A Categorical Replacement for Replacement

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The story beyond elementary toposes

It's 55 years since Lawvere and Tierney introduced elementary toposes as the categorical formulation of *bounded* Zermelo set theory.

Yet set theorists considered Ernst Zermelo's axioms to be incomplete, so Abraham Fraenkel added the Axiom Scheme of Replacement in 1922.

Do Category Theory and Mathematics in general (outside set theory) need this extra axiom (scheme)?

If so, how can we express it in the native language of category theory?

Category Theory does not need set-theoretic foundations and we do not intend to parrot set theory.

However, we will take and generalise some of its ideas, just as category theory has taken them from other subjects.

Why do we need it?

To make sense of a functor $F : \mathcal{I}^{\mathrm{op}} \longrightarrow \mathbf{Set}$ and turn it into a discrete fibration $P : C \longrightarrow \mathcal{I}$.

This transformation is the essence of the statement of Replacement in the "official" formulation of ZF set theory.

(But we could just use discrete fibrations directly to do sheaf theory.)

Infinitely (or transfinitely) iterated functors, such as in homological algebra.

The gluing construction (logical relations) and proofs of consistency or normalisation.

Please tell me (privately) if you know of other constructions that require Replacement (but not Universes).

Canonical example from set theory

Replacement is interesting because it builds skyscrapers from plans on the ground.

We do not assume Universes because that is like dropping building materials from a satellite.

The von Neumann hierarchy is the transfinite iteration of the covariant powerset functor. In particular,

$$V_{\omega\cdot 2} \quad \cong \quad \bigcup_{n\in\mathbb{N}} \mathcal{P}^n(\mathbb{N})$$

is (enough to construct) an elementary topos with $\mathbb{N}$, but cannot be constructed in elementary topos theory.

I will construct transfinite iterations of functors from my categorical proposal in a future lecture.

Today's lecture

Today I will just explain my proposal and

- how it uses a categorical generalisation of some set theoretic ideas,
- why ZF needs unbounded predicates for both Separation (Comprehension) and Replacement, and
- how to use Replacement.

Lots of mathematicians claim to use ZF,
but I have never seen one (outside set theory)
actually invoke Replacement as officially defined.
(Do you know any texts that do this?)

In fact the presheaf/fibration/colimit example does it.

But you know me — this lecture will still be
about category theory not set theory!

Big unions and families

The von Neumann hierarchy and transfinite iteration of functors *suggest* that the issue with Replacement is the existence of big unions.

Indeed, forming the colimit of a diagram $\mathcal{I} \to \mathbf{Set}$ will be a part of the story.

However, the presheaf/fibration example shows that, before invoking the colimit of a diagram, we have to construct the diagram itself.

The native language of category theory

The constructors in the language of
an elementary topos or other type theories,

such as (co)products, (co)equalisers, exponentials, Ω,
dependent products, tensor products, ...

are defined by universal properties, also known as
adjunctions, Kan extensions and by many other names.

These are formally equivalent to systems of
introduction, elimination, β and η rules in type theories.

Arguably the notion of adjunction
is the most important one in category theory.

Asserting new adjunctions

So if we want to postulate a new connective,

we specify its properties in the existing system, *e.g.*

$$X \times (-) \longrightarrow (=).$$

Then we assert that this functor have an adjoint,

$$(-) \longrightarrow (=)^X.$$

(*Nota bene il congiuntivo in inglese!*)

This is just the same as the way that
type theorists build up their systems of rules
from the simplest to the most powerful.

General adjoint functor theorem

Of course for a functor
to have a left adjoint it must preserve limits, and
to have a right adjoint it must preserve colimits.

That's good enough to obtain adjoint functions
between complete (semi)lattices,
but not for categories.

Is there some extra condition that guarantees it?

Please tell me if you know of any example
where an adjoint is given by some form of
General Adjoint Functor Theorem
but not by direct construction.

(I suspect that the GAFT is useless.)

Colimits and Replacement

If there is a left adjoint $F : \mathcal{X} \to \mathcal{A}$ then it must
preserve colimits and so the relevant colimits must exist.

Therefore, given any colimit diagram $X = \mathsf{colim} X_i$ in $\mathcal{X}$,
its image $FX = \mathsf{colim} FX_i$ must also exist in $\mathcal{A}$.

Replacement provides "set-indexed" colimits in **Set**,
as explained on the next slide

We assume that the categories we're considering
have faithful functors $\mathcal{A} \to \mathbf{Set}$ that create colimits.

So that's fine, so long as we can express
objects $X \in \mathcal{X}$ as colimits of "simpler" ones.

So finding the diagram (X_i) is the issue.

Directed rather than general colimits seem to be enough.

Replacement and set-indexed colimits in **Set**

A functor $F : \mathcal{I}^{\mathsf{op}} \longrightarrow \mathbf{Set}$

from a small (*i.e.* internal) category
directly provides data for Replacement.

Officially, ZF is defined as an untyped first order theory
with two relation symbols ($\in$ and $=$) satisfying certain axioms.

Replacement says that,
given a set u and a functional relation $\phi(x, y)$,

$$v \quad \equiv \quad \{y \mid \exists x \in u.\, \phi(x, y)\} \qquad \text{is a set.}$$

Applying this to the object and morphism parts of the functor F
we obtain the objects and morphisms of the "total category" $\mathcal{C}$
of the discrete fibration $P : \mathcal{C} \to \mathcal{I}$.

There is some easy category theory to fill in the details
of the construction of the discrete fibration
and from that the colimit, $\mathsf{colim}_i F(i)$.

Let's take some risks with the adjoints

We still need to explain where the diagrams come from.

How is each object expressed as a colimit of "simpler" ones
(without regressing into an "inductive conception of sets")?

The interesting and powerful case of an adjoint is
when it needs to be constructed by recursion.

The corresponding type-theoretic introduction rule
has a term of the new type as a premise,
for example adding a new element to an existing list.

But any recursion carries the risk of non-termination.

Strengthening a logic carries the risk of contradiction.

But nothing ventured, nothing gained!

Well-foundedness and extensional reflection

We handle the issues on the previous slide by working in a category of well founded structures.

Also, instead of a general adjoint situation, we consider a full replete subcategory and ask for a reflection functor (left adjoint to inclusion).

Maybe staying inside a subcategory is a tamer situation? — Not at all!

We will define our subcategory in terms of a generalisation of the axiom of extensionality.

Isn't that the most the most innocent of Zermelo's axioms? Again, not at all!

(I understand that set theorists have long been aware that Extensionality has a very powerful influence.)

Algebras and coalgebras for a functor

Well founded coalgebras for a functor $T : \mathcal{X} \to \mathcal{X}$ are approximations to the initial T-algebra.

Except that well founded coalgebras still exist even when there is no initial algebra.

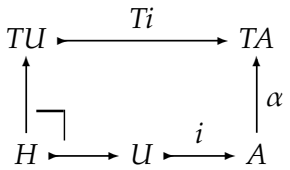
Leading example: $T \equiv \mathcal{P}$, the covariant powerset.

An initial algebra would have $\mathcal{P}A \cong A$, which is impossible.

A coalgebra $\alpha : A \to \mathcal{P}A \equiv \Omega^A$ is a binary relation on A, $\tilde{\alpha} : A \times A \to \Omega$ or $(\prec) \subset A \times A$.

Well founded coalgebras

A coalgebra $\alpha : A \to TA$ is well founded if: for any subobject $i : U \rightarrowtail A$ such that the pullback H factors through U,



in fact $i : U \cong A$.

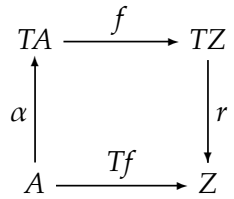
In the case of $T \equiv \mathcal{P}$, this happens (for all U) iff the relation $(\prec)$ corresponding to α obeys the induction scheme

$$\frac{\forall y. \quad (\forall z.\, z \prec y \Rightarrow \phi z) \Longrightarrow \phi y}{\forall x.\, \phi x}$$

for all predicates ϕ on A.

Well founded recursion

Under suitable conditions, any well founded coalgebra obeys the recursion scheme, *i.e.* for any algebra $r : TZ \to Z$ there is a unique coalgebra-to-algebra homomorphism



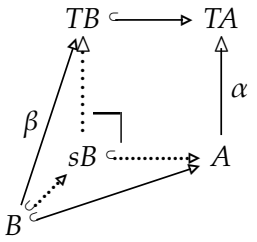
So this is the sense in which well founded coalgebras are approximations to the initial algebra.

The proof is in my paper on *Well founded coalgebras and recursion*.

The next two slides sketch the general idiom of induction.

Induction on sub-coalgebras

The proof is based on
Johann von Neumann's recursion theorem (1923, 1928),
which forms the union of attempts defined on sub-coalgebras.
There is a successor operation on them defined like this:



The coalgebra $A \xrightarrow{\alpha} TA$ is well founded iff the successor operation has a unique fixed point, namely A itself.

It also needs to be well powered, *i.e.* the sub-coalgebras are indexed by a set (object of a topos).
This can be formalised using fibred categories.

The order-theoretic fixed point theorem

In fact the system of sub-coalgebras

- has a least element $\emptyset$,
- has directed unions $\bigcup$,
- has an inflationary monotone endofunction s,
- such that $sB = B \subset C \implies B = C$.

Induction and recursion are valid
intuitionistically and without using ordinals.
The key idea was due to Dito Pataraia
(but significant development of it was done be me).

For recursion targeted at an algebra and several other results,
it is valid in the logic of an elementary topos
or *bounded* Zermelo set theory.

However, for our main theorem about the existence of an adjoint, we preform induction with a predicate that involves unbounded quantification.

Extensionality

The less-than-innocent first axiom of Zermelo set theory:

A binary relation $(\prec)$ is extensional if it satisfies

$$\frac{\forall z.\quad z \prec x \iff z \prec y}{x = y}$$

for all $x, y \in A$.

This is equivalent to $\alpha : A \rightarrowtail \mathcal{P}A$ being 1–1.

We generalise this to any functor $T : \mathcal{X} \to \mathcal{X}$ on any category by saying that α is mono
(and we go beyond this on the next slide).

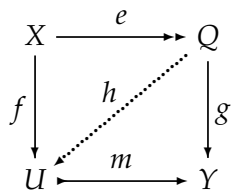
The initial algebra is
a (the terminal) extensional well founded coalgebra.

Factorisation systems

At last we come to the principal innovation.

We can replace the ordinary monos in the definition of extensionality by $\mathcal{M}$-maps of a factorisation system.

Recall that this has two classes $\mathcal{E}$ and $\mathcal{M}$
that each contain all isomorphisms,
such that every map factorises as $e\,;m$
and the classes are orthogonal in this sense:



There is a unique map h making both triangles commute.

For our application, the class $\mathcal{M}$ must also be preserved by the functor T and directed unions.

Extensional well founded coalgebras are like sets

Even with general functors and factorisation systems,

- there is at most one homomorphism
  between any two extensional well founded coalgebras;
- this homomorphism is in $\mathcal{M}$;
- there are directed unions;
- there are binary intersections, just like for sets;
- there are overlapping unions, just like for sets,
  so long as the base category has well behaved colimits.

So these properties of set-theoretic inclusion *etc.*,
which are really weird from a categorical point of view,
naturally arise from the general situation that we are considering.

See *Well founded coalgebras and recursion*.

Why use general factorisation systems?

We want to build skyscrapers.

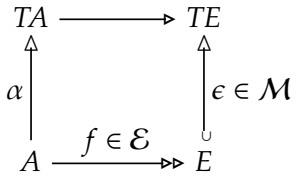
We imagine $\mathcal{M}$-maps to be inclusions that are
closed under rich structure.

Then the $\mathcal{E}$-maps that are orthogonal to these
(where the two form a Galois connection, varying contravariantly)
need only a few generators.

This is probably related to dilators,
although I don't understand how proof theorists use those
or how to define them constructively and categorically.

Characterising the extensional reflection

It's exactly a coalgebra homomorphism like this:



[⇒] If some homomorphism has the universal property,
factorise it as $f = e\,;m$,
then the intermediate coalgebra E' is extensional too,
so there's a homomorphism $n : E \rightarrowtail E'$ that splits m.

[⇐] Let $g : A \twoheadrightarrow F$ be another homomorphism
to an extensional coalgebra, wlog $g \in \mathcal{E}$;
then f and g factor through the intersection $E \cap F$,
so $E \cong E \cap F \cong F$. This is not trivial!

The unbounded predicate

The extensional reflection is an $\mathcal{E}$-homomorphism $A \twoheadrightarrow E$.

But, by the previous slide, it *suffices* that there be
any homomorphism $A \twoheadrightarrow E'$ to any extensional coalgebra.

So if A has one then so does TA, since T preserves $\mathcal{M}$.

Also, if a subcoalgebra $B \rightarrowtail A$ has one,
so does its successor $sB \rightarrowtail A$, since $sB \twoheadrightarrow TB$.

But this existential predicate on $\mathsf{Sub}(A)$
is not defined in the language of an elementary topos,
which is equivalent to *bounded* Zermelo set theory.

It's an unbounded predicate.

The induction

Now we can put these ideas together to prove that any well founded T-coalgebra $\alpha : A \to TA$ has an extensional reflection $A \longrightarrow\!\!\!\!\!\rightarrow E$ with respect to any factorisation system $(\mathcal{E}, \mathcal{M})$.

The subcoalgebras of A form a dcpo
with least element and inflationary monotone successor.

(We will discuss the colimits/joins on the next slide.)

We have an unbounded predicate,
so we're working in ZF, not an elementary topos.

$\emptyset$, successor and directed unions preserve this predicate.

So, by (my adaptation of) Pataraia's fixed point theorem,
the predicate holds for the greatest subcoalgebra,
namely the given coalgebra A itself.

What colimit diagrams have we used?

Replacement gives set-indexed colimits in **Set**,
but that is vague until we say for which diagrams.

The well powered hypothesis says that
$\mathsf{Sub}(A)$, the preorder of sub-coalgebras of A, is a set.

Pataraia's observation (that others should have made)
was that composition shows that
the poset of inflationary monotone endofunctions is directed.

It is for this specific diagram that we require colimits

- in **Set**, given by Replacement;
- inherited by the working category $\mathcal{X}$, by hypothesis;
- created in the categories of
  coalgebras, well founded ones, extensional ones,
  sub-coalgebras and endofunctions of them,
  by categorical calculation.

Example: plump ordinals

Let T : **Pos** $\to$ **Pos** be the lower sets functor,
$\mathcal{M}$ be the class of lower inclusions
and $\mathcal{E}$ be the class of cofinal functions.

Then the extensional well founded coalgebras
are the plump ordinals.

However, in the simplest non-classical topos $\mathbf{Set}^{\to}$
(presheaves of classical sets on $\bullet \to \bullet$)
plump $\omega \cdot 2$ is like $V_{\omega \cdot 2}$:
it is "too big" to be definable in an elementary topos.

That's all, folks

I will show how to do other categorical constructions,
such as transfinite iteration of functors like V_α,
in a future lecture.

My papers and slides on this subject are at
`www.paultaylor.eu/ordinals/`

My translations of various related historical papers are at
`www.paultaylor.eu/trans/`