solutions
which are coalgebras that are well founded or admit recursion.

The category of pointed endofunctors $\mathcal{W} \to \mathcal{W}$ is
a filtered diagram
that may or may not have a colimit
and so a terminal object $T : \mathcal{W} \to \mathcal{W}$.

With this construction of $\mathcal{W}$, the terminal pointed endofunctor
$T : \mathcal{W} \to \mathcal{W}$ gives the terminal object $T\bot$ of $\mathcal{W}$.

(Please ask me outside the lecture about this construction.)

Some curious features of set theory

From induction for predicates over well founded structures
we derive recursion for functions.

Although the well founded system has no free model,
von Neumann's recursion theorem is a least fixed point,
obtained a union of partial solutions.

Imposing extensionality turns a well founded relation
into a "set" — known as Mostowski's theorem.

Imposing transitivity turns it into an "ordinal" — the rank.

"Sets" have a weird "overlapping" (binary) union,
which is needed to prove transfinite recursion,
i.e. with successors and limits (unions).

This theory sometimes generalises to other functors.

Pretend your category is a topos

Category theory can take a familiar argument using sets
and generalise it to other categories,
by identifying which properties of **Set** it uses.

What are the most important categorical properties of a topos?

No, not Ω!

The relationship between colimits and limits (pullbacks)!

- stable effective quotients by equivalence relations,
- extensivity of coproducts,
- stable parametric recursion over lists.

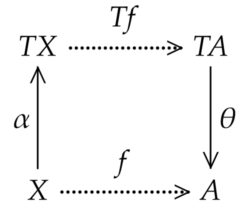
(This is André Joyal's notion of Arithmetic Universe.)

Other categories may or may not have these properties.

Partial structures: recursion

Simply adapt the universal property of the initial algebra to partial algebras.

A coalgebra $\alpha : X \to TX$ admits recursion if for any algebra $\theta : TA \to A$



there is a unique coalgebra to algebra homomorphism.

When $T \equiv \mathcal{P} : \mathbf{Set} \to \mathbf{Set}$ this says that a relation $\prec$ admits recursion if

$$fx \;=\; \theta\{fy \mid y \prec x\}$$

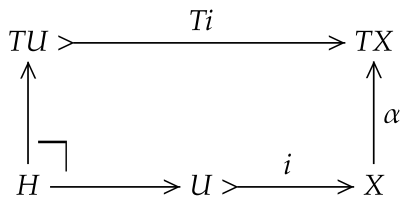
has a unique solution.

Well-foundedness

A relation $\prec$ is well founded if

$$\frac{(\forall y.\, y \prec x \Rightarrow \phi y) \Longrightarrow \phi x}{\forall x.\, \phi x}$$

A coalgebra $\alpha : X \to TX$ for an endofunctor T
is well founded if any pullback



has $i : U \cong X$.

The preorder of extensional well founded coalgebras

For any functor T that preserves monos,
the category of extensional well founded T-coalgebras and coalgebra homomorphisms
is like the "von Neumann hierarchy" in set theory:

- it is a preorder
  (at most one morphism between any two objects);
- the underlying function of that morphism is 1–1;
- $\emptyset$ is the least element;
- there are filtered/directed unions;
- the preorder has binary meets
  like set-theoretic "overlapping" intersection,
  given by "zipping together" the coalgebras.

Before the intersection in Zermelo set theory,
this gave the total order for the classical ordinals,
as Georg Cantor proved (1895).

Binary union for sets

If we also assume that

- the underlying category has "nice" pushouts
  (like those in a pretopos,
  where the pushout square is also a pullback) and
- the functor preserves inverse images

then the preorder of extensional well founded coalgebras
has binary joins that are like set-theoretic "overlapping" union.

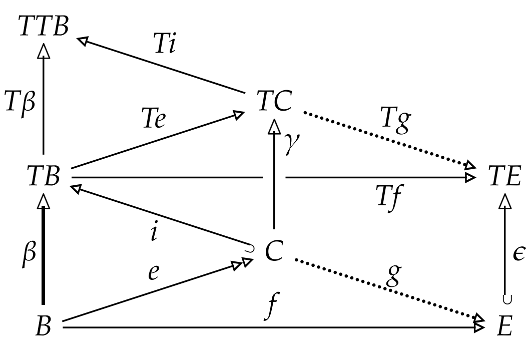
Rank without cardinals

- The functor T has a free algebra iff
- there is a terminal (extensional) well founded coalgebra iff
- the preorder of extensional well founded coalgebras
  is essentially small,
  i.e. equivalent to an internal preorder.

If you insist on using the word cardinal,
the rank of the functor is
the size of the preorder of extensional well founded coalgebras.

Mostowski's theorem

We can turn any (well founded) coalgebra into an extensional
one by repeatedly factorising its structure map:



For iteration of this to converge (using Pataraia's theorem),
the "epis" must

- have the cancellation property and
- be well co-powered.

Intuitionistic ordinals, 1977

A reminder of the history.
Robin Grayson observed that

if we define (thin) ordinals as transitive extensional well
founded relations, they satisfy

$$\beta^+ \in \alpha^+ \iff (\beta^+ \in \alpha \ \vee\ \beta^+ = \alpha) \implies \beta^+ \subseteq \alpha \iff \beta \in \alpha$$

$$\beta^+ \subset \alpha^+ \iff \beta \in \alpha^+ \iff (\beta \in \alpha \ \vee\ \beta = \alpha) \implies \beta \subseteq \alpha,$$

where the thin successor is $\alpha^+ \equiv \alpha \cup \{\alpha\}$.

Here there are two relevant poset orders, $(\subseteq)$ and

$$\beta \preceq \alpha \quad \equiv \quad (\beta \in \alpha \ \vee\ \beta = \alpha).$$

Can we find another notion of ordinal with a single poset order
that makes the $\Rightarrow$ reversible?

Intuitionistic ordinals, 1990s

André Joyal and Ieke Moerdijk, *Algebraic Set Theory* (1994) considered the free algebras for $\bigvee$ and s such that

- no condition: "sets" ($\in$-structures);
- $x \le sx$: thin ordinals;
- $x \le y \Rightarrow sx \le sy$: plump ordinals; or
- $s(x \vee y) = sx \vee sy$: directed ordinals,

where $\le$ is the order defined from $\bigvee$.

(André stresses that the notion of "smallness" using fibrations was the main point of this programme.)

In 1996 I published a symbolic account of these structures, introducing the name plump but with a difficult highly recursive definition.

Both orders, $(\prec)$ and $(\subseteq)$ or $(\preceq)$ are important!

Applying category theory to (systems of) ordinals

How can we apply these categorical methods to understand the behaviour of intuitionistic ordinals? Let's pretend that the category of posets $(X, \sqsubseteq)$ is a topos.

What could possibly go wrong?

Instead of the full powerset $\mathcal{P}$, we use the covariant down-sets functor $\mathcal{D} : \mathbf{Pos} \to \mathbf{Pos}$:

$$\begin{array}{lcl} \mathcal{D}(X, \sqsubseteq) & \equiv & \{U \mid U \subset_{\downarrow} X\} \\ \mathcal{D}fU & \equiv & \{y \in Y \mid \exists x \in U.\, y \sqsubseteq_Y f(x)\} \\ U \subset_{\downarrow} X & \equiv & \forall x, y \in X.\, x \sqsubseteq y \in U \Rightarrow x \in U \end{array}$$

Coalgebras for the lower sets functor

A coalgebra $\alpha : X \to \mathcal{D}X$ in **Pos** has two order relations $(\sqsubseteq)$ and $(\prec)$:

- $(X, \sqsubseteq)$ is the underlying poset and
- $y \prec x \iff y \in \alpha(x)$, as for $\mathcal{P}$.

They must be compatible:

- $z \sqsubseteq y \prec x \Longrightarrow z \prec x$ because $\alpha(x)$ is lower; and
- $z \prec y \sqsubseteq x \Longrightarrow z \prec x$ because α preserves order.

We write (X, α) and $(X, \sqsubseteq, \prec)$ interchangeably.

$(\sqsubseteq)$ could just be $(=)$, so the discrete $(\mathcal{P})$ case is embedded.

But more interesting is (abstract) transitivity:

$$y \prec x \Longrightarrow y \sqsubseteq x \quad \text{which is} \quad \alpha \le \eta_X : X \rightrightarrows \mathcal{D}X$$

Lots of different kinds of monos!

There are (at least) three factorisation systems on **Pos**:

$\mathcal{I}$ monos	1–1 on points, need not reflect order;
${}^{\perp}\mathcal{I}$ regepis	image generates the order on target;
$\mathcal{R}$ regmonos	inclusions with the induced order;
${}^{\perp}\mathcal{R}$ epis	onto on points;
$\mathcal{L}$ lower	lower subset inclusions;
${}^{\perp}\mathcal{L}$ cofinal	$\forall y.\, \exists x.\, y \sqsubseteq fx$.

Can we use these in place of 1–1 and onto functions in the theorems about extensionality?

Properties of the factorisation systems

Pretending that **Pos** is a topos,
how well can we use the factorisation systems
like 1–1 and onto functions?

There are lots of facts and fallacies to check:

	$\mathcal{I}$	${}^{\perp}\mathcal{I}$	$\mathcal{R}$	${}^{\perp}\mathcal{R}$	$\mathcal{L}$	${}^{\perp}\mathcal{L}$
$\mathcal{D}$ preserves	N		Y		Y	
inverse images exist	Y		Y		Y	
$\mathcal{D}$ pres inv image	N		N		Y	
cancellation	Y	Y	Y	Y	Y	N
well (co)powered	Y	Y	Y	Y	Y	N
nice pushouts	N		N		Y	

So for each of $\mathcal{I}$, $\mathcal{R}$ and $\mathcal{L}$, some things work, others don't.

(In fact there are lots more than this!)

Extensionality for $\mathcal{L}$: Plump Ordinals

From that table, $\mathcal{L}$ (lower inclusions) looks like the best bet.

A coalgebra $(X, \sqsubseteq, \prec)$ is $\mathcal{L}$-extensional,
i.e. its structure map $\alpha : X \to DX$ is in the class $\mathcal{L}$, iff

every subset $U \subset X$ that is
- $(\prec)$-bounded above, $\exists y \in X.\ \forall u \in U.\ u \prec y$, and
- $(\sqsubseteq)$-lower, $\forall y \in X.\ \forall u \in U.\ y \sqsubseteq u \Longrightarrow y \in U$

is represented by some unique $x \in X$: $U = \{u \mid u \prec x\}$.

Then a plump ordinal is an $\mathcal{L}$-extensional $\mathcal{D}$-coalgebra.

This is much simpler than the 1996 symbolic definition,
because we have treated the two relations independently.

Pretending that **Pos** is a topos

Since lower subsets behave nicely as "monos",
much of the theory transfers from sets to posets.

In particular there are "nice" pushouts,
giving "nice" binary joins of plump ordinals.

This is crucial to proving the transfinite recursion theorem,
i.e. the universal property with arbitrary (small) joins and
monotone successor.

Plump rank

But the "epis" corresponding to lower inclusions
are cofinal monotone functions.

They are far from being surjective.

Our "Mostowski" construction does not converge.

In fact, constructing plump $\omega \cdot 2$
in the simplest non-classical presheaf topos $\mathbf{Set}^{\rightarrow}$
requires the Axiom of Replacement.

Conversely, could transfinite iteration of functors
give a categorical replacement for Replacement?

Extensionality for $\mathcal{R}$

Next try $\mathcal{R}$, order-full inclusions.

A coalgebra $(X, \sqsubseteq, \prec)$ is $\mathcal{R}$-extensional iff

$$\forall yz. \quad \big(\forall x.\, x \prec y \Longrightarrow x \prec z\big) \iff (y \sqsubseteq z).$$

so $(\sqsubseteq)$ is set-theoretic inclusion, renamed $(\subseteq)$.

But for compatibility we require meta-transitivity:

$$\forall w, x, y. \quad (\forall z.\, z \prec y \Rightarrow z \prec x) \ \land \ (x \prec w) \Longrightarrow (y \prec w).$$

Any well founded meta-transitive relation is transitive in the usual sense, but not conversely.

So this looks a bit like the popular definition of ordinal as a transitive, extensional well founded relation.

Are $\mathcal{R}$-extensional ordinals better behaved?

Yes:
the "epis" are surjective monotone functions,
which obey cancellation and are well co-powered,
so iterated factorisation converges
and there is a rank functor.
That is, the inclusion of $\mathcal{R}$-extensional ordinals in all well founded $\mathcal{D}$-coalgebras has a reflection (left adjoint).

No:
pushouts in $\mathcal{R}$ and therefore unions of $\mathcal{R}$-ordinals
are badly behaved.
I cannot prove a transfinite recursion theorem for them
(whatever I use for the successor).

What happened to thin ordinals?

The notion of ordinals that is

- the simplest when defined symbolically (transitive extensional well founded relations)
- is the most difficult in this categorical setting.

This is because (plain) extensionality
is defined using (plain) monos,
which are not preserved by the down-sets functor.

It seems to be necessary to treat transitivity and extensionality separately, with transitivity first.

First, let's give more details about the category of $\mathcal{D}$-coalgebras over **Pos**.

Coalgebra homomorphisms

Recall that a function $f : (Y, \prec_Y) \to (X, \prec_X)$ is
a $\mathcal{P}$-coalgebra homomorphism iff it's a bisimulation

$$\forall x{:}X.\ \forall y{:}Y. \quad x \prec_X fy \iff \exists y'{:}Y.\ x = fy' \ \land \ y' \prec_Y y.$$

A function $f : (Y, \sqsubseteq_Y, \prec_Y) \to (X, \sqsubseteq_X, \prec_X)$ is
a $\mathcal{D}$-coalgebra homomorphism iff instead

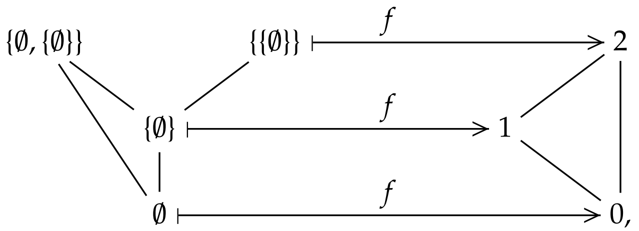
$$\forall x{:}X.\ \forall y{:}Y. \quad x \prec_X fy \iff \exists y'{:}Y.\ x \sqsubseteq_X fy' \ \land \ y' \prec_Y y$$

and $$\forall yy'{:}Y. \quad y' \sqsubseteq_Y y \Longrightarrow fy' \sqsubseteq_X fy.$$

There is no forgetful functor from
$\mathcal{D}$-coalgebra homomorphisms to $\mathcal{P}$-coalgebra homomorphisms.

A $\mathcal{D}$- but not $\mathcal{P}$-homomorphism

The rank function of the von Neumann hierarchy:



To make this a $\mathcal{D}$-homomorphism, we need $0 \sqsubseteq 1$.

This (counter)example arises over and over again in the theory.

The two coalgebras are $\mathcal{I}$-extensional,
but the function f between them is not in $\mathcal{I}$ (1–1),
indeed it's a split epi,
so (well founded) $\mathcal{I}$-extensional $\mathcal{D}$-coalgebras
don't form a preorder like the von Neumann hierarchy.

On the other hand, this is the universal way of
making the $\mathcal{D}$-coalgebra on the left transitive.

Thin ordinals as $\mathcal{D}$-coalgebras

What is the poset order on a thin ordinal?

There are two candidates for this.

They are a special case of "sets"
(extensional well founded $\mathcal{P}$-coalgebras),
which are embedded in $\mathcal{D}$-coalgebras
with the discrete order (=).

Alternatively, they can be made into $\mathcal{D}$-coalgebras directly,
using the thin order

$$\beta \preceq \alpha \quad \equiv \quad (\beta \prec \alpha \ \vee \ \beta = \alpha).$$

Such $\mathcal{D}$-coalgebras are (abstractly) transitive:

$$y \prec x \Longrightarrow y \sqsubseteq x.$$

We need this in order to make the rank function a
$\mathcal{D}$-homomorphism.

Thin ordinals as $\mathcal{D}$-coalgebras

Isn't the thin order

$$\beta \preceq \alpha \quad \equiv \quad (\beta \prec \alpha \ \vee \ \beta = \alpha)$$

just classical recidivism?

No, because every $\mathcal{D}$-homomorphism

$$f : (Y, \preceq_X, \prec_X) \to (X, \preceq_X, \prec_X)$$

is actually a $\mathcal{P}$-homomorphism if X is well founded.

Moreover, it's a lower inclusion (in $\mathcal{L}$),
even though the structure map α is not in $\mathcal{L}$.

Because of this, thin ordinals have nice unions
and transfinite recursion using the one-point successor,
which satisfies $x \preceq sx$ but not $y \preceq x \Rightarrow sy \preceq sx$.

The whole category of $\mathcal{D}$-coalgebras

(Preparing you for a scary diagram on the next slide.)

In the setting of the category of $\mathcal{D}$-coalgebras,
the various categories of interest,

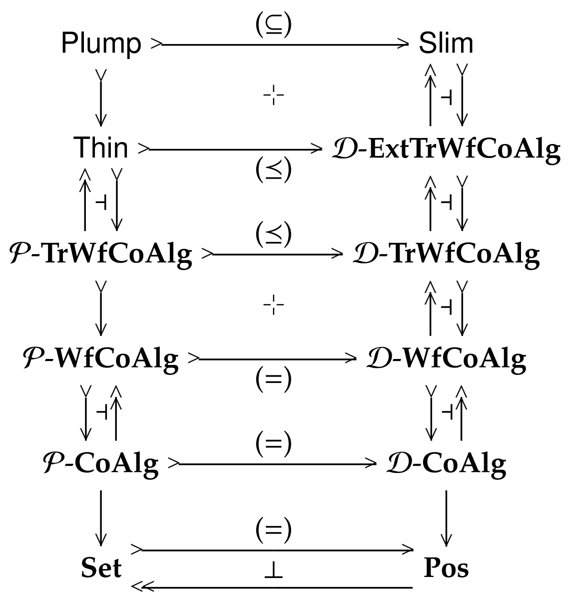
- well founded relations,
- extensional well founded relations ("sets"),
- thin ordinals,
- $\mathcal{R}$-extensional ordinals,
- plump ordinals,
- *etc.*

form full subcategories.

The "Mostowski" and "rank" constructions are reflection
functors into these.

The poset order is essential to allow "sets" and "ordinals" to
live within the same bigger category.

Summarising the categories



Some missing categorical techniques

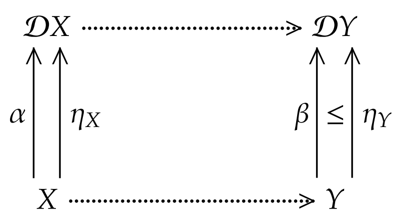
(Finally I am addressing my abstract!)

Since $\mathcal{D}$ does not preserve the class $\mathcal{I}$ of plain monos, some parts of the theory of extensional (well founded) coalgebras don't work, whilst other parts work more awkwardly.

Iterated factorisation does work, but as with the general application of Pataraia's theorem we have to cut down to the "well founded" elements. Since these are epis, we call them well projected. There is an exercise in order theory to characterise them, but I can't see what this characterisation is.

A more interesting question for 2-categorists

The transitive closure may be found by traditional methods. But what is its universal property?



It's not a co-inserter but something more complicated. When I asked a senior 2-categorist at CT23 he didn't know.

Where do we go from here?

This theory could be applied to many other categories, functors and factorisation systems.

"Plump" ordinals for binary semilattices (with $\vee$ but not necessarily $\bot$) give transfinite recursion with $s(x \vee y) = sx \vee sy$.

Polynomial functors instead of powersets should give more combinatorial notions of ordinal like those that proof theorists use.

What are "sub-structures" for

- algebraic theories with equations?
- Cartesian closed categories?
- more powerful type theories?

The systems of these ought to give "intrinsic" notions of ordinals for such theories.