On the Notion of Finite Set

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Abstract: The purpose of this note is to introduce a definition of a finite set and to demonstrate its equivalence with the definition given by Wacław Sierpiński.

W. Sierpiński has given a new definition of a finite set. This definition is distinguished essentially by the fact that it depends on neither the notion of natural number nor on the general notion of function, which usually enters into the definitions making use of the notion of correspondence.

The definition in question is the following:

> Consider the classes K of sets each of which satisfies the following conditions:
>
> 1. every set containing a single element belongs to the class K,
> 2. if A and B are two sets belonging to the class K their set-sum $A+B$ also belongs to K.
>
> We call a set *finite* if it belongs to all of the classes K that satisfy these conditions.

As is known, the set of all objects (if it exists) enjoys some paradoxical properties: contrary to a known theorem of G. Cantor, the power (cardinality) of this set would not be less than that of the class of all of its subsets. The same applies to the class formed from all the sets containing a single element; therefore, the classes K don't satisfy the theorem of Cantor. Taking account of this fact, one could also put in doubt the existence of the classes K.

Modifying Sierpiński's definition in order to avoid this inconvenience, we obtain the following definition:

> The set M is *finite* so long as the class of all of its (non-empty) subsets is the unique class satisfying the conditions:
>
> 1. its elements are (non-empty) subsets of M;
> 2. every set containing a single element of M belongs to this class;
> 3. if A and B are two sets belonging to this class, their set-sum $A+B$ also belongs to it.

Now we will demonstrate that a finite set according to this definition is also one in the ordinal sense, and conversely. In other words, for a set to be finite according to the definition proposed, it is necessary and sufficient that the number of its elements can be expressed by a natural number. (We will suppose that the notion of a natural number is understood.)

Indeed, let M be a set of which the number of elements can be expressed by a natural number; let Z be any class satisfying conditions 1–3. We will show that every subset of M belongs to Z. This is the case — by condition 2 — of the subsets consisting of a single element; in the same way, if it holds for the subsets with n elements, then it is also true of those that contain $n+1$ of them. Since the number of elements of each subset of M can be expressed by a natural number, it follows from this by induction that Z contains all of the subsets of M. Thus, since the class Z is necessarily identical to that of of all of the subsets of M, it is the unique class satisfying conditions 1–3. Therefore, every set of which the number of elements can be expressed by a natural number is a finite set in our sense.

Suppose, on the other hand, that the number of elements of a given set M cannot be expressed by a natural number. We write Z for the class of all of the subsets of M of which the number of elements can be expressed by a natural number. This class plainly satisfies the conditions 1–3; at the same time, by hypothesis, M doesn't belong to Z and it follows that Z is not the class of all of the subsets of M; hence, the class of all the subsets of M is not the unique class satisfying the conditions 1–3 and M is not finite in our sense. QED

The original paper, *Sur la notion d'ensemble fini*, appeared in 1920 on pages 129–131 of the first volume of *Fundamenta Mathematicae*, DOI: 10.4064/fm-1-1-129-131.

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The opening reference to Wacław Sierpiński is to *L'axiome de M. Zermelo et son rôle dans la Théories des Ensembles et l'Analyse* (Zermelo's Axiom, *i.e.* the Axiom of Choice, and its role in Set Theory and Analysis) in Bull. de l'Acad. des Scinces de Cracovie, 1918, page 106.

The full name of this journal is apparently *le Bulletin international de l'Académie des Sciences de Cracovie, Classe des sciences mathématiques et naturelles, Série A, Sciences mathématiques*, although I cannot locate it on the Web.

Kuratowski-finiteness is still in use, a century later, in the *constructive* logic of an elementary topos, without the need for excluded middle or a natural numbers object, although nowadays we would include the empty set.

So, an object X of a topos is *Kuratowski finite* if the smallest semilattice $K(X) \subset \mathcal{P}(X)$, that is, generated from $\emptyset$, singletons and binary unions, has $X \in K(X)$.

As Kuratowski says, any set that is enumerable by a natural number is finite in his sense. However, the converse depends on excluded middle and having a natural numbers object.

The key issue constructively is that *equality need not be decidable.* Therefore an object X may be Kuratowski finite, with $X = \{a, b\}$, where the truth-value of "$a = b$" need not be either $\bot$ or $\top$.

We may define an object to be *finite*, still without using a natural numbers object, if it is Kuratowski-finite and has decidable equality ($x = y \vee x \neq y$).

Given any Kuratowski-finite object, it is decidable whether it is empty or inhabited.